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ABSTRACT

The first part of Course III focuses on matrix algebra, graphs and functions, and combinatorics. Topics studied include: matrices and transformations, the solution of systems of linear equations, matrix multiplication, matrix inversion and a field of 2 x 2 matrices. The section on graphs and functions considers regions of the plane and translations, functions and solution of equations, operations on functions, and bounded functions and asymptotes. The chapter on combinatorics discusses such topics as a counting principle and permutations, the binomial theorem, and mathematical induction. (FL)

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Secondary School Mathematics

Curriculum Improvement Study

UNIFIED MODERN MATHEMATICS

COURSE III

PART I

Secondary School Mathematics

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UNIFIED MODERN MATHEMATICS

COURSE III

PART I

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Chapter 1

INTRODUCTION TO MATRICES

1.1 What is a Matrix?

Some things have an amazing number of uses. The wheel, for instance, reduces the force needed to move an ancient man's cart or a modern man's automobile; it is used as a steering wheel, in the gear system of a machine, in a roulette wheel, and so on. In mathematics, matrices also have many uses. First recognized and used 100 years ago by a British mathematician, Arthur Cayley (1821-1895), today they are useful to physicists, biologists, economists, agronomists, sociologists, psychologists, and many others.

Our complex society requires many numerical records. For instance, a manufacturing concern has three plants each making electronic equipment. The equipment requires 4 distinct parts called A, B, C, and D. Factory I uses 30 A-parts, 43 B-parts, 37 C-parts, and 16 D-parts daily; Factory II uses 25 A-parts, 15 B-parts, 30 C-parts and 12 D-parts daily; Factory III uses 61 A-parts, 50 B-parts, 55 C-parts and 30 D-parts daily. It is difficult to remember these data or compare them when presented in this manner. However, if we write them in a rectangular table, we obtain a compact summary of all the data.

		Factory		
		I	II	III
Part	A	30	25	61
	B	43	15	50
	C	37	30	55
	D	16	12	30

$$\begin{bmatrix} 30 & 25 & 61 \\ 43 & 15 & 50 \\ 37 & 30 & 55 \\ 16 & 12 & 30 \end{bmatrix}$$

Figure 1.1

If we separate the rectangular table from the headings and place it in brackets we obtain a matrix.

<u>Definition</u>. A <u>matrix</u> is a rectangular table of numbers arranged in rows (horizontal alignments) and columns (vertical alignments).

The matrix in Figure 1.1 has 4 rows and 3 columns. We say it has <u>dimension</u> 4 x 3, read "four by three." The number of rows is always given first in stating the dimension. The first row is the top row; the first column is at the left. The names of the factories and of the parts, given at the left in Figure 1.1, are not a part of the matrix; they merely describe the numbers which make up the matrix.

For a second example of a matrix, let us consider the problem of a traffic manager for a company with factories in Bridgeport, Conn., Newport, R.I., Salem, Mass., and Brattleboro, Vt. He must know the distance between any pair of factories. A chart provides him with easy access to the data (see Figure 1.2).

	Bridgeport	Newport	Salem	Brattleboro
Bridgeport	0	71	171	115
Newport	71	0	85	135
Salem	171	85	0	104
Brattleboro	115	135	104	0

$$\begin{bmatrix} 0 & 71 & 171 & 115 \\ 71 & 0 & 85 & 135 \\ 171 & 85 & 0 & 104 \\ 115 & 135 & 104 & 0 \end{bmatrix}$$

Figure 1.2

As you see, the matrix of this chart has dimension 4×4. The number in the first row, first column is 0. There is also a 0 in the second row, second column; also in the third row, third column; also in the fourth row, fourth column. This last statement can be abbreviated if we say: The number in the $\underline{i}$th row, $\underline{i}$th column is 0 for $i = 1,2,3,4$. Another interesting feature of this matrix is the fact that the number in the first row, second column and the number in the second row, first column is the same number (71). We can abbreviate this statement too, if we let a_{12} represent the number in the first row, second column, and a_{21} represent the number in the second row, first column, by saying $a_{12} = a_{21}$. Using similar representation we note: $a_{13} = a_{31}$, $a_{14} = a_{41}$, $a_{23} = a_{32}$, $a_{24} = a_{42}$, $a_{34} = a_{43}$. In fact we can abbreviate all these statements still further by writing $a_{ii} = 0$ for $i = 1,2,3,4$ and $a_{ij} = a_{ji}$ for $i,j = 1,2,3,4$.

For a third example of a matrix you are reminded of a table, used in Course I (see Figure 1.3), for the operational system $(Z_4, \cdot)$. (It is called a "Cayley Table," named after Arthur Cayley who first used the name "matrix.")

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

Figure 1.3

Like the preceding matrix this is a square matrix having the same number of rows as columns. We say it has order 4.

1.2 Exercises

1. The Department of Labor reported the following table to show, in percents, the educational level of workers in various occupations, for 1966.

	None	Elem.	H.S.	College	Graduate
Professional and technical	0	1.4	21.5	49.0	28.0
Farmers and farm managers	1.1	52.6	39.9	6.0	.3
Managers, except farms	.2	12.6	49.8	32.8	4.7
Clerical	.1	5.5	73.3	20.4	.8
Sales	.2	11.4	61.6	25.3	1.4
Craftsmen and foremen	.2	26.5	64.4	8.6	.3
Operatives	.8	32.7	61.6	4.6	.2
Service	.8	33.5	57.8	7.7	.2
Farm laborers and foremen	5.1	50.9	39.3	4.7	0
Laborers, except farm and mine	1.9	44.0	49.7	4.1	.2

(a) What are the dimensions of the matrix of this table?

(b) What is a_{35}? a_{53}?

(c) What is the set $\{a_{ij} : i = j, j \leq 5\}$? Compare this with the set $\{a_{ii} : i \leq 5\}$.

(d) What is the greatest entry in the first row? What does it signify?

(e) What is the greatest entry in the first column? What does it signify?

(f) What are the greatest and least numbers in the fifth row? What do they signify?

(g) What are the greatest and least numbers in the fifth column? What do they signify?

2. (a) Obtain an example of a matrix that appears in a newspaper or another similar source. What are the dimensions of the matrix?

(b) Does the stock market report that appears daily in a newspaper contain a matrix? Support your answer.

3. Study the table below, which lists the continents (except Antarctica).

	Area in sq. mi.	Highest point in ft.	Lowest point in ft.*	Highest temp. in degrees F	Lowest temp. in degrees F
Asia	16,900,000	29,028	-1,296	127.1	-89.9
Africa	11,500,000	19,340	-436	136.0	-11.4
N. America	8,440,000	20,320	-228	134.0	-81.0
S. America	6,800,000	22,834	-131	120.0	-27.4
Europe	3,750,000	18,481	-92	122.0	-67.0
Australia	2,945,000	7,316	-39	127.5	-8.0

* The negative numbers in this column indicate "below sea level" measures

(a) What are the dimensions of the matrix of this table?

(b) Let a_{ij} be the number in the ith row, jth column. Find a_{12}, a_{21}, a_{45}.

(c) List the set of numbers $\{a_{ij} : i = 2, j \leq 5\}$

(d) List the set of numbers $\{a_{ii} : i \leq 5\}$

(e) List the set of numbers $\{a_{ij} : i = j + 1, j \leq 5\}$

(f) List the set of numbers $\{a_{ij} : j = i + 1$ for all possible values of $i\}$

(g) List the set of numbers $\{a_{ij} : i = 2j, j \leq 3\}$

1.3 Using Matrices to Describe Complex Situations

This is an important use for matrices as our examples will show. For our first example we take what is called a pay-off matrix, used in Game Theory. Suppose Joe and Pete play a game in which each tosses a coin. They agree on the following rules: if both coins fall heads, Joe pays Pete 3 cents; if both fall tails, Joe pays Pete 4 cents; if Pete's coin falls heads and Joe's coin falls tails, Pete pays Joe 2 cents; finally if Pete's coin falls tails and Joe's coin falls heads then Pete pays Joe 5 cents.

Indeed, for some, these rules may be bewildering. How much clearer they become when organized as a matrix (see Figure 1.4).

$$\begin{array}{ccc|cc} & & & \multicolumn{2}{c}{\text{Joe}} \\ & & & H & T \\ \hline \text{Pete} & H & & 3 & -2 \\ & T & & -5 & 4 \end{array} \qquad\qquad \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$$

Figure 1.4

The numbers in this matrix tell how much Pete receives. If the number is positive he gains; if negative he "receives" a negative amount, which of course means he loses and forfeits an amount to Joe.

For our second example consider four cities, A, B, C, and D that are connected, if at all, by two-way bus routes, as shown in Figure 1.5.

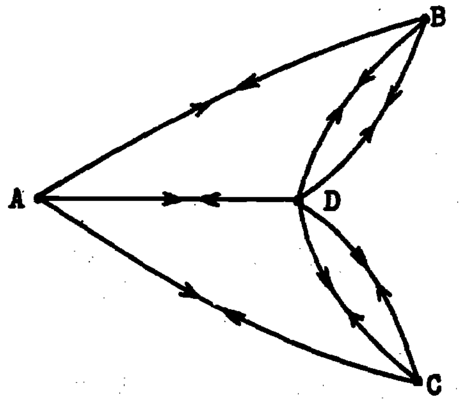


Figure 1.5

As you see there are three bus routes out of A, one of them to B, one to D, and one to C. Out of B there are three routes, two of them to D, the other to A. A complete description of this network of routes may overwhelm some readers. How much clearer to put the description in matrix form (see Figure 1.6).

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 1 & 1 \\ B & 1 & 0 & 0 & 2 \\ C & 1 & 0 & 0 & 2 \\ D & 1 & 2 & 2 & 0 \end{array} \qquad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$$

Figure 1.6

In this matrix we write a "1" for a_{12} to show one bus route between A and B; we write a "2" in a_{42} to show two bus routes between D and B, and so on. For all i, $a_{ii} = 0$ to show no bus routes between a town and itself. Note that $a_{ij} = a_{ji}$ for all i and j. Why is this so?

Our third example shows how to use matrices to describe a pair of linear equations in two variables. Suppose the pair of equations is:

$$\begin{aligned} 3x + 2y &= 8 \\ 4x - y &= -2 \end{aligned}$$

If we detach the coefficients from x and y, leaving each in its position we get a <u>coefficient matrix</u>, namely $\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$.

The brackets about the coefficients are there to denote a matrix. This matrix, you see, is written without an explanatory column or row. These are omitted on the agreement that the first row displays the coefficients of the first equation and the second row displays those of the second equation, while the first column displays the coefficients of x and the second those of y.

If we write a third column giving the constants that appear at the right of the equal sign, then we obtain a 2x3 matrix (see Figure 1.7).

$$\begin{bmatrix} 3 & 2 & 8 \\ 4 & -1 & -2 \end{bmatrix}$$

Figure 1.7

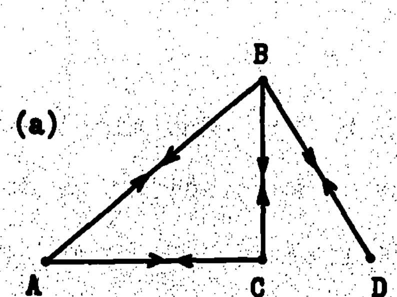
1.4 Exercises

1. Three people, A,B, and C play a game. By the rules, if A beats B, A gets 40 cents from B; if A beats C, A gets 30 cents from C; if B beats A, B gets 35 cents from A; if B beats C, B gets 25 cents from C; if C beats A, C gets 38 cents from A; if C beats B, C gets 32 cents from B. Display these pay-offs as a matrix in which the winner is read at the left of each row, and the loser at the top of each column. For a_{ii} write 0. Is $a_{ij} = a_{ji}$ for any values of i or j?

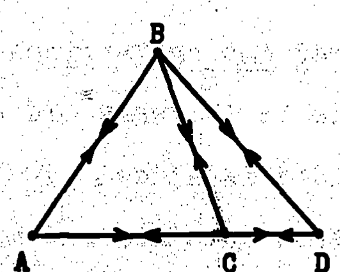
2. A and B play a game in which each rolls a single die (having six faces showing numerals 1,2,3,4,5,6). If the sum of the numbers appearing on the top faces is even B pays A that number of dollars. If the sum is odd, then A pays that number of dollars to B. Using positive and negative numbers display these pay-offs in a 6 x 6 matrix. (See the first example in Section 1.3 for a suggestion.)

3. The diagrams below represent two way bus routes connecting towns A,B,C, and D. Describe, in a 4 x 4 matrix, the number of routes between each pair of towns.

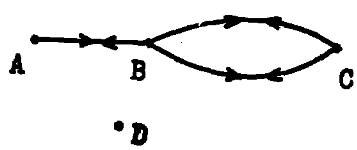
(a)



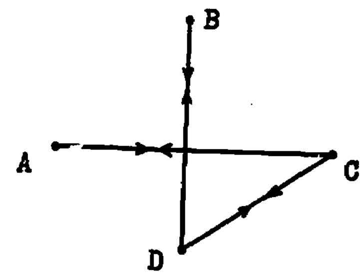
(b)



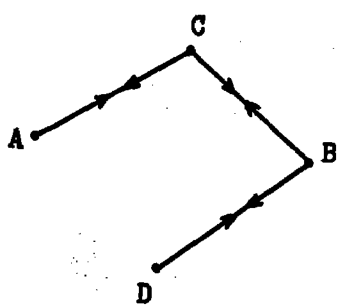
(c)



(d)



(e)



4. For each set of equations listed below write a matrix of coefficients and constants.

(a) $3x + 5y = 8$

$4x - 2y = 0$

(b) $3x + 2y - z = 3$

$2x - 3y + z = 5$

(c) $2x + 3y = 4$

$x + 2y = 8$

$x - 4y = -4$

(d) $x + y + z = 3$

$x + y = 2$

$y + z = 1$

(e) $2x - y = 5$

(f) $2x - 4y + z = 8$

1.5 Operations on Matrices

Alice is 13 years old and Daniel is 10. One might ask: How much older is Alice? The operation of subtraction is designed to answer this question and many others like it. In fact, the purpose of operations on numbers is to gain knowledge about number situations over and beyond what the numbers themselves tell.

So too with matrices and operations on matrices. In this section we see, using an example, how three operations involving matrices are designed to give information beyond that given by each matrix. This example is about a man who contracts to build two models of homes that we call A and B. He operates in three towns, Huntington, Smithtown, and Merrick. Matrix P, shown in Figure 1.8, tells how many homes of each model he built in each town in 1966. Matrix Q tells the same story for 1967.

$$
\begin{array}{l}
\\
\\
\text{Huntington} \\
\text{Smithtown} \\
\text{Merrick}
\end{array}
\quad
\begin{array}{c}
1966 \\
\begin{array}{cc} A & B \end{array} \\
\begin{bmatrix} 8 & 3 \\ 4 & 5 \\ 3 & 3 \end{bmatrix} \\
\text{Matrix P}
\end{array}
\qquad\qquad
\begin{array}{c}
1967 \\
\begin{array}{cc} A & B \end{array} \\
\begin{bmatrix} 6 & 3 \\ 2 & 7 \\ 4 & 3 \end{bmatrix} \\
\text{Matrix Q}
\end{array}
$$

Figure 1.8

One might ask: How many of each model home, in each town, did he build in both years? To find the answer it is natural to add the entries in P and Q that occupy corresponding places, and to write the answer in the same space of a third matrix which we call R in Figure 1.9.

$$
\begin{array}{ccccccccc}
\begin{bmatrix} 8 & 3 \\ 4 & 5 \\ 3 & 3 \end{bmatrix} & + & \begin{bmatrix} 6 & 3 \\ 2 & 7 \\ 4 & 3 \end{bmatrix} & = & \begin{bmatrix} 8+6 & 3+3 \\ 4+2 & 5+7 \\ 3+4 & 3+3 \end{bmatrix} & = & \begin{bmatrix} 14 & 6 \\ 6 & 12 \\ 7 & 6 \end{bmatrix} \\
P & + & Q & = & R & = & R
\end{array}
$$

Figure 1.9

Addition of Matrices

It is important to note that P, Q, and R have the same dimensions, namely 3 x 2. Matrices can be added only when they have the same number of rows and the same number of columns, that is the same dimensions.

A second question might be asked. How many of each model should the man build in 1968, in each town, to double his 1966 production? It is natural, in answering this question, to double each number in P. And it is also natural to call this newly formed matrix 2P. We illustrate with Figure 1.10.

$$2P = 2\begin{bmatrix} 8 & 3 \\ 4 & 5 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 16 & 6 \\ 8 & 10 \\ 6 & 6 \end{bmatrix}$$

Figure 1.10

Multiplication of a Matrix by a Scalar

Note we have multiplied a matrix by another kind of object, the real number 2. In this context we call the real number a <u>scalar</u>, and the operation is called <u>multiplying a matrix by a scalar</u>.

Continuing our example, suppose the model A home requires 6 doors and 8 windows, while the model B home requires 5 doors and 7 windows. This information can be easily displayed in tabular form (see Figure 1.11). We call the matrix of this table S.

	Doors	Windows
A	6	8
B	5	7

$$S = \begin{bmatrix} 6 & 8 \\ 5 & 7 \end{bmatrix}$$

Figure 1.11

A third question is: How many doors did the man use in each town in 1967, and how many windows? To obtain the answer we use:

$$Q = \begin{bmatrix} 6 & 3 \\ 2 & 7 \\ 4 & 3 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 6 & 8 \\ 5 & 7 \end{bmatrix}$$

A common sense way to find the answer is to calculate as follows: For Huntington we need $6 \cdot 6$ doors for model A and $3 \cdot 5$ doors for model B, making a total of $6 \cdot 6 + 3 \cdot 5$ or 51 doors. For Smithtown we need $2 \cdot 6$ for model A and $7 \cdot 5$ doors for model B, making a total of $2 \cdot 6 + 7 \cdot 5$ or 47 doors. For Merrick we need $4 \cdot 6 + 3 \cdot 5$ or a total of 39 doors.

A similar calculation finds the numbers of windows.

For Huntington we need $6 \cdot 8 + 3 \cdot 7$ or 69 windows.

For Smithtown we need $2 \cdot 8 + 7 \cdot 7$ or 65 windows.

For Merrick we need $4 \cdot 8 + 3 \cdot 7$ or 53 windows.

Putting these results together in matrix form, we get

$$\begin{array}{cc} & \begin{array}{cc} D & W \end{array} \\ \begin{array}{c} H \\ S \\ M \end{array} & \begin{bmatrix} 51 & 69 \\ 47 & 65 \\ 39 & 53 \end{bmatrix} \end{array} \qquad \text{We call this matrix T.}$$

These calculations involve multiplications and additions on scalars. But we regard the entire calculation as our matrix operation, called <u>multiplication on matrices</u>. The operation, without explanation, is shown in Figure 1.12.

$$\begin{bmatrix} 6 & 3 \\ 2 & 7 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 & 8 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 6\cdot 6 + 3\cdot 5 & 6\cdot 8 + 3\cdot 7 \\ 2\cdot 6 + 7\cdot 5 & 2\cdot 8 + 7\cdot 7 \\ 4\cdot 6 + 3\cdot 5 & 4\cdot 8 + 3\cdot 7 \end{bmatrix} = \begin{bmatrix} 51 & 69 \\ 47 & 65 \\ 39 & 53 \end{bmatrix}$$

$$Q \cdot S = T = T$$

Figure 1.12

This matrix multiplication is possible because the number of columns in Q is the same as the number of rows in S, and the product matrix T has as many rows as Q and as many columns as S. Multiplying matrices may seen strange and complicated. With experience it becomes familiar and easy. This will happen sooner if you see a pattern in the operation. Study the three partial multiplications in Figure 1.13 and try to find that pattern.

$$\begin{array}{ccccc} R & & S & & T \\ \begin{bmatrix} 6 & 3 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} & \cdot & \begin{bmatrix} 6 & \cdot \\ 5 & \cdot \end{bmatrix} & = & \begin{bmatrix} 6\cdot 6+3\cdot 5 & \cdot \\ & \cdot \\ & \cdot \end{bmatrix} \end{array}$$

first row first column first row, first column entry

$$\begin{bmatrix} 6 & 3 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \cdot \begin{bmatrix} \cdot & 8 \\ \cdot & 7 \end{bmatrix} = \begin{bmatrix} \cdot & 6\cdot 8+3\cdot 7 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

first row second column first row, second column entry

$$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} \cdot & 8 \\ \cdot & 7 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & 4\cdot 8+3\cdot 7 \end{bmatrix}$$

third row second column third row, second column entry

Figure 1.13

In general, to find the entry in the product matrix for the $\underline{i}$th row, $\underline{j}$th column, multiply in pairs, the first number in the $\underline{i}$th row of the first matrix and the first number in the $\underline{j}$th column of the second matrix, do the same for the second numbers, the third numbers, and so on. Then add these products.

$$\begin{array}{l|l} & \text{Cost} \\ \hline \text{Door} & 8 \\ \text{Window} & 10 \end{array} \qquad\qquad C = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

Figure 1.14

Continuing our example, suppose doors cost $8 each and windows cost $10 each. This information can be displayed in a 2x1 matrix, shown in Figure 1.14 and named C. We ask another question: What, in 1967, for each town, was the total cost of doors and windows? It is a happy fact that the three answers are found by the matrix multiplication illustrated as Figure 1.15.

$$\begin{bmatrix} 51 & 69 \\ 47 & 65 \\ 39 & 53 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 51 \cdot 8 + 69 \cdot 10 \\ 47 \cdot 8 + 65 \cdot 10 \\ 39 \cdot 8 + 53 \cdot 10 \end{bmatrix} = \begin{array}{c} \text{Cost} \\ \begin{bmatrix} 1098 \\ 1026 \\ 842 \end{bmatrix} \end{array} \begin{array}{c} \\ H \\ S \\ M \end{array}$$

$$T \quad \cdot \quad C \quad = \quad D \quad = \quad D$$

Figure 1.15

Total Cost Matrix D

Note the dimension of each matrix in this multiplication.

T: 3x2 , C: 2x1 , D: 3x1.

The essential features concerning dimensions are:

(1) The number of columns of T is equal to the number of rows of C.

(2) The dimensions of D are the number of rows of T and the number of columns of C.

1.6 Exercises

1. A man builds 3 model homes A,B, and C, in two towns P and Q. His contruction program for two years is given below and the associated matrices are named D and E.

$$\begin{array}{cc} & 1967 \\ & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} P \\ Q \end{array} & \begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & 1968 \\ & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} P \\ Q \end{array} & \begin{bmatrix} 2 & 1 & 2 \\ 3 & 5 & 0 \end{bmatrix} \end{array}$$

Matrix D Matrix E

This need for doors and windows for each model is given by Matrix F,

$$\begin{array}{cc} & \begin{array}{cc} \text{Door} & \text{Window} \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 4 & 5 \\ 5 & 6 \\ 5 & 7 \end{bmatrix} \end{array}$$

Matrix F

and the cost in dollars of doors and windows are given in Matrix G.

$$\begin{array}{cc} & \text{Cost} \\ \begin{array}{c} D \\ W \end{array} & \begin{bmatrix} 5 \\ 6 \end{bmatrix} \end{array}$$

Matrix G

(a) Give the dimensions of each of the Matrices D,E,F,G.

(b) Using an operation on matrices, find the number of each model home built in both years in each town.

(c) Can one add D and F? Explain.

(d) Using an operation on matrices find how many doors and how many windows were used in each town in 1967.

(e) Can one multiply D and E? Explain.

(f) Interpret the meaning of E·F, of F·G.

(g) Interpret the meaning of (E·F)·G and express it as a single matrix. Do the same for E·(F·G)

(h) Find the matrix that displays the 1969 construction program for each town if the 1969 program is three times the 1967 program.

2. If possible, add. If not explain why not.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 6 & 8 \end{bmatrix} + \begin{bmatrix} -3 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} a & b \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 2 \\ 4 & 5 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ -4 & -5 \\ 2 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. If possible multiply. If not possible explain why not.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(g) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ (h) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

(i) $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 6 & 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (j) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(k) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (l) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

4. If possible express each of the following as a single matrix. If not possible explain why not.

(a) $4\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ (b) $\frac{1}{2}\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$ (c) $0\begin{bmatrix} 2 & 0 & 4 \\ -2 & 0 & -4 \end{bmatrix}$

(d) $2\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} + 3\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ (e) $3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(f) $a\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ (g) $3\begin{bmatrix} 2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 5 & 6 \end{bmatrix}$

(h) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a \\ c \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (i) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

5. Express $\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$ as a single matrix.

Do you think the product will be the same if matrices are commuted? Try it and see.

6. Is the product $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ the same as the product $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$? Try it and see.

7. Find $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot A$ if A is the matrix:

(a) $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

8. Using the data in Exercise 7 find $A \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for each A.

1.7 Matrices and Coded Messages

A simple way to code a message is to substitute for each letter in the message a numeral, as given, for instance, in Figure 1.16.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26.

Figure 1.16

Thus the message GOOD LUCK would be sent as 7-15-15-4 12-21-3-11. The recipient of the message then decodes using the inverse substitution in Figure 1.16. An outsider can easily decode a message of this type by noting the frequency of numerals. One would expect, in general, the most frequent numeral to correspond to E, the next frequent numeral to T, and so on. To make it more difficult for an outsider to decode a message one can use a <u>coding matrix</u> in conjunction with the substitution transformation described above. After using the substitution determined by Figure 1.16 the numerals are arranged in 2x2 matrices. For GOOD LUCK this gives

$$\begin{bmatrix} 7 & 15 \\ 15 & 4 \end{bmatrix} \qquad \begin{bmatrix} 12 & 21 \\ 3 & 11 \end{bmatrix}.$$

Then we introduce a coding matrix, say $C = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, multiplying (on the right) each matrix in the message by C.

$$\begin{bmatrix} 7 & 15 \\ 15 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 + 15 & 21 + 30 \\ 30 + 4 & 45 + 8 \end{bmatrix} = \begin{bmatrix} 29 & 51 \\ 34 & 53 \end{bmatrix}.$$

$$\begin{bmatrix} 12 & 21 \\ 3 & 11 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 24 + 21 & 36 + 42 \\ 6 + 11 & 9 + 22 \end{bmatrix} = \begin{bmatrix} 45 & 78 \\ 17 & 31 \end{bmatrix}.$$

The coded message is 29-51-34-53 45-78-17-31. The recipient of this message has the problem of decoding it. First he restores the matrices and then multiplies each restored matrix by a <u>decoding</u> matrix, which in this case is $D = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

The process then is the following:

$$\begin{bmatrix} 29 & 51 \\ 34 & 53 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 58 + (-51) & (-87) + 102 \\ 68 + (-53) & (-102) + 106 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 15 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 45 & 78 \\ 17 & 31 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 90 + (-78) & (-135)+156 \\ 34 + (-31) & (-51)+ 62 \end{bmatrix} = \begin{bmatrix} 12 & 21 \\ 3 & 11 \end{bmatrix}$$

Finally the inverse substitution, according to Figure 1.16, reveals the message GOOD LUCK.

The choice of coding and decoding matrices involves some mathematics that we will consider in Chapter 3.

1.8 Exercises

1. Using the coding method described above with the coding matrix $C = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, code each of the following messages:
   (a) COME HOME
   (b) WHERE ARE YOU. (Group as follows: WHER|EARE|YOUX. The X fills the empty space in the last 2x2 matrix).
2. Using the decoding matrix $D = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ decode the following messages.
   (a) 58-97-27-53 25-49-27-53
   (b) 30-51-52-89 35-65-51-87
3. In this exercise we describe a method for solving a pair of linear equations in two variables whose coefficient matrix is C, the coding matrix in Exercise 1. The equations are

$$2x + 3y = 12$$
$$x + 2y = 7$$

Multiply $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ 7 \end{bmatrix}$, where the first matrix is the decoding matrix that decodes messages coded by C, and the second matrix consists of the constants in the equations. The product matrix $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ tells us $x = 3$, $y = 2$. Check. In the next chapter we shall explain some of the mathematics involved in this solution. Using this method solve each of the following pairs of equations and check.

(a) $2x + 3y = 5$, $x + 2y = 4$ (b) $2x + 3y = -5$, $x + 2y = -2$ (c) $2x + 3y = 12$, $x + 2y = 6$

(d) $2x + 3y = 0$, $x + 2y = 5$ (e) $2x + 3y = 0$, $x + 2y = 0$

4. In this exercise the coefficient matrix is the decoding matrix of Exercise 2. What matrix do you think should be used to solve each of the following equations? Solve and check.

(a) $2x - 3y = 5$, $-x + 2y = -2$ (b) $2x - 3y = 7$, $-x + 2y = -2$ (c) $2x - 3y = 4$, $-x + 2y = -2$

5. Note that in coding the message GOOD LUCK in Section 1.7 the first O became 51 and the second O became 34. (One letter became two different numbers.) Can you write a message in which two different letters, coded in this fashion, become the same number?

6. The recipient of the secret message in 2(a) did his multiplication with the decoding matrix at the left (instead of at the right). Did the message get through?

1.9 Matrices and Transformations

In Course II we used coordinate rules in connection with transformations of the plane onto itself. To help you recall them we list some of them and one or two others in Figure 1.17. The last column gives the coordinate rule in convenient matrix form. We are using a rectangular coordinate system with origin at O. The matrix gives both x and y coefficients. When these are missing in the coordinate rule we give the missing variable the coefficient 0. For example, in the first row of Figure 1.17 the rule $\begin{matrix} x' = x \\ y' = -y \end{matrix}$ may be written $\begin{matrix} x' = 1x + 0y \\ y' = 0x - 1y \end{matrix}$

1.	R_x	Reflection in x-axis.	$\left\{\begin{matrix} x' = x \\ y' = -y \end{matrix}\right\}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
2.	R_y	Reflection in y-axis.	$\left\{\begin{matrix} x' = -x \\ y' = y \end{matrix}\right\}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
3.	H_O	Half-turn about O.	$\left\{\begin{matrix} x' = -x \\ y' = -y \end{matrix}\right\}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
4.	D_3	Dilation with center O and scale factor 3.	$\left\{\begin{matrix} x' = 3x \\ y' = 3y \end{matrix}\right\}$	$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$
5.	r_{90}	Rotation about O through 90°.	$\left\{\begin{matrix} x' = -y \\ y' = x \end{matrix}\right\}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
6.	R_ℓ	Reflection in ℓ, the line with equation y=x.	$\left\{\begin{matrix} x' = y \\ y' = x \end{matrix}\right\}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Figure 1.17

Now consider the image of (3,5) under R_x. We can use the matrix of R_x if we write (3,5) as the 2x1 matrix $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

A multiplication yields the image as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

and $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ is interpreted as the point (3,-5). Why this works will be explained in Chapter 3.

For another example let us find the image of (3,-2) under r_{90}. The calculation is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$ or (2,3). Check by

plotting (3,-2) and (2,3) and see if the results are reasonable.

Now suppose we are to find the image of (4,-1) under a composition of D_3 followed by H_o. The computation is:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ -3 \end{bmatrix} = \begin{bmatrix} -12 \\ 3 \end{bmatrix}$$

The image is (-12,3).

Did you wonder whether we could have multiplied the first two matrices first, and then this product with the third matrix? Let us try.

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 3 \end{bmatrix}$$

Again the answer is (-12,3). It would seem that multiplication on matrices is associative. We consider this question further in Chapter 3. Meanwhile, it seems that the composition of the dilation D_3 followed by the half-turn H_o can be effected by a single matrix that is found by multiplying matrices in the correct order.

This discussion suggests that 2x2 matrices may determine other transformations. Let us investigate $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, by noting how it maps O(0,0), B(0,1), A(1,0) and D(1,1).

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(See Figure 1.18.) In general $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a + b \\ b \end{bmatrix}$.

We write this: $x' = x + y$, $y' = y$.

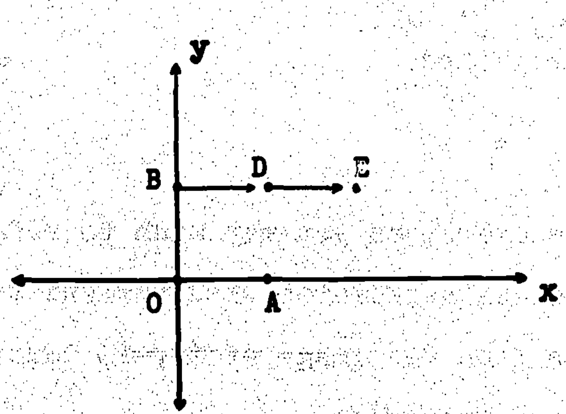


Figure 1.18

This means that the y-coordinate of a point is unchanged in the image, while the x-coordinate is increased by the y-coordinate. This is a new transformation to us, it is an example of a <u>shear</u>. It maps the square OADB onto the parallelogram OAED.

Using matrices we can compose the shear, with matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ followed by r_{90}, whose matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. A multiplication of their matrices, in correct order, gives the matrix of the composition.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

If we reverse the order of the matrices we get

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

We end this section with the observation that $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ may be regarded as $3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and also that $\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ may be regarded as $-1\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This shows how the operations on matrices are intertwined.

1.10 Exercises

In doing these exercises you may wish to refer to Figure 1.17 to recall matrices associated with various transformations.

1. Using matrices find the image of (3,-2) under each of the following transformations:

   (a) R_x (b) R_y (c) R_ℓ, where ℓ has equation $y = x$

   (d) r_{90} (e) H_O (f) D_3

2. Using matrices find the image of (-2,0) under each of the compositions listed below. (ℓ has the equation $y = x$.)

(a) $R_x \circ H_O$ (b) $R_\ell \circ H_O$ (c) $r_{90} \circ H_O$

(d) $H_O \circ r_{90}$ (e) $D_4 \circ R_\ell$ (f) $R_\ell \circ D_4$

3. Express with a single matrix the matrix of each of the following composition:

(a) The shear with matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ followed by H_O.

(b) H_O followed by the shear in (a).

(c) D_{-2} followed by the shear in (a). (D_{-2} is the dilation with center O and scale factor -2.)

(d) The shear in (a) followed by D_{-2}.

(e) D_{-2} followed by D_{-2}.

4. Find the images of (0,0), (1,0), (0,1), and (1,1), under each of the transformations whose matrix is given below. Then, if you think you have sufficient information, describe the transformation.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (g) $2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

5. Determine whether or not the mapping with matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is a transformation. (Hint: Find the images of (3,2) and (2,3).)

6. Investigate (describe) the transformation whose matrix is $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

7. Investigate the transformation with matrix $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

*8. Investigate (describe) the transformation whose matrix is the coding matrix $C = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, and the transformation whose matrix is the decoding matrix $D = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. Then describe the transformations whose matrices are the products C•D and D•C.

9. We have confined ourselves in this section to plane transformations whose matrices are 2x2. Suggest a kind of matrix that might be used for space transformations.

1.11 Transition Matrices

As the word "transition" implies, transition matrices describe how a set of circumstances change from one state to another. As an example of a transition let us consider people moving from a city to its suburbs and back, and as this happens population totals change. Suppose, to keep the example simple, we disregard numbers of deaths and births and assume that, in one year, 90% of city people stay in the city and 10% move to the suburbs, while 20% of suburb people move back to the city and 80% remain there. This data is conveniently displayed in Figure 1.19

		To	
		City	Suburb
From	City	.9	.1
	Suburb	.2	.8

Figure 1.19

Transition Matrix

Further suppose that at the end of 1963 the city had a population of 5 million and that the suburbs had a population of 2 million. We can calculate what the population should be at the end of 1964 as follows.

From city to city	.9x5,000,000	=	4,500,000
From suburb to city	.2x2,000,000	=	400,000
Total to city			4,900,000
From city to suburbs	.1x5,000,000	=	500,000
From suburb to suburb	.8x2,000,000	=	1,600,000
Total to suburb			2,100,000

As you no doubt recognize this calculation resembles what happens when we multiply matrices. We try then to set up a product of two matrices that calls for this calculation. After some effort we hit on

$$\underset{\text{1963 population}}{\begin{array}{cc} \text{City} & \text{Suburb} \\ \end{array} \atop \begin{bmatrix} 5{,}000{,}000 & 2{,}000{,}000 \end{bmatrix}} \cdot \underset{\text{transition matrix}}{\begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix}} = \underset{\text{1964 population}}{\begin{array}{cc} \text{City} & \text{Suburb} \\ \end{array} \atop \begin{bmatrix} 4{,}900{,}000 & 2{,}100{,}000 \end{bmatrix}}$$

You might try $\begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix} \cdot \begin{bmatrix} 5{,}000{,}000 \\ 2{,}000{,}000 \end{bmatrix}$ to see that this product does not yield the desired result.

Since the procedure for calculating the population at the end of 1965 (on the basis of the same assumptions) is no different we can write

$$\underset{\text{1964 population}}{\begin{array}{cc} \text{City} & \text{Suburb} \\ \end{array} \atop \begin{bmatrix} 4{,}900{,}000 & 2{,}100{,}000 \end{bmatrix}} \cdot \underset{\text{transition matrix}}{\begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix}} = \underset{\text{1965 population}}{\begin{array}{cc} \text{City} & \text{Suburb} \\ \end{array} \atop \begin{bmatrix} 4{,}830{,}000 & 2{,}170{,}000 \end{bmatrix}}$$

We might also have written this product as follows.

$$\begin{array}{cc} \text{City} & \text{Suburb} \\ \end{array}$$
$$\begin{bmatrix} 5{,}000{,}000 & 2{,}000{,}000 \end{bmatrix} \cdot \begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix} \cdot \begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix} = \begin{bmatrix} 4{,}830{,}000 & 2{,}170{,}000 \end{bmatrix}$$
$$\begin{array}{cc} \text{City} & \text{Suburb} \\ \end{array}$$

1963 population transition matrix 1965 population

Assuming multiplication of matrices is associative, we have a choice of two neighboring matrices in the left member. If we group the two transition matrices, then we can write

$$\begin{array}{cc} \text{City} & \text{Suburb} \\ \end{array}$$
$$\begin{bmatrix} 5{,}000{,}000 & 2{,}000{,}000 \end{bmatrix} \cdot \left(\begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix} \right)^2 = \begin{bmatrix} 4{,}830{,}000 & 2{,}170{,}000 \end{bmatrix}$$
$$\begin{array}{cc} \text{City} & \text{Suburb} \\ \end{array}$$

1963 population transition matrix 1965 population

and $\left(\begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix} \right)^2$ can be interpreted as a transition matrix for a 2 year period. Do you see how this can be extended for a 3 year period? Or an n year period?

As you see, the changes in population during the second year were not as great as those of the first year. In fact, the changes are less and less, and the population tends to become stable.

For a second example of a transition matrix consider some water in a closed tank and the water vapor that naturally comes from it. Assume that 2% of the water evaporates in one hour while 1% of the vapor condenses to water. Study the transition matrix in Figure 1.20 to see how this data is displayed in matrix form. You are asked in exercises to use this matrix to calculate amounts of water and vapor at the end of hourly periods.

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{\text{To}} \\ & & \text{water} & \text{vapor} \\ \hline \text{From} & \text{water} & .98 & .02 \\ & \text{vapor} & .01 & .99 \end{array} \qquad \begin{bmatrix} .98 & .02 \\ .01 & .99 \end{bmatrix}$$

Transition Matrix

Figure 1.20

1.12 <u>Exercises</u>

1. (a) Using the transition matrix $\begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix}$ calculate the 1966 population if the 1965 city population was 4,830,000 and the suburb population was 2,170,000.

   (b) Compare the changes in the city population for the years 1963, 1964, 1965, 1966. Also for the suburb population. Explain how these changes seem to indicate that the population in each place tends to become stable.

2. Using the transition matrix $\begin{bmatrix} .98 & .02 \\ .01 & .99 \end{bmatrix}$ for water-vapor states, and starting with 100 units of water and 0 units of vapor (when the units are suitably chosen) calculate the amounts of water and vapor at the end of (a) one hour. (b) two hours. (c) three hours.

3. Using your data found in Exercise 2, discuss the question of a stability between water and vapor.

4. (a) What is the single transition matrix that determines population changes over a two year period, for the situation discussed in Section 1.11?

   (b) What is the single transition matrix that describes the water-vapor changes in a 2 hour period in Section 1.11?

5. Without supplying the details, explain how you would go about finding the transition matrix needed to find the 1962 population from the 1963 population.

1.13 Summary

In this chapter we discussed

1. the prevalence of matrices as they occur in charts and tables of numbers.
2. how matrices can be used to display clearly a set of complex data such as pay-offs and bus route networks.
3. how matrices can be used to code and decode messages, and how they can handle some problems of the builder of homes, and related economic problems.
4. how matrices help in the study of plane transformations, and in solving a pair of linear equations in two variables.
5. how matrices can be used to describe transition from one state to another.

In the course of this discussion three operations on matrices were introduced, namely, addition, multiplication by a scalar, and multiplication of matrices. This raised a number of mathematical questions concerning the properties of these operations. Some answers will be suggested in Chapter 3.

1.14 Review Exercises

1. Express as a single matrix A·B and B·A for each pair of matrices listed below.

(a) $A = \begin{bmatrix} 3 & 8 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ 2 & 6 \end{bmatrix}$,

(b) $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$,

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$,

(d) $A = \begin{bmatrix} 1 & 0 \\ -10 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 5 \end{bmatrix}$.

2. For each pair of matrices listed in Exercise 1 express, as a single matrix, A+B and B+A.

3. Express as a single matrix 2A+2B, when A and B are the matrices in Exercise 1(a).

4. Express as a single matrix.

(a) $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ -6 & -4 \end{bmatrix}$

5. (a) Using $C = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ as a coding matrix and the substitution mapping of Figure 1.16 code the following message: WILL COME SOON.

(b) Using $D = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ as a decoding matrix and the inverse mapping of Figure 1.16, decode the message that was coded in (a).

6. Using the appropriate matrix C or D of Exercise 5 solve the following equations and check:

(a) $3x + y = 9$
$5x + 2y = 16$

(b) $3x + y = 3$
$5x + 2y = 4$

(c) $2x - y = -1$
$-5x + 3y = 1$

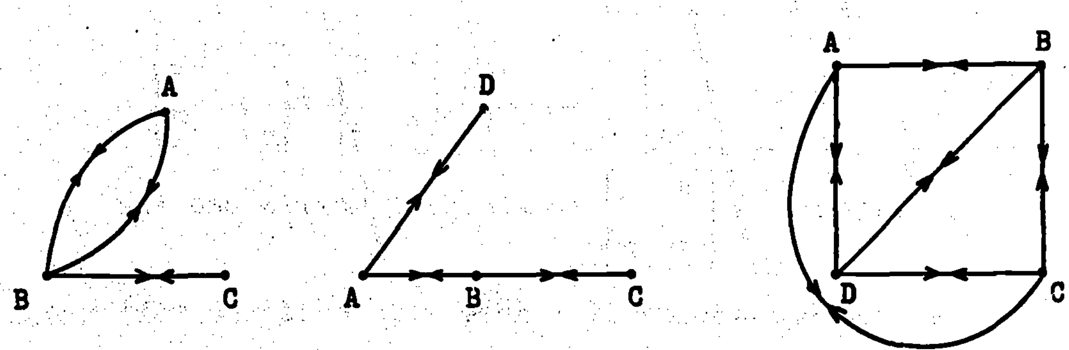
(d) $2x - y = 0$
$-5x + 3y = 0$

7. Express as a single matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Describe the change on the second matrix resulting from multiplication.

8. Describe each of the following two-way bus routes between towns by a matrix.



(a) (b) (c)

9. The population of a city at the end of 1968 is 3,000,000, and that of its suburbs is also 3,000,000. Assume that 70% of the city people in any year remain in the city and 30% of them move to the suburb, while 80% of the suburban population remain in the suburbs and 20% of them move to the city. Using matrices, calculate the population in both places at the end of

(a) 1969

(b) 1970

(c) 1971.

Chapter 2

LINEAR EQUATIONS AND MATRICES

2.1 Linear Combinations of Equations

You may recall working with equations of lines, and that they had the form $ax + by = c$, where not both a and b are zero. To find the coordinates of the point of intersection of two intersecting lines, we solved a system of two such equations. This need to solve a system of linear equations occurs frequently in mathematical situations. In this chapter we examine a method to solve these systems which leads to a procedure that can be programmed efficiently for automatic computation.

This method depends on two basic operations which we illustrate in this section. We work with linear equations of the form $ax + by = c$. At the right are three examples of linear equations that have this form.

$$A_1:\ 2x + 3y = 6$$
$$B_1:\ x - 4y = -\frac{1}{2}$$
$$C_1:\ \frac{3}{4}x + \frac{1}{2}y = 12$$

The first operation is multiplying the coefficients of x and y (the variables) and the constant term by a non-zero number. If the multiplier for equation A_1 is 3, the resulting equation is called $3A_1$. Study the equations at the right noting the multiplier for each.

$$3A_1:\ 6x + 9y = 18$$
$$-2B_1:\ -2x + 8y = 1$$
$$4C_1:\ 3x + 2y = 48$$

It is natural to ask how the solution set of an equation is affected when it is multiplied in this manner. To get a suggestion of the answer let us see if $(x,y) = (0,2)$ satisfies both A_1 and $3A_1$. If x is replaced by 0 and y by 2 in A_1 then we get $2(0) + 3(2) = 6$. This is a true statement. Therefore (0,2) satisfies A_1. If we make the same replacements in $3A_1$ we get $6(0) + 9(2) = 18$. This too is a true statement. On the other hand, (1,2) does not satisfy A_1 for $2(1) + 3(2) = 6$ is a false statement. Neither does it satisfy $3A_1$, for $6(1) + 9(2) = 18$ is also a false statement. This illustrates a little theorem:

> If A_1 represents the linear equation $ax + by = c$ and $m \neq 0$, then A_1: $ax + by = c$ and mA_1: $max + mby = mc$ have the same solution set.

Can you prove this theorem?

This little theorem is useful in converting a coefficient in a linear equation to 1 (or any non-zero number), without disturbing the solution set of the equation. For instance, if we want the coefficient of y in $2x + 3y = 6$ to be 1 we take $m = \frac{1}{3}$, yielding $\frac{2}{3}x + y = 2$. Now suppose we have a system A of linear equations, such as A_1 and A_2.

$$A \begin{cases} 2x + 3y = 6 & A_1 \\ x - 4y = -\frac{1}{2} & A_2 \end{cases}$$

(Observe the name of the system and of its component equations.) The system has a solution if and only if that solution satisfies each of its component equations. Suppose that A_1 is replaced by $\frac{1}{2}A_1$ and A_2 remains unchanged. A new system is formed. Let us call it B.

$$B \begin{cases} x + \frac{3}{2}y = 3 & B_1 = (\frac{1}{2})A_1 \\ x - 4y = -\frac{1}{2} & B_2 = A_2 \end{cases}$$

Its component equations are $B_1 = \frac{1}{2}A_1$ and $B_2 = A_2$. How do the solution sets of systems A and B compare? Clearly all the solutions that satisfy both equations A_1 and A_2 must also satisfy B_1 and B_2, since by the little theorem above, no solutions are either gained or lost when A_1 is replaced by B_1.

<u>Theorem</u> 1. If an equation in a system of linear equations is replaced by a non-zero multiple of itself, the new system and the original system have the same solution set.

The second operation replaces an equation in a system by the sum of itself and a constant multiple of another equation. For instance, in the illustration below, A_1 is replaced by $B_1 = A_1 + (-2)A_2$.

$$A \begin{cases} 2x + 3y = 6 & A_1 \\ x - 4y = -\frac{1}{2} & A_2 \end{cases}$$

$$B \begin{cases} 0x + 11y = 7 & B_1 = A_1 + (-2)A_2 \\ x - 4y = -\frac{1}{2} & B_2 = A_2 \end{cases}$$

The actual work in finding $A_1 + (-2)A_2$ is:

$$(2x + 3y = 6) + (-2)(x - 4y = -\tfrac{1}{2})$$
$$(2x + 3y = 6) + (-2x + 8y = 1)$$
$$0x + 11y = 7$$

Definition 1. The two operations, called Elementary Operations, are:

(1) replacing an equation in a system by a non-zero constant multiple of itself

(2) replacing an equation in a system by the sum of itself and a constant multiple of another.

As one or both of the elementary operations are performed, a new system of equations is generated. These operations may be repeated, thus generating a sequence of systems.

Definition 2. Two systems of equations are equivalent if an only if one can be obtained from the other by a finite sequence of elementary operations.

Example 1.

$$\text{System A} \begin{cases} 3x + 6y = 9 & A_1 \\ 2x - 3y = -1 & A_2 \end{cases}$$

$$\text{System B} \begin{cases} x + 2y = 3 & B_1 = (\frac{1}{3})A_1 \\ 0x - 7y = -7 & B_2 = A_2 + (-2)B_1 \end{cases}$$

$$\text{System C} \begin{cases} x + 2y = 3 & C_1 = B_1 \\ 0x + y = 1 & C_2 = (-\frac{1}{7})B_2 \end{cases}$$

Systems A, B, and C are equivalent.

Example 1 uses the elementary operations. The first operation was used to obtain equation B_1, giving x the coefficient 1.

It was also used to get equation C_2, giving y the coefficient 1. The second operation was used to obtain equation $B_2 = A_2 + (-2)B_1$, giving x the coefficient 0. These two strategies are crucial in solving a system of equations. But before we can use them we have to satisfy ourselves that equivalent systems have the same solution set.

Let us examine the three systems above. From $C_2 (0x + y = 1)$ it is clear that $y = 1$. This is used in C_1 to replace y, yielding $x + (2)1 = 3$. From this $x = 1$. It is clear that $(x,y) = (1,1)$ satisfies C, for C_1 and C_2 are both true when $(x,y) = (1,1)$. Let us see if (1,1) also satisfies B_1 and B_2. Since $B_1 = C_1$ there is no need to check B_1, and for B_2 $(0(1) - 7(1) = -7)$ is true. Now to check A. On replacing (x,y) by (1,1) A_1 becomes $3(1) + 6(1) = 9$, a true statement, and A_2 becomes $2(1) - 3(1) = -1$, also a true statement. This suggests the next theorem, which is a special case for a system of two linear equations in two variables. But it is true for any number of equations in any number of variables.

<u>Theorem</u> 2. If System A has linear equations A_1 and A_2, in two variables, and System B has linear equations $B_1 = A_1$ and $B_2 = A_2 + mA_1$, with $m \neq 0$, then A and B (equivalent systems) have the same solution set.

Can you verify this theorem for a special case?

Examine this verification, and explain how each equation is obtained.

$$A \begin{cases} 3x - 2y = -1 & A_1 \\ x + y = 3 & A_2 \end{cases}$$

$$B \begin{cases} 3x - 2y = -1 & B_1 = A_1 \\ 0x + \frac{5}{3}y = \frac{10}{3} & B_2 = A_2 + (-\frac{1}{3})A_1 \end{cases}$$

From B_2, show that $y = 2$. Using this and B_1, show $x = 1$. Now check $(x,y) = (1,2)$ in systems A and B.

While Theorem 2 applies to two systems having two equations, it is possible to apply it to two systems having a different number of equations, and two or more variables as we show in Examples 2 and 3. In studying these examples you should do all details not shown.

Example 2.

$$A \begin{cases} x + 2y = 5 & A_1 \\ 2x - y = 5 & A_2 \\ -x + y = -2 & A_3 \end{cases}$$

$$B \begin{cases} x + 2y = 5 & B_1 = A_1 \\ 0x - 5y = -5 & B_2 = A_2 + (-2)B_1 \\ 0x + 3y = 3 & B_3 = A_3 + B_1 \end{cases}$$

Find y from B_2. Does it agree with what you obtain for y in B_3? Find x from B_1. Now show $(x,y) = (3,1)$ satisfies all equations in Systems A and B.

Example 3.

$$A \begin{cases} 3x + 2y - z = 4 & A_1 \\ x + y + z = 1 & A_2 \\ -4x - y = -7 & A_3 \end{cases}$$

$$B \begin{cases} 0x - y - 4z = 1 & B_1 = A_1 + (-3)A_2 \\ x + y + z = 1 & B_2 = A_2 \\ 0x + 3y + 4z = -3 & B_3 = A_3 + (4)A_2 \end{cases}$$

Did you supply the details explaining how B_1 and B_3 are obtained? Show that $(x,y,z) = (2,-1,0)$ satisfies both systems A and B.

Example 4.

Given

$$A \begin{cases} 3x + y = 2 & A_1 \\ 2x - 3y = -3 & A_2 \end{cases}$$

(a) Find k such that in kA_1 the x-coefficient will be 1.

(b) Find m such that in $A_2 + mA_1$, the y-coefficient will be 0.

Solutions.

(a) $k = \frac{1}{3}$, the multiplicative inverse of the coefficient of x.

(b) $m = 3$, for in $A_2 + 3(A_1)$, = $(2x - 3y = -3) + (9x + 3y = 6)$ or $11x + 0y = 3$. If the coefficient of y in A_1 is 1, (as it is), m is the additive inverse of the y-coefficient in A_2.

2.2 Exercises

1. Given equation A_1: $5x + y = 3$. Form the equation defined by (a) $2A_1$ (b) $\frac{1}{5}A_1$ (c) $-\frac{1}{3}A_1$

2. Given equation B_1: $3x - 2y = -6$. Form the equivalent equation whose

   (a) x-coefficient is 1. (b) y-coefficient is 1.

3. Given equations A_1: $2x - y = 7$ and A_2: $x + 3y = -7$.

   Form the linear equation defined by:

   (a) $A_1 + A_2$ (c) $A_1 + (-2)A_2$ (e) $A_1 + \frac{1}{3}A_2$

   (b) $2A_1 + A_2$ (d) $\frac{1}{2}A_1 + (-1)A_2$ (f) $3A_1 + A_2$

4. Using the given equations A_1 and A_2 of Exercise 3, find m such that in $A_2 + mA_1$

   (a) the y-coefficient is zero.

   (b) Find n such that in $A_1 + nA_2$ the x-coefficient is 0.

5. Given equations A_1: $x + 2y + z = -2$ and A_2: $2x - y + 3z = 4$.

   (a) Find m so that in $A_2 + mA_1$ the x-coefficient is zero.

   (b) Find n so that in $A_1 + nA_2$ the y-coefficient is zero.

   (c) Find k so that in $A_2 + kA_1$ the z-coefficient is zero.

*6. Given $A \begin{cases} ax + by = c & A_1 \\ a'x + b'y = c' & A_2 \end{cases}$.

   By elementary operations obtain the equivalent system:

$$B \begin{cases} 1 \cdot x + 0y = \dfrac{b'c - bc'}{ab' - a'b} \\ \\ 0x + 1y = \dfrac{ac' - a'c}{ab' - a'b} \end{cases}$$

   if $ab' \neq a'b$. Discuss what happens if $a'b = a'b$.

2.3 Pivotal Operations.

It has probably occurred to you that we can generate a sequence of equivalent systems, in which the last system has coefficients 1 and 0 only. In that case it would be a simple matter to see the solution of a system, if there were one. In some of the exercises in the preceding section we saw how this can be done by a judicious choice of elementary operations. In our next example we illustrate how this can be done in general.

Example 1. Solve

$$A \begin{cases} 2x + 3y = 1 & A_1 \\ x + 2y = 0 & A_2 \end{cases}$$

Solution. Let us choose system B as follows. $B_2 = A_2$ and $B_1 = A_1 + (-2)B_2$.

$$B \begin{cases} 0x - y = 1 & B_1 = A_1 + (-2)B_2 \\ x + 2y = 0 & B_2 = A_2 \end{cases}$$

For B_1 we obtain

$$(2x + 3y = 1) + (-2)(x + 2y = 0)$$

or $(2x + 3y = 1) + (-2x - 4y = 0)$

or $-y = 1$ which implies $y = -1$.

Using this value in B_2 we get

$$x + 2(-1) = 0 \text{ or } x = 2.$$

So the solution is $(x,y) = (2,-1)$.

Does this check in system A? in system B?

Example 2. Solve

$$A \begin{cases} 2x + 3y = 1 & A_1 \\ x + 2y = 0 & A_2 \end{cases}$$

Solution. We have already seen how this system can be transformed by elementary operations to

$$B \begin{cases} 0x - y = 1 & B_1 = A_1 + (-2)B_2 \\ x + 2y = 0 & B_2 = A_2 \end{cases}$$

This introduced a 0 coefficient in B_1, and this was quite helpful. We can also introduce a 0 coefficient in B_2 by transforming system B to system C as follows:

$$C \begin{cases} 0x + y = -1 & C_1 = (-1)B_1 \\ x + 0y = 2 & C_2 = B_2 + (-2)C_1 \end{cases}$$

Check this carefully.

From C it is a simple matter to see that the solution of A is $(x,y) = (2,-1)$.

Can you see how the following table shortens and highlights all the operations we have gone through?

A	$2x + 3y = 1$	A_1	
	$x + 2y = 0$	A_2	
B	$0x - y = 1$	$B_1 = A_1 + (-2)B_2$	(Note that B_2 was obtained before B_1.)
	$x + 2y = 0$	$B_2 = A_2$	
C	$0x + y = -1$	$C_1 = (-1)B_1$	
	$x + 0y = 2$	$C_2 = B_2 + (-2)C_1$	

solution $(x,y) = (2,-1)$

Before we make precise the details of the method of Example 2, let us examine another example. We shall use the table form we discussed in Example 2. If you have trouble with it lay it out in detail as we did in the previous example. circle around a coefficient in the table is explained later.

<u>Example</u> 3. Solve: $2x + 3y = 1$

$$3x - 2y = 8$$

<u>Solution</u>: This can also be written

$$2x + 3y - 1 = 0$$

$$3x - 2y - 8 = 0$$

A	$2x + 3y - 1 = 0$	A_1
	$3x - 2y - 8 = 0$	A_2
B	$x + \frac{3}{2}y - \frac{1}{2} = 0$	$B_1 = \frac{1}{2}A_1$
	$0x - \frac{13}{2}y - \frac{13}{2} = 0$	$B_2 = A_2 + (-3)B_1$
C	$x + 0y - 2 = 0$	$C_1 = B_1 + (-\frac{3}{2})C_2$
	$0x + y + 1 = 0$	$C_2 = (-\frac{2}{13})B_2$

$$(x,y) = (2,-1)$$

Check A_1: $2(2) + 3(-1) - 1 = 0$ is true.

A_2: $3(2) - 2(-1) - 8 = 0$ is true.

Note that in this case C_2 was obtained prior to C_1 and in fact was used to obtain C_1.

In the first operation we chose to convert the coefficient 2 in $2x$ of A_1 to 1 in B_1. This choice is indicated by the circle around the 2. Another such choice, also marked by a circle, was made in B_2 to convert $-\frac{13}{2}$ to 1 in C_2. This choice, to convert a coefficient to 1, is the first step in an operation called <u>pivoting</u>. The numbers 2 and $-\frac{13}{2}$ are called the <u>pivots</u>. Having chosen a pivot, say an x-coefficient, we then try to convert the x-coefficients of the other equations to zero. The two steps in pivoting are the elementary operations. In this way, if possible, we end with a system in which one x-coefficient is 1, and all others are zero. By pivoting on a y-coefficient we can also try to convert one y-coefficient to 1 and all others to zero. This can be done, as Example 3 shows, **without disturbing** the effects of pivoting on x-coefficients. In this manner we arrive, if possible, at an equivalent system whose solution set is obvious.

To summarize, the pivotal operations on a non-zero pivot consist of elementary operations which replace a given system by an equivalent system in which each pivot is converted to 1 and all other coefficients of the variable of the pivot are converted to zeros. When pivotal operations are performed as far as they can go, the last system is called the Gauss-Jordan reduced form, or simply the Gauss-Jordan form. If there is a solution the Gauss-Jordan form shows what it is.

The name Gauss-Jordan form is after Gauss who invented the pivoting operations, and Jordan who used it to get only 1 and 0 coefficients as far as possible. Because this method is used extensively, we write only what is essential, namely coefficients and constants. This leads to a sequence of matrices which, together with the instructions, we call a tableau. Example 3 is rewritten in both equation and tableau form so that you may see what is stripped off and what remains. The middle column explains what happens in both forms. We continue to draw lines between successive systems but we do not name them any more.

Equation Form	Instructions	Tableau Form: x	y	-1	
$2x + 3y - 1 = 0$	A_1	2	3	1	= 0
$3x - 2y - 8 = 0$	A_2	3	-2	8	= 0
$x + \frac{3}{2}y - \frac{1}{2} = 0$	$B_1 = (\frac{1}{2})A_1$	1	$\frac{3}{2}$	$\frac{1}{2}$	= 0
$0x + -\frac{13}{2}y - \frac{13}{2} = 0$	$B_2 = A_2 + (-3)B_1$	0	$-\frac{13}{2}$	$\frac{13}{2}$	= 0
$x + 0y - 2 = 0$	$C_1 = B_1 + (-\frac{3}{2})C_2$	1	0	2	= 0
$0x + y + 1 = 0$	$C_2 = (-\frac{2}{13})C_1$	0	1	-1	= 0

<u>Comment</u> 1. Note the headings x, y, and -1 in the tableau. They are there to help us retrieve an equation from the tableau. To retrieve C_1, for example, multiply each number in row C_1 by its heading and set the sum of the products equal to zero. System C is, by this method of retrieval, $0x + y + 1 = 0$ and $x + 0y - 2 = 0$.

<u>Comment</u> 2. The -1 heading may startle you. It actually saves you the effort of solving $y + 1 = 0$ and

$x - 2 = 0$. Simply read the solutions in the -1 column. Can you explain how the -1 heading does this?

Comment 3. Note, in passing, that the first two columns in system A of the tableau comprise the coefficient matrix of the original system of equations.

The pivotal method is not restricted to solving a system of two equations in two variables. It can also be used to solve a system containing any number of linear equations in any number of variables, if there is a solution. Our next example shows how this is done, in tableau form, for three equations in three unknowns. You may expect three pivoting operations. The row operations are explained in the last column.

Example 4. Solve

$$2x + 3y + z = 4$$
$$x + y - z = 1$$
$$x - 2y + 2z = 7$$

Solution. Writing the equations in the form $ax + by + cz + d = 0$ we get in tableau form:

	x	y	z	-1		
	2	-3	1	4	= 0	A_1
A	1	1	-1	1	= 0	A_2
	1	-2	2	7	= 0	A_3
	0	-5	3	2	= 0	$B_1 = A_1 + (-2)B_2$
B	1	1	-1	1	= 0	$B_2 = A_2$
	0	-3	3	6	= 0	$B_3 = A_3 + (-1)B_2$
	0	0	-2	-8	= 0	$C_1 = B_1 + 5C_3$
C	1	0	0	3	= 0	$C_2 = B_2 + (-1)C_3$
	0	1	-1	-2	= 0	$C_3 = (-\frac{1}{3})B_3$
	0	0	1	4	= 0	$D_1 = (-\frac{1}{2})C_1$
D	1	0	0	3	= 0	$D_2 = C_2 + 0D_1$
	0	1	0	2	= 0	$D_3 = C_3 + D_1$

Now system D can be rewritten as:

$$0x + 0y + z - 4 = 0$$
$$x + 0y + 0z - 3 = 0$$
$$0x + y + 0z - 2 = 0$$

It is evident from the -1 column of system D that

$$(x,y,z) = (3,2,4)$$

Check that this solution satisfies system A.

Can you write out in full detail how we obtain system B from system A?

Can you write it out in equation form?

Can you explain how system C is obtained from system B? Can you write it out in equation form?

Do the same in going from system C to system D.

2.4 Exercises

Solve the systems of equations in Exercises 1-14 by the pivotal method, using only the tableau form. Check all your solutions.

1. $x + 3y = 10$
   $2x + 5y = 16$

2. $2x + 3y = 10$
   $y + 2x = 6$

3. $5x - 3y - 12 = 0$
   $2x - y - 5 = 0$

4. $5u + 3v - 27 = 0$
   $6u - 2v - 10 = 0$

5. $2r + 4s = 1$
   $4r - 3s = 1$

6. $3x = 13 - 4y$
   $y = 5x + 4$

7. $x + y - z = -2$
   $x - 2y - 2z = 1$
   $2x + 3y + z = 1$

8. $x + 4z = 4$
   $2x + y + z = 3$
   $-x + y + z = 1$

9. $x + y - z - 6 = 0$
   $2y + z - 20 = 0$
   $5x - y - 2z + 3 = 0$

10. $x - 3y + 2z + 1 = 0$
    $2y - 3z - 3 = 0$
    $3x + 5z + 2 = 0$

11. $x_1 + 4x_2 + 2x_3 - 19 = 0$
    $2x_1 + x_2 + 2x_3 - 19 = 0$
    $2x_1 + 3x_2 + x_3 - 18 = 0$

12. $3x - 4z = 0$
    $6x + 4y = -1$
    $8y + 2z = 5$

13. $x - y + z = 3$
    $3x + 2y - z = 1$
    $4x - 2y - 3z = -2$

14. $x + y + z + w = 5$
    $2x + y - z + w = 4$
    $x + y - w = 5$

2.5 Solving Systems of Linear Equations: Continued

In this section we consider two basic questions.

1. Do all linear equations having as many equations as variables have solutions?
2. Can we solve a system of m linear equations in n variables if $m \neq n$?

The pivotal method helps to answer both questions. Consider, by way of answering the first question, the system in Example 1.

Example 1. Solve: $3x + 2y = 8$

$6x + 4y = 9$

Solution.

$$\begin{array}{c|ccc|cc}
 & x & y & -1 & & \\
\hline
A & 3 & 2 & 8 & = 0 & A_1 \\
 & 6 & 4 & 9 & = 0 & A_2 \\
\hline
B & 1 & \frac{2}{3} & \frac{8}{3} & = 0 & B_1 = \frac{1}{3}A_1 \\
 & 0 & 0 & -7 & = 0 & B_2 = A_2 + (-6)B_1
\end{array}$$

The last row represents the equation $0x + 0y = -7$. Clearly, there are no values of (x,y) that satisfy this equation. Inasmuch as $0x + 0y = 0 \neq 7$ for all values of (x,y). Since there are no solutions for this equation (B_2), there can be none for the system A.

Let us try solving another pair of linear equations, one that closely resembles the first pair.

Example 2. Solve $3x + 2y = 8$

$6x + 4y = 16$

Solution.

$$\begin{array}{c|ccc|l}
 & x & y & -1 & \\
\hline
\multirow{2}{*}{A} & 3 & 2 & 8 & = 0 \quad A_1 \\
 & 6 & 4 & 16 & = 0 \quad A_2 \\
\hline
\multirow{2}{*}{B} & 1 & \frac{2}{3} & \frac{8}{3} & = 0 \quad B_1 = \frac{1}{3}A_1 \\
 & 0 & 0 & 0 & = 0 \quad B_2 = A_2 + (-6)B_1 \\
\hline
\end{array}$$

This time the last row represents the equation $0x + 0y = 0$ which is satisfied by all values of (x,y). Hence whatever satisfies B_1 also satisfies B_2 and the original system consisting of A_1, A_2. Equation B_1, which is $x + \frac{2}{3}y - \frac{8}{3} = 0$ is equivalent to $x = -\frac{2}{3}y + \frac{8}{3}$. Any value assigned to y yields a value for x. For instance,

$$\text{if } y = 3, \quad x = -2 + \frac{8}{3} = \frac{2}{3},$$
$$\text{if } y = -50, \quad x = \frac{100}{3} + \frac{8}{3} = 36,$$
$$\text{if } y = 0, \quad x = \frac{8}{3}.$$

Thus $(\frac{2}{3},3)$ $(36,-50)$, $(\frac{8}{3}, 0)$ are among an infinite number of values of (x,y) that satisfy the original equations. Check this for $(\frac{2}{3},3)$ and $(\frac{8}{3},0)$. The entire solution set may be designated

$$\{(x,y)\colon\ x = -\frac{2}{3}s + \frac{8}{3},\ y = s,\ s \in R\}$$

or more compactly

$$\{(-\frac{2}{3}s + \frac{8}{3}, s)\colon\ s \in R\}.$$

Summarizing the results of Examples 1 and 2 we see that there are no solutions if a row in the Gauss-Jordan form contains only zeros except for the last number. If a row contains only zeros, we may delete it and work with the remaining row or rows in the tableau.

Below are two more examples, this time for three linear equations in three variables. Note that their coefficient matrices are the same. This makes it possible to use the same pivotal operations for both.

<u>Example</u> 3.

Solve: $x + 2y - 3z = 4$
$2x - 3y + 5z = 5$
$3x - y + 2z = 10$

<u>Solution</u>.

x	y	z	-1			
1	2	-3	4	= 0	A_1	
2	-3	5	5	= 0	A_2	
3	-1	2	10	= 0	A_3	
1	2	-3	4	= 0	B_1	$= A_1$
0	-7	11	-3	= 0	B_2	$= A_2+(-2)B_1$
0	-7	11	-2	= 0	B_3	$= A_3+(-3)B_1$
1	0	$\frac{1}{7}$	$\frac{22}{7}$	= 0	C_1	$= B_1+(-2)C_2$
0	1	$-\frac{11}{7}$	$\frac{3}{7}$	= 0	C_2	$= (-\frac{1}{7})B_2$
0	0	0	1	= 0	C_3	$= B_3+7C_2$

<u>Example</u> 4.

Solve: $x + 2y - 3z = 4$
$2x - 3y + 5z = 5$
$3x - y + 2z = 9$

<u>Solution</u>

x	y	z	-1	
1	2	-3	4	= 0
2	-3	5	5	= 0
3	-1	2	9	= 0
1	2	-3	4	= 0
0	-7	11	-3	= 0
0	-7	11	-3	= 0
1	0	$\frac{1}{7}$	$\frac{22}{7}$	= 0
0	1	$-\frac{11}{7}$	$\frac{3}{7}$	= 0
0	0	0	0	= 0

There is no solution to the system of Example 3. In Example 4 delete the last row. The remaining rows represent $y - \frac{11}{7}z - \frac{3}{7} = 0$ and $x + \frac{1}{7}z - \frac{22}{7} = 0$

or $x = -\frac{1}{7}z + \frac{22}{7}$, $y = \frac{11}{7}z + \frac{3}{7}$

Any value of z produces one value for x and one for y. For instance, if $z = 7$, $x = \frac{15}{7}$, $y = \frac{80}{7}$

$$\text{if } z = 0,\ x = \frac{22}{7},\ y = \frac{3}{7}$$

$$\text{if } z = 1,\ x = 3,\ y = 2.$$

The solution set may be designated

$$\{(x,y,z):\ x = -\frac{1}{7}s + \frac{22}{7},\ y = \frac{11}{7}s + \frac{3}{7},\ z = s,\ s \in R\},$$

or briefly as $\{(-\frac{1}{7}s + \frac{22}{7}, \frac{11}{7}s + \frac{3}{7}, s);\ s \in R\}$.

This answers the first of the two questions, and also part of the second, since we solved two linear equations in three variables in Example 4. Now to see how pivotal operations handle three linear equations in two variables. Again we use two examples having the same coefficient matrix.

Example 5.

Solve: $x + 2y = 5$

$2x - y = 5$

$3x + 4y = 13$

Solution

x	y	-1			
1	2	5	= 0	A_1	
2	-1	5	= 0	A_2	
3	4	13	= 0	A_3	
1	2	5	= 0	B_1	$= A_1$
0	-5	-5	= 0	B_2	$= A_2+(-2)B_1$
0	-2	-2	= 0	B_3	$= A_3+(-3)B_1$
1	0	3	= 0	C_1	$= B_1+(-2)C_2$
0	1	1	= 0	C_2	$= (-\frac{1}{5})B_2$
0	0	0	= 0	C_3	$= B_3+2C_2$

$(x,y) = (3,1)$

Example 6.

Solve: $x + 2y = 5$

$2x - y = 5$

$3x + 4y = 11$

Solution

x	y	-1	
1	2	5	= 0
2	-1	5	= 0
3	4	11	= 0
1	2	5	= 0
0	-5	-5	= 0
0	-2	-4	= 0
1	0	3	= 0
0	1	1	= 0
0	0	-2	= 0

no solutions

The solution $(x,y) = (3,1)$ checks in all of the three original equations of Example 5.

Our last example illustrates how to handle one linear equation in three variables.

Example 6. Solve: $x + 2y - 3z = 5$

Solution For all (x,y,z), $x = -2y + 3z + 5$. If we assign a value to y and one to z, not necessarily the same as the one we assign to y, we find a value that corresponds to (y,z). For instance,

$$\text{if } y = 3,\ z = 1, \text{ then } x = -6 + 3 + 5 = 2,$$

$$\text{if } y = 0,\ z = 20, \text{ then } x = 0 + 60 + 5 = 65.$$

Thus $(2,3,1)$ and $(65,0,20)$ are in the solution set.

The entire set can be designated by

$$\{(x,y,z)\colon\ x = -2s + 3t + 5,\ y = s, z = t, \text{ and } s,t \in R\},$$

or $\{(-2s + 3t + 5,\ s,\ t)\colon\ s,t \in R\}$.

2.6 Exercises

Solve and check. If the solution set contains an infinite number of solutions represent it in set notation.

1. $2x + 5y = 3$
   $4x + 10y = 7$
2. $3x - 2y = 3$
   $6x - 4y = 6$
3. $x + 2y + z = 1$
   $2x + y = 3$
   $3x + 4y + 2z = 4$
4. $x + y - 2z = 1$
   $2x - y + z = 1$
   $x + 2z + 2y = 2$
5. $2x_1 + x_2 + 2x_3 = 4$
   $2x_1 + 2x_2 + x_3 = 7$
   $x_2 - x_3 = 3$

6. $2x_1 + x_2 + 3x_3 = -3$

   $3x_1 + 4x_2 = 24$

7. $x_1 + x_2 = 5$

   $2x_1 - 3x_2 = 15$

   $5x_1 + 2x_2 = 28$

8. $x + y = 5$

   $2x - 3y = 15$

   $3x - 2y = 10$

9. $3r + s - 4t = 6$

10. $2u - 7v = 4$

11. $5a + 2b = 14 - c$

    $2a - 3c = 14 + b$

12. $x + 3z = 5$

    $x + 5z = 3$

    $x + 9z = -1$

13. $x + 2y + 3z = 5$

    $x + 3y + 5z = 3$

    $x + 5y + 9z = -1$

14. $x + 2y + 3z = 5$

    $x + 3y + 5z = 3$

2.7 Homogeneous Linear Equations

Homogeneous linear equations have constant terms which are zero. They present no special problem that cannot be solved by the pivotal method. We make special mention of them because they occur quite frequently and occupy an important place in mathematical theory.

The special thing to notice is that the -1 column of the related tableaus contains nothing but zeros. This follows from the fact that results of elementary operations on zeros are zero. Hence we can omit the -1 column when solving homogeneous equations and work only with the coefficient matrix.

Let us agree, from now on, to omit the "=0" that follows each row. Let it be understood hereafter, also, for systems that are not homogeneous.

Example 1.

Solve: $x + 2y + z = 0$

$2x - y - 3z = 0$

$3x + 4y + z = 0$

$$
\begin{array}{ccc|l}
x & y & z & \\
\hline
1 & 2 & 1 & A_1 \\
2 & -1 & -3 & A_2 \\
3 & 4 & 1 & A_3 \\
\hline
1 & 2 & 1 & B_1 = A_1 \\
0 & -5 & -5 & B_2 = A_2+(-2)B_1 \\
0 & -2 & -2 & B_3 = A_3+(-3)B_1 \\
\hline
1 & 0 & -1 & C_1 = B_1+(-2)C_2 \\
0 & 1 & 1 & C_2 = (-\frac{1}{5})B_2 \\
0 & 0 & 0 & C_3 = B_3+2C_2
\end{array}
$$

$y + z = 0$ or $y = -z$

$x - z = 0$ or $x = z$

let $z = S$

$(x,y,z) = (S,-S,S)$

Example 2.

Solve: $x + 2y + z = 0$

$2x - y - 3z = 0$

$3x + 4y + 2z = 0$

$$
\begin{array}{ccc|l}
x & y & z & \\
\hline
1 & 2 & 1 & \\
2 & -1 & -3 & \\
3 & 4 & 2 & \\
\hline
1 & 2 & 1 & \\
0 & -5 & -5 & \\
0 & -2 & -1 & \\
\hline
1 & 0 & -1 & \\
0 & 1 & 1 & \\
0 & 0 & 1 & \\
\hline
1 & 0 & 0 & D_1 = C_1+D_3 \\
0 & 1 & 0 & D_2 = C_2+(-1)D_3 \\
0 & 0 & 1 & D_3 = C_3
\end{array}
$$

$(x,y,z) = (0,0,0)$

Perhaps you anticipated that all homogeneous systems having 3 equations in 3 unknowns necessarily had the solution (0,0,0) only. Indeed this happened in Example 2. But in Example 1 there are an infinite number of solutions, including not only (0,0,0), but also (2,-2,2), ($\sqrt{5}$, $-\sqrt{5}$, $\sqrt{5}$) and so on. Had you known that one of the equations in Example 1 is the sum of multiples of the other two, you might have guessed otherwise. Indeed, $A_3 = \frac{11}{5}A_1 + \frac{2}{5}A_2$. (Verify this). Thus the system in Example 1 is equivalent to the system containing only A_1 and A_2. On the other hand, try as you will, you will not be able to express an equation in Example 2 as the sum of multiples of the other two. This is an essential difference between the two systems.

2.8 Exercises

In Exercises 1-4, solve by the pivotal method, and check.

1. $2x + 3y = 0$
   $x + 2y = 0$

2. $2x + 3y = 0$
   $3x + 2y = 0$

3. $2x + 3y = 0$
   $x + \frac{3}{2}y = 0$

4. $2x + 3y = 0$
   $3x - 2y = 0$

5. Prove: If $\frac{a}{c} = \frac{b}{d}$, where a,b,c,d are non-zero, then the system $ax + by = 0$ has an infinite solution set.
   $cx + dy = 0$

   Express its solution set in set notation.

   In Exercises 6-11 solve by the pivotal method

6. $x - 3y + 2z = 0$
   $x - 2y - z = 0$
   $2x - y + 3z = 0$

7. $x_1 + x_3 = 0$
   $2x_1 + 3x_2 - x_3 = 0$
   $4x_1 + 5x_2 + x_3 = 0$

8. $2x + 7y + 4z = 0$
   $x + y + z = 0$
   $3x - 2y + z = 0$

9. $2a + 3b - 5c = 0$
   $a - 2b + c = 0$
   $4a + 13b - 17c = 0$

10. $x_1 + x_2 + x_3 = 0$
    $2x_1 + x_2 - x_3 = 0$
    $x_1 - x_2 + x_3 = 0$

11. $x_1 + x_2 + x_3 + x_4 = 0$
    $x_1 + 4x_2 + 3x_3 + 2x_4 = 0$
    $x_1 - x_2 - x_4 = 0$
    $4x_1 + x_2 + 2x_3 + 3x_4 = 0$

12. Let a system of three equations in three variables be such that one of the equations can be obtained from the other two via elementary operations.
Prove that the system has an infinite solution set.

2.9 Matrix Multiplication Derived from Linear Equations in Matrix Notation

We return to the system $\begin{cases} 2x + 3y = 1 \\ x + 2y = 0 \end{cases}$, that we have previously examined. We write this system in terms of matrices, as follows:

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In doing this we have detached the coefficients from their variables, leaving a 2 x 2 coefficient matrix , and instead of writing x y at the top of a tableau we wrote $\begin{bmatrix} x \\ y \end{bmatrix}$ as a 2 x 1 matrix to be multiplied by the coefficient matrix. Note that, if we perform the multiplication ,as we did in Chapter 1, then

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ x + 2y \end{bmatrix}, \text{ a } 2 \times 1 \text{ matrix.}$$

Then when we set $\begin{bmatrix} 2x + 3y \\ x + 2y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we have corresponding components equal (as required by the definition of matrix equality), and we have retrieved the two original equations.

It often happens that people who use systems of linear equations to solve problems in science, industry, and other activities, have to solve many systems that have the same coefficient matrix. To illustrate with our simple coding matrix, suppose one has to solve

$$\begin{aligned} 2x_1 + 3x_2 &= 1 \\ x_1 + 2x_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 2y_1 + 3y_2 &= 0 \\ y_1 + 2y_2 &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} 2z_1 + 3z_2 &= 4 \\ z_1 + 2z_2 &= 3 \end{aligned}$$

We can write them as follows:

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Now the astonishing thing is that all three of these matrix equations can be combined into one, as follows:

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix},$$

provided we accept the definition of multiplication of matrices as suggested in Chapter 1. Perhaps you recall that we multiplied row terms by corresponding column terms and added, to get the terms in the product. For the last multiplication this would be:

$$\begin{bmatrix} 2x_1 + 3x_2 & 2y_1 + 3y_2 & 2z_1 + 3z_2 \\ x_1 + 2x_1 & y_1 + 2y_2 & z_2 + 2z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

Do you see, when corresponding terms in the two matrices are equated, we retrieve the six equations we started with?

2.10 Exercises

1. Write a matrix equation for each of the following systems:

(a) $\begin{aligned} 3x + 5y &= 8 \\ x + 2y &= 3 \end{aligned}$ (b) $\begin{aligned} 3x - 5y &= 2 \\ x - 3y &= 4 \end{aligned}$ (c) $\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$

2. Write one matrix equation for all of the following systems whose coefficient matrices are the same:

$3x + 5y = 8$ $\quad$ $3x + 5y = 3$ $\quad$ $3x + 5y = 1$

$x + 2y = 3$ $\quad$ $x + 2y = 1$ $\quad$ $x + 2y = 0$

2.11 Matrix Inversion

In this section we show how to determine whether or not a (square) matrix has an inverse, and if it does, how to find it. We illustrate the method with $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

Let $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be the inverse. Then $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

This is equivalent to

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In tableau form these can be written:

$$\begin{array}{ccc|l} x & z & -1 & \\ \hline 2 & 3 & 1 & = 0 \\ 1 & 2 & 0 & = 0 \end{array} \quad \text{and} \quad \begin{array}{ccc|l} y & w & -1 & \\ \hline 2 & 3 & 0 & = 0 \\ 1 & 2 & 1 & = 0 \end{array}$$

Since the coefficient matrices are the same in both tableaus, we shall be performing the same pivotal operations. Hence we can combine them into one tableau with two -1 columns, if we are careful to read the first -1 column for variables x and z, and the second for y and w.

$$\begin{array}{c|cc|cc|l}
 & & & -1 & -1 & \\
\hline
A & 2 & 3 & 1 & 0 & A_1\\
 & 1 & 2 & 0 & 1 & A_2\\
\hline
B & 1 & \frac{3}{2} & \frac{1}{2} & 0 & B_1 = \frac{1}{2}A_1\\
 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & B_2 = A_2 + (-1)B_1\\
\hline
C & 1 & 0 & 2 & -3 & C_1 = B_1 + (-\frac{3}{2})C_2\\
 & 0 & 1 & -1 & 2 & C_2 = 2B_2
\end{array}$$

We see from C that there are indeed unique solutions for x,y,z and w. Hence $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ has a (unique) inverse. When the identity matrix emerges at the left of C we read the inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ in the right half of C.

Note that we started with a tableau

$$\boxed{A \;\big|\; I_2}$$

where A is the matrix whose inverse we seek, and ended with

$$\boxed{I_2 \;\big|\; A^{-1}}$$

We can therefore describe this method as the application of pivotal operations on A that ultimately produce I_2. If this can be done then these operations will transform I_2 into A^{-1}. We illustrate the procedure by trying to find the inverse, if any, of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

1	2	0	1	0	0	A_1
0	1	0	0	1	0	A_2
0	3	1	0	0	1	A_3
1	0	0	1	-2	0	$B_1 = A_1 + (-2)B_2$
0	1	0	0	1	0	$B_2 = A_2$
0	0	1	0	-3	1	$B_3 = A_3 + (-3)B_2$

Note that the first and third columns have one 1 and two zeros. Hence we need only pivot on the second column. We see that

$$A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}.$$

Verify this by showing that $A \cdot A^{-1} = A^{-1} \cdot A = I_3$.

In the next example we try to find the inverse of a matrix that has no inverse. Can you anticipate how this will show itself? Let

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -2 \\ 3 & 1 & 1 \end{bmatrix}.$$

Before starting note that the third row is the sum of the first two. Does this arouse any suspicions?

1	2	3	1	0	0	A_1
2	-1	-2	0	1	0	A_2
3	1	1	0	0	1	A_3
1	2	3	1	0	0	$B_1 = A_1$
0	-5	-8	-2	1	0	$B_2 = A_2 + (-2)B_1$
0	-5	-8	-3	0	1	$B_3 = A_3 + (-3)B_1$
1	0	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	0	$C_1 = B_1 + (-2)C_2$
0	1	$\frac{8}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	0	$C_2 = (-\frac{1}{5})B_2$
0	0	0	-1	-1	1	$C_3 = B_3 + 5C_2$

It is hopeless. The three zeros in the last row proclaim that we shall never obtain I_3 via pivotal operations. Can you explain why it is futile to continue? (Can you pivot on such a row?) If we try to retrieve the equations implicit in the last row they would be:

$$0x_1 + 0x_2 + 0x_3 = -1$$
$$0x_4 + 0x_5 + 0x_6 = -1$$
$$0x_7 + 0x_8 + 0x_9 = 1$$

Why? (See the beginning of this section.)

Clearly there are no solutions for $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$.

Hence M has no inverse.

2.12 Exercises

In Exercises 1-12, find the inverse, if it exists, of the matrices listed.

1. $\begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 8 & 2 \\ 7 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 8 & -4 \\ -2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 1 & -4 \\ -1 & 2 & -5 \\ -1 & 0 & 3 \end{bmatrix}$

7. $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ $(abc \neq 0)$

8. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & -2 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 2 & -4 & 1 \\ 1 & -3 & -2 \\ 3 & 1 & 19 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & 9 & -4 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

13. (a) Solve for matrix X:

$$\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \cdot X = I_3$$

(b) Verify that

$$X \cdot \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} = I_3$$

2.13 Word Problems

Problems in the real world do not come to us in the form of equations or inequalities. To solve these problems we first have to formulate them in words, and then translate these words into the language of mathematics. This gives a problem the form of an equation or inequality. If we are able to solve these we are led to a solution of the original problem. In this section we illustrate how this works. "Real" problems are usually too complex to illustrate simply, so we avail ourselves of "puzzle" problems that resemble "real" problems in some ways.

Problem 1. Mr. Ross said to his son Peter "For every exam you pass this term I will give you 50 cents. But for every exam you fail you must forfeit 20 cents." At the end of the term, Mr. Ross erroneously interchanged the number of exams passed and failed and paid Peter 40 cents. Peter objected, claiming $3.20. How many exams did Peter pass and how many did he fail? Assume that Mr. Ross and Peter made no arithmetic mistakes.

Solution. Part I.

Our first goal is to express the conditions in this problem in the form of equations. We cannot do this without using a symbol that represents the number of exams passed and one for the number failed. So we start with: Let p represent the number of exams passed. (We choose p to remind us that it represents the number of exams passed.) Let f represent the number of exams failed. To determine how much he was to receive Peter multiplied 50 and p, then 20 and f, and finally subtracted.

This can be represented by $50p - 20f$.

Peter claimed \$3.20. This should be written 320 because the 50 and 20 are in cents. Thus

$$50p - 20f = 320.$$

On the other hand, Mr. Ross interchanged p and f. For him then

$$50f - 20p = 40.$$

Rearranging for matrix solution:

$$-20f + 50p = 320$$
$$50f - 20p = 40$$

It is convenient to divide each member by 10. This reduces the coefficients and constants without changing the solution set.

$$-2f + 5p = 32$$
$$5f - 2p = 4$$

This completes the first part of the solution. We have succeeded in describing the conditions of the problem as equations.

<u>Part II</u>

Now to solve these equations. We use the pivotal operation method.

$$\begin{array}{ccc|l} f & p & -1 & \\ \hline -2 & 5 & 32 & A_1 \\ 5 & -2 & 4 & A_2 \\ \hline 1 & -\frac{5}{2} & -16 & B_1 = (-\frac{1}{2})A_1 \\ 0 & \frac{21}{2} & 84 & B_2 = A_2 + (-5)B_1 \\ \hline 1 & 0 & 4 & C_1 = B_1 + \frac{5}{2}C_2 \\ 0 & 1 & 8 & C_2 = \frac{2}{21}B_2 \end{array}$$

$(f,p) = (4,8)$

Therefore Peter failed 4 exams and passed 8. These satisfy the conditions of the problem.

Problem 2. Next semester the terms of the agreement were revised. For each grade of 90 or better, reported as E (excellent), Peter was to receive 50 cents. For each grade between 70 and 90 reported as P (passing), including 70, he was to receive 10 cents. For each failing grade reported as F, he was to forfeit 30 cents. At the end of the term Peter (the best mathematician in the family) claimed $2.20. His father, as usual, reversed the number of E, P, and F reports and claimed a forfeit of 20 cents. They both appealed to Mrs. Ross, who erroneously interchanged the number of E and P reports and said Peter should receive $1.40. How many of each type of report did Peter earn? Assume no arithmetic mistakes were made.

Solution. Part I

Let E represent the number of excellent reports, let P represent the number of passing reports, and let F represent the number of failing reports. By Peter's calculation

$$50E + 10P - 30F = 220$$

By Mr. Ross' calculation

$$50F + 10P - 30E = -20.$$

By Mrs. Ross' calculation

$$50P + 10E - 30F = 140.$$

Arranging these for solutions, after dividing by 10,

$$5E + P - 3F = 22$$

$$-3E + P + 5F = -2$$

$$E + 5P - 3F = 14.$$

Part II

This system can now be solved by the pivotal method. The solution is

$$(E,P,F) = (5,3,2).$$

Does this agree with the conditions of the original problem?

In doing the exercises that follow try to see your solution as consisting of two parts, as above. You may find it necessary to read some problems several times before you understand the conditions well enough to write equations that describe them. This will be the more difficult part. Do not get discouraged. Good luck.

2.14 Exercises

1. A classroom has 36 desks, some single, others double

(seating two). The seating capacity is 42. How many desks of each kind are there?

2. I bought 15 postage stamps paying 72 cents, some 4 cent stamps, the others 6 cent stamps. How many of each did I buy?

3. For $1.06 I bought some 4 cent stamps, some 5 cent stamps and some 6 cent stamps, 21 stamps altogether. Had the price of 5 and 6 cent stamps been increased 1 cent I would have paid $1.20. How many of each stamp did I buy?

4. A grocer wants to mix two brands of coffee, one selling at 70 cents per pound, the other at 80 cents per pound. He wants 20 pounds of mixture to sell at 76 cents per pound, and he wants the net revenue from sales to be the same whether the coffee is sold mixed or unmixed. How many pounds of each brand should be mixed?

5. A collection of dimes and quarters amounts to $2.95. If the dimes were quarters and the quarters dimes, the amount collected would be 30 cents less. How many of each coin are there in the collection?

6. A collection of nickels, dimes, and quarters, 13 coins in all, amounts to $2.40. If the dimes were nickels, the quarters dimes, and the nickels quarters, the collection would amount to $1.45. How many of each kind are there?

7. A club has 28 members. Its junior members pay monthly dues of 25 cents, and all others pay monthly dues of 35 cents. During one month when all paid dues, the collection

was \$8.70. How many junior members are there?

*8. A manufacturer makes two kinds of toys, A and B, using three machines M_1, M_2, M_3 in the manufacturing process of each toy. The table displays the number of minutes needed on each machine for one toy of each kind. In making a batch of toys M_1 was used 4 hours 20 minutes;

	M_1	M_2	M_3
A	4	8	6
B	6	5	3

M_2 was used 5 hours 10 minutes, and M_3 was used 3 hours 30 minutes. How many toys of each kind were there in the batch?

9. Relative to a coordinate system, an equation of a line is $ax + by = 7$. The line contains points with coordinates $(-2,3)$ and $(4,5)$. Find a and b.

10. Relative to a space coordinate system, a plane has equation $ax + by + cz = 12$. Find a, b, c if the plane contains points with coordinates $(1,2,-3)$, $(1,-3,2)$, $(3,1,-2)$.

11. A contractor employed 12 men; some he paid \$15 per day, some \$18 per day, and the rest \$20 per day, expecting to pay a total of \$219 per day. His bookkeeper erroneously interchanged the number of men earning the least with the number earning the most and prepared a payroll of \$204. How many were hired at each rate?

12. There were twice as many men on a bus as women. At the next stop four men got off and five women got on. Then there were as many women as men. How many men and women were there at first?
13. There were a total of 46 passengers on a bus, consisting of men, women, and children. At the next stop two men got off. Then there were as many adults as children. At the next stop 12 children got off. Then the number of children was equal to the difference between the numbers of women and men. How many men, women, and children were there at first?
14. Said a young boy: "I am thinking of two numbers. Whether I take four times the first minus the second, plus 2; or twice the first plus the second, plus 4; or three times the first minus the second, plus 1, I always get zero." What numbers did the young boy have in mind?
15. The sum of the ages of man, wife, and son is 64 years. In 6 years the father will be three times as old as the son. Four years ago the mother was 12 times as old as the son. How old is each now?
16. A contractor plans to spend $295,000 to build three types of houses, 16 in all. It costs $15,000 to build one house of the first type, $20,000 to build one house of the second type, and $25,000 to build one house of the third type. How many of each type should he build if

there are to be as many of the first type as the other two together?

17. A restaurant owner plans to use x tables each seating 4, y tables each seating 6, and z tables each seating 8; altogether 20 tables. If fully occupied, the tables will seat 108 customers. If only $\frac{1}{2}$ of the x tables, $\frac{1}{2}$ of the y tables and $\frac{1}{4}$ of the z tables are used, each fully occupied, then 46 customers will be seated. Find (x,y,z).

2.15 Summary

This chapter presented the pivotal method for solving a system of m linear equations in n variables, $m \leq 3$ and $n \leq 3$. (The method can be used for any m and any n.) This involved

(a) The notion of pivotal operations on equations, and equivalent systems of linear equations.

(b) Two elementary operations on equations in a system of equations; the first replaces an equation with one in which a coefficient is 1; the other replaces an equation with one in which a coefficient is 0.

(c) These pivotal operations are repeated as far as possible. The last system then has the Gauss-Jordan reduced form, in which each column has zeros and possibly one 1.

(d) The convenience of a tableau arrangement that records the operation results and equivalent systems.

The pivotal operation method enables us to

(a) solve systems that have one, none, or an infinite number of solutions,

(b) solve a set of systems of linear equations that have the same coefficient matrix,

(c) find the inverse of a square matrix, if it has one.

The chapter ended with word problems, that were solved with the aid of systems of linear equations.

2.16 Review Exercises

Using the pivotal method in tableau form solve the systems in Exercises 1-6. Express infinite sets, if any, in set notation.

1. $x - 4y = a$
   $-x + 3y = b$

2. $x + 3y + 2z = 1$
   $x + 2y + 2z = 3$
   $3x + 7y + 5z = 6$

3. $-x - y + 2z = 10$
   $-x + y = 11$
   $x + 2y - z = 9$

4. $\begin{bmatrix} 3 & 2 \\ 5 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

6. $3x + 2y = 8$

Find the inverses of each of the matrices in Exercises 7-10, if any.

7. $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 7 & 9 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

12. Solve the systems listed below in one step, after writing a matrix equation. (Hint: use the result of Exercise 7.)

$3x + 2y = 4$ $3x + 2y = 7$ $3x + 2y = 0$

$7x + 5y = 11$ $7x + 5y = 17$ $7x + 5y = 0$

13. Without solving, show that

$$3x + y - z = 0$$
$$x - 2y + z = 0$$
$$4x - y = 0$$

has an infinite number of solutions.

14. The value of $ax^2 + bx + c$ is 0 when $x = 1$; 5 when $x = 2$; and 13 when $x = 3$. Find (a,b,c).

15. A dealer puts up pens and pencils in two kinds of packages, 4 pencils and 3 pens, in one, 3 pencils and 5 pens in the other. How many of each package should one buy to obtain a total of 38 pencils and 45 pens?

16. To be admitted to a concert, elementary school students pay 25 cents each, high school students pay 50 cents each, and college students pay one dollar each. One hundred students paid 63 dollars and 75 cents. Had the price of admission for high school students been reduced 15 cents and that for college students 25 cents, the receipts would have been 48 dollars and 50 cents. How many students at each level attended the concert?

Chapter 3

THE ALGEBRA OF MATRICES

3.1 The World of Matrices

In Chapters 1 and 2 of this course, we have come across a new kind of entity--the matrix (plural, matrices). We have seen them arising in a variety of circumstances and have observed how matrices can be used to organize and express complex sets of facts easily, simply, and clearly. Furthermore, we have gone through various processes and activities in analyzing and solving the problems that we expressed by means of these matrices. These activities may have reminded us of activities that we used in many areas in mathematics such as addition and multiplication of numbers.

In this chapter we will organize what we have learned about matrices in a mathematical way and will explore to see what structures we are led to. In this study we will proceed by means of definitions, theorems, and proofs.

Definition 1. Let m and n be natural numbers. A rectangular array (arrangement) of mn elements chosen from a set S, and arranged in m rows and n columns is an m x n S matrix, or an m x n matrix over the set S, or simply, if the set S is clearly understood, an m x n matrix. The elements of S are called scalars.

<u>Definition</u> 2. m and n are called the dimensions of the matrix. If m = n, the matrix is called a <u>square</u> matrix and m is its <u>order</u>.

<u>Notation</u>. We use capital letters A,B,C,... to name matrices and lower case letters a,b,c,... to name scalars. The scalar in the <u>i</u>th row <u>j</u>th column of A is denoted a_{ij}, where capital A and lower case a correspond. Thus the scalar in the <u>i</u>th row <u>j</u>th column of matrix B is b_{ij}, and so on.

For example, the 2 X 3 matrix B should be:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

and the m x n matrix A would be:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

<u>Definition</u> 3. Two matrices A and B are <u>equal</u>, written A = B, if and only if they have the same dimensions, and for all i and j, $a_{ij} = b_{ij}$.

<u>Theorem</u> 1. Equality of matrices is an equivalence relation. To prove this we must prove three things:

(a) For all matrices A, A = A (the reflexive property).

(b) If $A = B$, then $B = A$ (the symmetric property).

(c) If $A = B$ and $B = C$, then $A = C$ (the transitive property).

<u>Proof.</u> The proof of each part depends on the corresponding properties of equality for scalars. You are asked to supply the proofs in an exercise.

We have not yet specified the nature of S, the set of scalars. Let us agree, from now on, unless otherwise specified, that S is the field of real numbers, R. In some exercises the field may be $(Z_5,+,\cdot)$ or other finite fields.

3.2 Exercises

1. Consider

$$A = \begin{bmatrix} 1 & 7 & 5 & 1 & 0 \\ 2 & -1 & 6 & 1 & 0 \\ 3 & 2 & 8 & 1 & -7 \\ 4 & 0 & -1 & 1 & 0 \end{bmatrix}$$

   (a) What are the dimensions of A?

   (b) What is a_{13}? a_{41}? a_{35}? a_{22}? a_{43}?

   (c) For what values of i, j is $a_{ij} = 0$?

2. Write the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ if $a_{ij} = 3i - 2j + 2$.

3. Solve the following equations:

a) $\begin{bmatrix} x + 3 \\ 2 - y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} x^2 & y \\ x & y^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

4. Write the matrix whose entries are the sums of the corresponding entries of the matrices:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \\ 5 & -2 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$

5. Prove that equality of matrices is

(a) reflexive: for all A, $A = A$

(b) symmetric: if $A = B$, then $B = A$

(c) transitive: if $A = B$ and $B = C$, then $A = C$

3.3 The Addition of Matrices

We have already seen that addition of two matrices of the same dimensions by adding corresponding elements of the two matrices is a quite natural operation and lends itself to useful applications.

Definition 4. Let A and B be $m \times n$ matrices. By the sum $A + B$ is meant the matrix C where

$c_{ij} = a_{ij} + b_{ij}$.

<u>Example</u> 1. $\begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$

<u>Example</u> 2. $\begin{bmatrix} 5 & 3 & -1 \\ -2 & 1 & 4 \\ 7 & -3 & 2 \end{bmatrix} + \begin{bmatrix} -5 & -3 & 1 \\ 2 & -1 & -4 \\ -7 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

To add $\begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ makes no sense because our definition for addition applies only to matrices having the same dimensions.

<u>Definition</u> 5. A matrix is called a <u>zero</u> matrix if each of its entries is 0. The zero matrix is denoted $\bar{0}$. (The bar indicates its matrix nature and distinguishes it from the letter O and the numeral 0.) To emphasize its dimensions m x n, we write $\bar{0}_{mn}$, or if it is a square matrix of order m, we write $\bar{0}_m$.

<u>Example</u> 3. $\bar{0}_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\bar{0}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

<u>Definition</u> 6. A matrix B is called the <u>additive</u> <u>inverse</u> of A (write as -A) if each element of B is the opposite of the corresponding element of A.

<u>Example</u> 4. If $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \\ -4 & 0 \end{bmatrix}$ then $B = \begin{bmatrix} -2 & 3 \\ -1 & -2 \\ 4 & 0 \end{bmatrix} = -A$.

<u>Theorem</u> 2. Let M be the set of m x n matrices. Then (M,+) is an abelian (commutative) group.

<u>Proof.</u> The proof of this theorem has five parts, four of them to prove the group properties, and one to prove it commutative:

(a) (M,+) is an operational system. That is, the sum of any two matrices in M is in M.

(b) For any two matrices A, B $\in$ M, A + B = B + A. (commutativity)

(c) For any three matrices A, B, C $\in$ M, (A + B) + C = A + (B + C). (associativity)

(d) There exists a matrix Z in M, such that for all A in M, A + Z = Z + A = A. (existence of identity) (Z of course is $\mho_{mn}$.)

(e) For each A in M there exists a B in M, such that A + B = B + A = $\mho$. (we denote B as -A or A as -B) (existence of inverse element)

Proofs of each of these parts are based on the field properties of the set of scalars. We prove (a) to show how this is done, and you are asked to prove the other properties as an exercise.

<u>Proof of (a)</u> Let A and B be in M. Then, for each i = 1, 2, ..., m and each j = 1, 2, ..., n the elements a_{ij} and b_{ij} are scalars in R and therefore $a_{ij} + b_{ij}$ is in R. We conclude that A + B is in M. Hence (M,+) is an operational system.

The existence of an additive inverse (property (e)) makes possible the operation of subtraction of matrices.

Definition 7. If A and C are m x n matrices then $A - C = A + (-C)$.

Just as there is a unique solution for the equation $x + a = b$ in the group $(R,+)$ (namely $x = b + -a$), and a unique solution for the equation $ax = b$, $a \neq 0$, in the group $(R/\{0\},\cdot)$ (namely, $x = b \cdot \frac{1}{a}$) so there is a unique solution for the equation

$$X + A = B$$

where A and B are in M, namely

$$X = B + (-A) = B - A. \qquad \text{Why?}$$

Example 5. Solve $X + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$.

Solution. $X = \begin{bmatrix} e & f \\ g & h \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$= \begin{bmatrix} e - a & f - b \\ g - c & h - d \end{bmatrix}.$$

The check is left for you.

3.4 Exercises

1. Add, if possible.

(a) $\begin{bmatrix} 3 & 2 & 0 \\ 5 & 6 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 3 \\ -2 & -6 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 & 0 \\ 5 & 6 & -2 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 2 & -6 \\ 3 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

2. Subtract, if possible.

(a) $\begin{bmatrix} 3 & 2 & 8 \\ 5 & -6 & 0 \end{bmatrix} - \begin{bmatrix} 2\frac{1}{2} & 3 & 8 \\ \sqrt{2} & 4 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \end{bmatrix}$

3. Let A and B be matrices having the same dimensions.

Prove:

(a) $-(A + B) = (-A) + (-B)$

(b) $-(-A) = A$

(c) $-\overline{0} = \overline{0}$

4. Find values of a, b, c, and d that satisfy:

(a) $\begin{bmatrix} a - 2 & 2b + 1 \\ a + 3 & 16 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ c & 3d - 2 \end{bmatrix}$

(b) $\begin{bmatrix} 3a & 10 \\ 2a + c & 2b - d \end{bmatrix} = \begin{bmatrix} 15 & 2b \\ 10 & 0 \end{bmatrix}$

5. Let M be the set of m x n matrices. Prove:

(a) For all A and B in M, $A + B = B + A$.

(b) For all A, B, C in M, $(A + B) + C = A + (B + C)$.

(c) For all A in M, $A + \overline{0}_{mn} = \overline{0}_{mn} + A = A$.

(d) For each A in M, there is a B in M such that

$$A + B = B + A = \overline{0}_{mn}$$

6. Express each of the following as a single matrix.

(a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix} + \begin{bmatrix} \frac{1}{6} & \frac{1}{7} \\ \frac{1}{8} & \frac{1}{9} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 2 & -2 \\ \frac{1}{3} & 3 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 \\ -3 & \frac{1}{3} & 3 \end{bmatrix}$

(c) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2a & 3b \\ 4a & a+b \end{bmatrix} + \begin{bmatrix} -a & -2b \\ b & a-b \end{bmatrix}$

3.5 Multiplication by a Scalar

From our definition of the addition of matrices we have:

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -2 & 0 \end{bmatrix}$$

We can express this in another form:

$$2\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 \times 3 & 2 \times 2 \\ 2 \times -1 & 2 \times 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -2 & 0 \end{bmatrix}$$

We define formally a new operation on matrices:

Definition 8. If A is an m x n matrix and k is a scalar, then by kA is meant the m x n matrix C where $c_{ij} = ka_{ij}$.

Example 1. $3\begin{bmatrix} 4 & 2 \\ 1 & 0 \\ 3 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 12 & 6 \\ 3 & 0 \\ 9 & 1 \end{bmatrix}$

We should notice that this new operation of multiplying a matrix by a scalar is different from the operations we have seen. In various systems (including the addition of matrices) we have one set of elements and we combine members of the set to obtain new elements of the same set. Here we have two sets - a set of scalars and a set of matrices. We combine them to obtain new members of one of the sets - the set of matrices. Why does it not make sense to talk of closure in connection with this operation?

However, though this is a new kind of operation, it has many properties similar to properties we have studied before.

Theorem 3. Let A and B be m x n matrices, and let k and ℓ be scalars.

(a) $k(A + B) = kA + kB$ (distributive law)

(b) $(k + \ell)A = kA + \ell A$ (another distributive law)

(c) $k(\ell A) = (k\ell)A$ (a kind of associativity)

(d) $kA = \overline{O}$ if and only if $k = 0$ or $A = \overline{O}$

(e) $1 \cdot A = A$

(f) If $kA = kB$ and $k \neq 0$ then $A = B$ (scalar cancellation).

Proof of (a). Let $C = A + B$, then for all i, j:

$c_{ij} = a_{ij} + b_{ij}$ (definition of +).

$kc_{ij} = ka_{ij} + kb_{ij}$ (distributive property of $(R,+,\cdot)$).

$kC = kA + kB$ (definition of +).

$k(A + B) = kA + kB$ (SPE, replacing A + B for C)

The proofs of the remaining parts are based on the properties of $(R,+,\cdot)$ and are left for you in an exercise.

Theorem 4. Let A be a specific matrix in the set M of m x n matrices. Then the set of matrices $\{kA : k \in R\}$ is a subgroup of $(M,+)$.

Comment. Let $m = n = 2$. To illustrate what $\{kA : k \in R\}$ means, let $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$.

For $k = 2$, $kA = 2\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$

For $k = \sqrt{2}$, $kA = \sqrt{2}\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & 2\sqrt{2} \\ \sqrt{2} & 4\sqrt{2} \end{bmatrix}$

For $k = 0$, $kA = \overline{O}_2$

For $k = 1$, $kA = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$.

Similarly for each $k \in R$ we obtain a matrix in $\{kA\}$.

Proof. Remember, (Course II, Chapter 2) to prove that a subset of a group is a subgroup we need prove only:

(a) the subset is an operational system under the operation of the group.

(b) for every element of the subset, its group inverse is in the subset.

Proof of (a). Let k_1 and k_2 be scalars. Then $k_1A + k_2A = (k_1 + k_2)A$ by (b) of Theorem 3. Since $k_1 + k_2 \in R$, $(k_1 + k_2)A$ is in the subset.

You are asked to prove (b) in an exercise.

Theorem 5. (a) The subset $B = \{qA : q \in Q\}$ is a subgroup of $\{kA\}$.

(b) The subset $C = \{iA : i \in Z\}$ is a subgroup of B.

(c) The subset $\{\overline{O}\}$ is a subgroup of C.

<u>Example</u> 2. With the aid of Theorem 3 we can solve a matrix equation such as

$$-3\left(X + \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}\right) = 2X + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

where X is a 2 X 2 matrix.

<u>Solution</u>. By Theorem 3(a) the left member can be written

$$-3X + \begin{bmatrix} -9 & -6 \\ 3 & -12 \end{bmatrix} = 2X + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Adding 3X to both members,

$$3X - 3X + \begin{bmatrix} -9 & -6 \\ 3 & -12 \end{bmatrix} = 3X + 2X + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

By Theorem 3(b)

$$\begin{bmatrix} -9 & -6 \\ 3 & -12 \end{bmatrix} = (3 + 2)X + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

So $\begin{bmatrix} -9 & -6 \\ 3 & -12 \end{bmatrix} = 5X + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Adding $\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$, the additive inverse of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ to both members gives

$$\begin{bmatrix} -10 & -7 \\ 2 & -12 \end{bmatrix} = 5X + \overline{O} = 5X.$$

Multiplying both members by $\frac{1}{5}$, we get by Theorem 3(c) and 3(e)

$$\begin{bmatrix} -2 & -\frac{7}{5} \\ \frac{2}{5} & -\frac{12}{5} \end{bmatrix} = \frac{1}{5}(5X) = X.$$

This completes the solution. See if it checks.

3.6 Exercises

1. Let $A = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & \sqrt{3} \end{bmatrix}$. Express kA as a single matrix, if k is equal to:

   (a) 2 (b) 3 (c) -2 (d) 0
   (e) $\sqrt{3}$ (f) $2 + \sqrt{3}$ (g) $\frac{1}{3}$ (h) .2

2. Let $A = \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 \\ 0 & -4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$. Express each of the following as a single matrix.

   (a) $2A + B - C$ (b) $3A + 2B - 4C$
   (c) $2(A + 2C) - 3C$ (d) $\sqrt{2}(A + B + C) - A$

3. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   Express each of the following as a single matrix.

   (a) $A + B + C$ (b) $A + B - C$ (c) $A - (B + C)$
   (d) $2A + 3B + 4C$ (e) $3(A - B) + 2C$

4. (a) Verify the statement:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Express $\begin{bmatrix} 3 & -2 \\ 0 & 2 \end{bmatrix}$ as the sum of four 2 X 2 matrices, each containing three zeros and one 1, and each multiplied by a suitable scalar.

(c) Express $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ as the sum of six 2 X 3 matrices whose entries are all zeros except for one 1, each multiplied by a suitable scalar.

5. Let A, B, C be the matrices of Exercise 2. Solve each of the following equations for X.

(a) $A + X = B + C$ (b) $A + 2X = B - C$

(c) $\frac{1}{2}(A + X) = 3X + 2B$ (d) $3(B - X) = 2(X - C) - B$

6. Let A and B be m x n matrices and let k and ℓ be scalars. Prove:

(a) $(k + \ell)A = kA + \ell A$ (c) $kA = \overline{0}$ iff $k = 0$ or $A = \overline{0}$.

(b) $(k\ell)A = k(\ell A)$ (d) $1 \cdot A = A$

(e) $kA = kB$ and $k \neq 0$ imply $A = B$.

7. Let A be a specific matrix in the set M of m x n matrices, and form the set $\{kA : k \in R\}$. Prove that every matrix of $\{kA\}$ has an additive inverse in $\{kA\}$.

8. Consider the field $\{Z_3, +, \cdot\}$ and the set P of 2 X 2 matrices over Z_3.

   (a) How many matrices are there in P?

   (b) Is (P,+) a group? Why?

   (c) List the members of $\{kA\}$ if $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $k \in Z_3$. Is $\{kA\}$ a group? Why?

   (d) List the members of $\{kB\}$ where $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $k \in Z_3$. Is $\{kB\}$ a group? Why? Find $\{kA\} \cap \{kB\}$.

   (e) Show that $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ form a subgroup of P.

   (f) Solve in $\{Z_3, +, \cdot\}$ the matrix equation:

$$X + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = 2\left(X + \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right)$$

9. Is $\{kA : k \in Z_4$ and A is a specific 2 X 2 matrix with entries in $Z_4\}$ a group under addition? Be prepared to support your answer.

3.7 Multiplication of Matrices

In this section we concentrate mainly on 2 X 2 matrices. The reason for this concentration lies in the fact that multiplication of matrices is more complicated than either addition or multiplication of a matrix by a scalar, and it is easier to unravel these complications for the relatively simpler 2 X 2 matrices.

We have seen in Chapters 1 and 2 how matrices are multiplied. Recall that the entry in the product matrix is found by multiplying numbers in a row of the first matrix by numbers in a column of the second, and adding the products. You may wish to remember this procedure as: "multiply row by column."

Example 1. As you follow this example, note the dimensions written below each matrix.

$$\underset{2 \times 3}{\begin{bmatrix} 3 & 4 & 0 \\ 2 & -1 & 3 \end{bmatrix}} \cdot \underset{3 \times 1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = \begin{bmatrix} 3\cdot1 + 4\cdot2 + 0\cdot3 \\ 2\cdot1 + (-1)\cdot2 + 3\cdot3 \end{bmatrix} = \underset{2 \times 1}{\begin{bmatrix} 11 \\ 9 \end{bmatrix}}$$

An m x n matrix times an n x p matrix produces an m x p matrix.

Example 2. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$

Example 2 may serve as the definition for multiplication of 2 X 2 matrices. The general definition follows.

Definition 9. Let A be an m x n matrix and B an n x p matrix. The product AB = C is the matrix whose entry in the *i*th row *j*th column is the sum of the products formed by multiplying the *k*th number in the *i*th row of A by the *k*th number in the *j*th column of B, where k = 1, 2,..., n, i = 1, 2,..., m, and j = 1, 2,..., p. (See Figure 3.1.)

$$\underset{A}{\underline{i}\text{th row}\ \begin{bmatrix} \\ a_{i1} \ a_{i2} \cdots a_{ik} \cdots a_{in} \\ \\ \end{bmatrix}}
\underset{B}{\begin{array}{c} \underline{j}\text{th column} \\ \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{kj} \\ \vdots \\ b_{nj} \end{bmatrix} \end{array}}
=
\underset{C}{\begin{array}{c} \underline{i}\text{th row} \\ \underline{j}\text{th column} \\ \begin{bmatrix} \vdots \\ \cdots c_{ij} \cdots \\ \vdots \end{bmatrix} \end{array}}$$

with $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$

Figure 3.1

Let us examine in some detail the set M_2 of 2 X 2 matrices under multiplication. Our first question is: Is multiplication of such matrices commutative? Perhaps you noted in Chapter 1 that it is not. To show that it is not we need exhibit but one counter-example. To this end let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ while $BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

and we see that $AB \neq BA$.

Is multiplication in M_2 associative? We can determine the answer by working with

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad D = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, \quad E = \begin{bmatrix} i & j \\ k & \ell \end{bmatrix},$$

and observe whether or not the product matrix of (CD)E is the same as that of C(DE). You will find, if you carry out the details of these multiplications, that the two products are indeed the same. We urge you to find these products yourself as a profitable exercise and thus prove

<u>Theorem</u> 6. Multiplication in M_2 is associative.

Is there a multiplicative identity in M_2? We easily show

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let us denote $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by I_2. This leads to our next theorem.

<u>Theorem</u> 7. For any matrix A in M_2, $AI_2 = I_2A = A$.

Is there perhaps another matrix in M_2 that behaves like I_2 in this respect? If there were, say $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$, then

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ would equal $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. But $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} =$ $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Hence $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ equals $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, that is, I_2. This proves our next theorem and permits the definition that follows it.

<u>Theorem</u> 8. I_2 is the only matrix in M_2 such that for all A in M_2

$$AI_2 = I_2A = A$$

Definition 10. The multiplicative identity (or unit matrix in M_2 is $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Having established a unique identity matrix in M_2 we go on to investigate whether or not for every matrix in M_2 there is a multiplicative inverse in M_2. We can exhibit a matrix in M_2 that has no such inverse. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. If B is its inverse, let it be represented by $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x + 2z & y + 2w \\ x + 2z & y + 2w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or

(1) $x + 2z = 1$ (2) $y + 2w = 0$

(3) $x + 2z = 0$ (4) $y + 2w = 1$

Look at (1) and (3). Are there any values of x and z for which both (1) and (3) are simultaneously true? Clearly not. Therefore A has no multiplicative inverse.

On the other hand some matrices in M_2 do have inverses in M_2. For instance $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$, the coding and decoding matrices appearing in Chapter 1, Section 1.7 are inverses of each other. Verify.

To summarize what we have said about $(M_2, \cdot)$:

1. It is an operational system. This follows directly from the definition of multiplication.
2. It is not commutative.
3. It is associative.
4. It has a unique identity I_2.
5. Some matrices in M_2 do not have inverses in M_2; some do.

3.8 Exercises

1. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

   Compute each of the following.

   (a) AB (b) AC (c) BC

   (d) BA (e) CA (f) CB.

2. For A, B, C in Exercise 1, determine whether or not $AB = -BA$, $AC = -CA$, $BC = -CB$. Do you think that for all matrices D and E in M_2, $DE = -ED$? If not, exhibit two matrices for which this is not true.

3. Let $A = \begin{bmatrix} -2 & 3 \\ 2 & 1 \end{bmatrix}$. By A^2 we mean $A \cdot A$. Find A^2.

   Suggest what A^3 means and find it.

4. Show that $\begin{bmatrix} a & a \\ b & b \end{bmatrix}$ has no multiplicative inverse in M_2, no matter what values a and b take on.

5. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   Determine whether each of the following statements is true or false.

   (a) $A(B + C) = AB + AC$ (c) $A(B + C) = AB + CA$

   (b) $(B + C)A = AB + AC$ (d) $A(B + C) = BA + CA$.

6. Let $E = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $F = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$.

   (a) Prove $EF = FE$.

   (b) Note for E that $e_{11} = e_{22}$ and $e_{12} = -e_{21}$; also note a similar statement for the elements of F. Show that EF has the same property.

   (c) Let $b = 0$. Show $EF = aF$.

7. Show by a direct substitution that:

   (a) $B - 2I_2 = \bar{0}_2$, when $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

   (b) $A^2 - 2A - 3I_2 = \bar{0}_2$ when $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$.

   (c) $A^2 - 2A + 2I_2 = \bar{0}_2$ when $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

8. Let $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

   Show by direct substitution that:

   (a) $(A + B)(A - B) \neq A^2 - B^2$.

   (b) $(A + B)(A + B) \neq A^2 + 2AB + B^2$.

   (c) Explain why the statements in (a) and (b) are inequalities that are true while $(a + b)(a - b) =$

$a^2 - b^2$ and $(a + b)(a + b) = a^2 + 2ab + b^2$ are equalities when a and b are real numbers.

9. Show that $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ satisfies $X^2 = \bar{0}$. Find another matrix that satisfies this equation.

10. Find the following products:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

How many square roots does the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ have? Are there any others?

Discuss the solutions of the equation

$$X_2^2 - I_2 = \bar{0}_2$$

[X_2 is a 2 X 2 matrix.]

3.9 Multiplicative Inverses in M_2

In this section we present a test by which one can determine whether or not a matrix in M_2 has an inverse. If it does, then we want to know whether or not it is unique, and how to find it.

We have seen in Section 3.7 that $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ has no inverse. On the other hand we have seen that $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ (our friend, the coding matrix in Chapter 1) does have an inverse. How do we go about finding that inverse? Let us assume that $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is that inverse. Then

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} \text{ should equal } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

That is, $\begin{bmatrix} 2x + 3z & 2y + 3w \\ x + 2z & y + 2w \end{bmatrix}$ should equal $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

This equality between the two matrices demands equality of corresponding elements. That is,

(1) $2x + 3z = 1$ (2) $2y + 3w = 0$

(3) $x + 2z = 0$ (4) $y + 2w = 1$

Observe that equations (1) and (3) have the same variables, x and z. We have solved systems of equations before.

(1) $2x + 3z = 1$

(3) $x + 2z = 0$

(1') $-2x - 3z = -1$ (multiplying each member of (1) by -1)

(3') $2x + 4z = 0$ (multiplying each member of (3) by 2)

$z = -1$ (adding members of (1') and (3'))

When we replace z with -1 in (3) we readily find $x = 2$.

Now $(x,z) = (2,-1)$ satisfies both equations (1) and (3).

By the same method used on equations (2) and (4) we get $(y,w) = (-3, 2)$.

Thus we find $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ (the decoding matrix).

Finally we note that $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ does indeed equal I_2 and $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = I_2$, and our search is ended.

The inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ is $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

Let us broaden our investigation to include any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in M_2. As above, assume its inverse is $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

This leads to four equations

(1) $ax + bz = 1$ (2) $ay + bw = 0$

(3) $cx + dz = 0$ (4) $cy + dw = 1$

Let us assume that $a \neq 0$.

x	z	-1		
a	b	1	$= 0$	A_1
c	d	0	$= 0$	A_2
1	$\frac{b}{a}$	$\frac{1}{a}$	$= 0$	$B_1 = (\frac{1}{a})A_1$
0	$\frac{ad-bc}{a}$	$-\frac{c}{a}$	$= 0$	$B_2 = A_2 + (-c)B_1$
1	0	$\frac{d}{ad-bc}$	$= 0$	$C_1 = B_1 + (-\frac{b}{a})C_2$
0	1	$\frac{-c}{ad-bc}$	$= 0$	$C_2 = (\frac{a}{ad-bc})B_2$

Let $ad - bc = h$, and assume $h \neq 0$.

Then $x = \frac{d}{h}$, $z = \frac{-c}{h}$.

Similarly $y = \frac{-b}{h}$, $w = \frac{a}{h}$. (Question: What if $a = 0$?)

Thus equations (1) - (4) can be solved for x, y, w, z (hence an inverse matrix can be found) if and only if $ad - bc = h \neq 0$.

<u>Theorem</u> 9. The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse in M_2

iff $ad - bc \neq 0$.

<u>Notation.</u> We denote a multiplicative inverse of A by A^{-1}.

<u>Definition</u> 11. A matrix that has no inverse is said to be <u>singular.</u> A matrix that has an inverse is called <u>non-singular</u> or <u>invertible</u>.

Continuing our investigation to find what the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is, we assume $ad - bc \neq 0$, and check $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix}$. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix} = \begin{bmatrix} \frac{ad - bc}{h} & \frac{-ab + ab}{h} \\ \frac{cd - cd}{h} & \frac{-bc + ad}{h} \end{bmatrix}$.

Keeping in mind $h = ad - bc$, the last matrix is seen to be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, or I_2.

One more point. Will $\begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ also be I_2? Try it. You will find it is. So comes our next theorem.

<u>Theorem</u> 10. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $h = ad - bc \neq 0$, then

$$A^{-1} = \begin{bmatrix} \frac{d}{h} & \frac{-b}{h} \\ \frac{-c}{h} & \frac{a}{h} \end{bmatrix} \text{ and } AA^{-1} = A^{-1}A = I_2.$$

It is convenient to write A^{-1} as $\frac{1}{h}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. In this form the formula for an inverse in M_2 is easily remembered.

<u>Theorem</u> 11. A^{-1} is unique.

You will be asked to prove this as an exercise.

<u>Example</u>. Solve: $3x + 2y = 6$

$x + 4y = 5$

<u>Solution</u>. Let A be the coefficient matrix $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$.

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $C = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$.

The equation can then be written

$AX = C$ (check).

Since in A, $h = 3 \cdot 4 - 2 \cdot 1 = 10 \neq 0$, A has an inverse.

It is $A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

Then $A^{-1}AX = A^{-1}C$ (left operation) and we can easily show

$X = A^{-1}C$.

To find what X is we need only obtain the product $A^{-1}C$, as follows:

$$\frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 14 \\ 9 \end{bmatrix} = \begin{bmatrix} \frac{14}{10} \\ \frac{9}{10} \end{bmatrix}$$

Finally we write $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{14}{10} \\ \frac{9}{10} \end{bmatrix}$.

Check: $3(\frac{14}{10}) + 2(\frac{9}{10}) = \frac{42 + 18}{10} = 6$

$(\frac{14}{10}) + 4(\frac{9}{10}) = \frac{14 + 36}{10} = 5$

(Compare this method of solution with that in Chapter 2.)

3.10 Exercises

1. Determine whether or not each of the following matrices has an inverse. If so, find it.

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 9 \\ 2 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} -1 & 0 \\ 3 & 4 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$

(g) $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} a & b \\ \frac{1}{b} & \frac{1}{a} \end{bmatrix}$ $(ab \neq 0)$

2. For what value(s) of x will each of the following matrices be singular (non-invertible)?

(a) $\begin{bmatrix} x & 1 \\ 6 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} x & 9 \\ 4 & x \end{bmatrix}$ (c) $\begin{bmatrix} x & 4 \\ 2 & x-2 \end{bmatrix}$ (d) $\begin{bmatrix} x-1 & 2 \\ 1 & x-2 \end{bmatrix}$

3. (a) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Prove $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

(b) Let $B = kI_2$, k is a non-zero scalar.
Prove $B^{-1} = \frac{1}{k}I_2$.

4. Investigate this question: Is I_2 the only matrix in M_2 that is its own inverse?

5. Prove that $\overline{0}_2$ is a singular matrix.

6. Let A and B be non-zero matrices in M_2 such that $AB = \overline{0}$. Prove that neither A nor B is invertible. (Hint: Use an indirect proof.)

7. Let A be an invertible matrix and B a singular matrix, both in M_2. Determine whether AB and BA are invertible or singular. Support your answer with examples (or a proof if you can find one).

8. (a) Let A and B be invertible matrices in M_2. Show that AB and BA are also invertible by displaying some examples.

   (b) Prove: $(AB)^{-1} = B^{-1} \cdot A^{-1}$ if A and B are invertible matrices in M_2.

9. Using the method of multiplication by inverses, solve each of the following pairs of equations, and check.

(a) $x + 3y = 5$
$2x + 5y = 8$

(b) $3x + 2y = 5$
$2x + y = 3$

(c) $5x + 3y = 13$
$2x + y = 5$

(d) $2x - 7y = 3$
$x - 3y = 2$

(e) $3x + y = 14$
$4x + 2y = 20$

(f) $4x + 3y = 26$
$5x - y = 4$

(g) $3r + 4s = 1$
$5r - 7s = -12$

(h) $5u - 3y = 27$
$6u + 2y = 10$

(i) $ax + by = a$
$bx + ay = b$
$a^2 \neq b^2$

(j) $ax + by = 1$
$bx + ay = 2$
$a^2 \neq b^2$

(k) $ax - by = b$
$bx - ay = a$
$a^2 \neq b^2$

(l) $\frac{1}{2}x + \frac{1}{3}y = 20$
$\frac{1}{3}x + \frac{1}{4}y = 14$

10. Prove Theorem 11.

3.11 The Ring of 2 X 2 Matrices

The set of all matrices is so rich that we find it advisable, both for possible applications as well as further mathematical study, to restrict our investigation to various subsets of the set of all matrices.

We have already found that the set of m x n matrices, $M_{m \times n}$, for a given m and n, and with addition as we defined it, constitutes a commutative (abelian) group. For $m \neq n$ we cannot define a multiplication for this subset of matrices, though we can define a multiplication of m x n by n x p matrices. But in this latter case if A is an m x n matrix and B an n x p matrix, we can multiply them to obtain AB and the result is an m x p matrix, but BA is a meaningless expression unless p = m. (Why?) If m = n = p we have square matrices. Then both AB and BA are defined, but in general $AB \neq BA$. If we restrict our investigation to square matrices - and we will further restrict this to considering 2 X 2 matrices, elements of M_2 - we find that we have a richer structure than a group since we have two operations, addition and multiplication. We will symbolize this structure as $(M_2,+,\cdot)$. We know that $(M_2,+)$ is an abelian group, and that $(M_2,\cdot)$ is an operational system in which multiplication is associative. Moreover, it is not hard to see that in $(M,+,\cdot)$ multiplication distributes over addition (both from the left and the right).

A structure like $(M_2,+,\cdot)$ with the properties given above is called a <u>ring</u>.

Definition 12. A system $(B,+,\cdot)$ is called a ring if:

(a) $(B,+)$ is an abelian (commutative) group;

(b) $(B,\cdot)$ is an operational system in which $a\cdot(b\cdot c) = (a\cdot b)\cdot c$

(c) In $(B,+,\cdot)$

(i) $a\cdot(b + c) = a\cdot b + a\cdot c$

(ii) $(b + c)\cdot a = b\cdot a + c\cdot a$

Theorem 12. The system $(M_2,+,\cdot)$ is a ring.

We found earlier in this chapter that the set M_2 has an identity element under multiplication. This property is not an essential characteristic of a ring. When a ring does have a multiplicative identity element, usually called a unity of the ring, we call the ring a ring with unity.

Theorem 13. The set $(M_2,+,\cdot)$ is a ring with unity.

We will see in the exercises that there are rings which do not have multiplicative identity elements.

In Section 3.13 we will study a subset of 2 X 2 matrices which includes all invertible 2 X 2 matrices. One interesting property of this set is:

Theorem 14. The set of invertible matrices of order 2 is a group (non-commutative) under multiplication.

3.12 Exercises

1. Give a complete formal proof that $(M_2,+,\cdot)$ is a ring with unity.

2. Give a complete formal proof that the set of invertible matrices of order 2 is a group under multiplication.
3. Investigate the set of all integers to see if it is a ring under addition and multiplication. Discuss commutativity and a unity element.
4. Investigate the set of even integers $(E,+,\cdot)$. Discuss commutativity and a unity element.
5. Investigate $(R,+,\cdot)$ to see if it is a ring. Commutativity? Unity? Group property?
6. Investigate the set of integers mod 7, $(Z_7,+,\cdot)$. Ring? Unity? Field? Group under multiplication?
7. Investigate the set of integers mod 6, $(Z_6,+,\cdot)$. Ring? Identity? Field? If $a\cdot b = 0$, what can you say about a or b or both? If $a \neq 0$ and $b \neq 0$ and $a\cdot b = 0$, then a and b are called divisors of zero.
8. Consider the set of matrices:

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) Construct a table of all possible products

$$e_{ij}\cdot e_{k\ell}.$$

(b) Discuss the table. Ring structure? Divisors of zero?

(c) Discuss $e_{11} + e_{12}$.

3.13 A Field of 2 X 2 Matrices

In our previous experience in mathematics we have met many instances of an algebraic structure called a field. Let us recall the definition of a field. The ring of 2 X 2 matrices is not a field, because multiplication is not commutative. The subset of invertible matrices is not a field for the same reason, and also because this subset does not contain the identity element for addition, namely

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Are there subsets of 2 X 2 matrices that are fields? What conditions must we satisfy to get such a subset?

If we take the set of invertible 2 X 2 matrices and add to them the identity element for addition we will have a set which may have a subset in which multiplication is commutative.

Consider the set Y of matrices of the form

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix},$$ where $x, y \in R$. This set contains

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (x = 0,\ y = 0), \quad \text{and also} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (x = 1,\ y = 0).$$

It is not hard to verify that $(Y,+)$ is an abelian group. Since Y contains I_2 we know that for every $A \in Y$, $A \cdot I_2 = A = I_2 \cdot A$. We also know that $x^2 + y^2 = h$ is either zero or positive. It is positive for all elements of Y except $\overline{0}_2$. Therefore, by Theorem 9, for every $A \in Y$, $A \neq \overline{0}_2$, we have an A^{-1} such that

$$A \cdot A^{-1} = I_2 = A^{-1} \cdot A.$$

It remains to prove that for every $Y_1, Y_2 \in Y$

(1) $Y_1 \cdot Y_2 \in Y$

(2) $Y_1 \cdot Y_2 = Y_2 \cdot Y_1$

Let $Y_1 = \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix}$, $Y_2 = \begin{bmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{bmatrix}$. Calculate $Y_1 \cdot Y_2$ and $Y_2 \cdot Y_1$ and verify points (1) and (2) directly above. We therefore have:

Theorem 15. The system $(Y,+,\cdot)$ is a field.

3.14 Exercises

1. Which of the following matrices belong to Y?

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

(i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (j) $\begin{bmatrix} \sqrt{2}+1 & 1-\sqrt{2} \\ \sqrt{2}-1 & \sqrt{2}+1 \end{bmatrix}$

2. Find the inverses of those matrices in Exercise 1 which do belong to Y.

3. Give a complete and formal proof that Y is a field.

4. Study the subset of Y consisting of matrices $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ for which $x^2 + y^2 = 1$.

*5. Consider the transformation whose matrix is

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $a^2 + b^2 = 1$. Prove that under this mapping every point of the unit circle maps into a point of the unit circle.

3.15 Summary

1. Matrices, the equality of matrices, and their addition were defined formally.
   - (a) Equality of matrices is an equivalence relation.
   - (b) There exists an additive identity.
   - (c) The set of m x n matrices is a commutative group under addition.
2. Scalar multiplication is a novel mapping which maps a pair consisting of an element from a set of scalars and one element from a set of matrices into the set of matrices.
   - (a) There are two sets involved in this operation.
   - (b) Scalar multiplication has two distributive properties and one associative one.
   - (c) The set of all scalar multiples of a given matrix is an abelian group.
3. A definition of multiplication of matrices was made formally.
   - (a) It can be performed only if the first matrix has as many columns as the second has rows.

(b) Therefore, square matrices of the same order can always be multiplied.

(c) It is associative when it is possible.

4. For multiplication in the set M_2 of 2 X 2 matrices, we found

(a) $(M_2,\cdot)$ is an operational system.

(b) It is not commutative.

(c) It is associative.

(d) It has a unique identity, I_2.

(e) Some matrices in M_2 do not have inverses; if they do, the inverse is unique.

(f) A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in M_2 has an inverse iff $h = ad - bc \neq 0$. Then the inverse $A^{-1} = \frac{1}{h}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

5. We defined a new algebraic structure called a ring.

(a) The set $(M_2,+,\cdot)$ is a ring with unity.

(b) The set of invertible matrices of order 2 is a non-commutative group under multiplication.

(c) We found rings with and without commutativity; with <u>and</u> without a unity; with <u>and</u> without divisors of zero.

6. It is possible to find subsets of M_2 which are fields.

3.16 Review Exercises

1. Solve for matrix X and check.

$$2X + \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} = 3\left(X - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

2. Prove that $(Z_4, +, \cdot)$ is a ring.

3. For each of the following either give its inverse or explain why it has no inverse.

(a) $\begin{bmatrix} 4 & 1 \\ 11 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}$

4. Express as a single matrix:

$$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

(Hint: It is a 1 X 1 matrix.)

5. Show that $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ satisfies $A^2 - 4A - 5I_2 = \overline{0}_2$.

6. Verify that $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

Does this mean that $\begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}$ is a multiplicative identity? Explain your answer.

7. Construct a 5 X 4 matrix whose elements a_{ij} are given by $a_{ij} = \min(i, j)$.

8. If $x^2 + x - 1 = 0$, show that

$$\begin{bmatrix} x^2 + x & -1 \\ x & 0 \end{bmatrix} = \begin{bmatrix} 1 & -x^2 - x \\ x & 0 \end{bmatrix}$$

9. If we switch around the elements of a matrix so that its rows become columns and its columns become rows (in the same order), we obtain a second matrix called the transpose of the original matrix.

If $A = \begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}$, construct the transpose of A. What is the transpose of $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$?

10. Show that the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

satisfies the equation $A^2 = 0$. Can you find other matrices in M_2 that satisfy this equation?

11. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find AB and BA.

What do you observe? Can you find other matrices that behave this way with each other - or with A or B?

12. Determine which of the following sets are rings under addition and multiplication:

(a) the set of numbers of the form $a + b\sqrt{3}$, where a and b are integers;

(b) the set of numbers $\frac{a}{2}$, where a is an integer.

13. Show that if $A \in M_2$, $B \in M_2$, $B \neq \bar{0}_2$, and $AB = \bar{0}_2$, then A cannot have an inverse. Can B have an inverse?

Chapter 4

GRAPHS AND FUNCTIONS

4.1 Conditions and Graphs

In this chapter we will study many questions and problems which involve graphs. You have constructed graphs already in several situations: (1) lattice point graphs where only points with integer coordinates were used, (2) graphs in coordinate geometry where oblique coordinate axes were used much of the time, and (3) graphs of functions where perpendicular coordinate axes with equal units were used. In this chapter we will consider only graphs in a rectangular coordinate system.

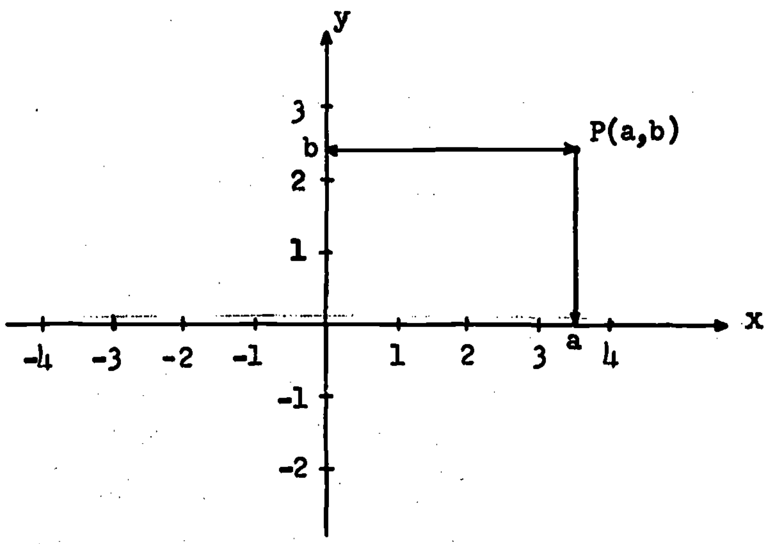


Figure 4.1

Recall that for each ordered pair $(a,b) \in R \times R$, $P(a,b)$ is the point of the plane whose x-coordinate is a and whose y-coordinate is b. (See Figure 4.1.) Since the assignment of ordered pairs of real numbers to points of the plane is a one-to-one correspondence, we often talk about the point (a,b) when we mean the point with coordinates (a,b).

<u>Example</u> 1. Given the condition $2x + y \leq 3$, what is its solution set? What is its graph?

One way to write the solution set is, of course, $S = \{(x,y): 2x + y \leq 3\}$. (Unless the contrary is stated we take $x, y \in R$ to be understood.)

Since

$$2x + y \leq 3 \text{ iff } y \leq -2x + 3 \text{ iff } y = -2x + 3 \text{ or } y < -2x + 3,$$

we can write,

$$S = \{(x,y): y = -2x + 3 \ \underline{\text{or}}\ y < -2x + 3\}$$
$$= \{(x,y): y = -2x + 3\} \cup \{(x,y): y < -2x + 3\}$$

This is about as far as we can go in this direction in examining the solution set, S. Now let us see what the graph, T, of ~~S looks like~~.

The graph of $\{(x,y): y = -2x + 3\}$ is easy to draw since it is a line with slope -2 and must intersect the y axis at (0,3). Now, the line $y = -2x + 3$ divides the plane into 3 subsets: (1) the line itself, (2) the open halfplane "above" the line, and (3) the open halfplane "below" the line. Take first a point (u,v) in the open halfplane below the line (see Figure 4.2).

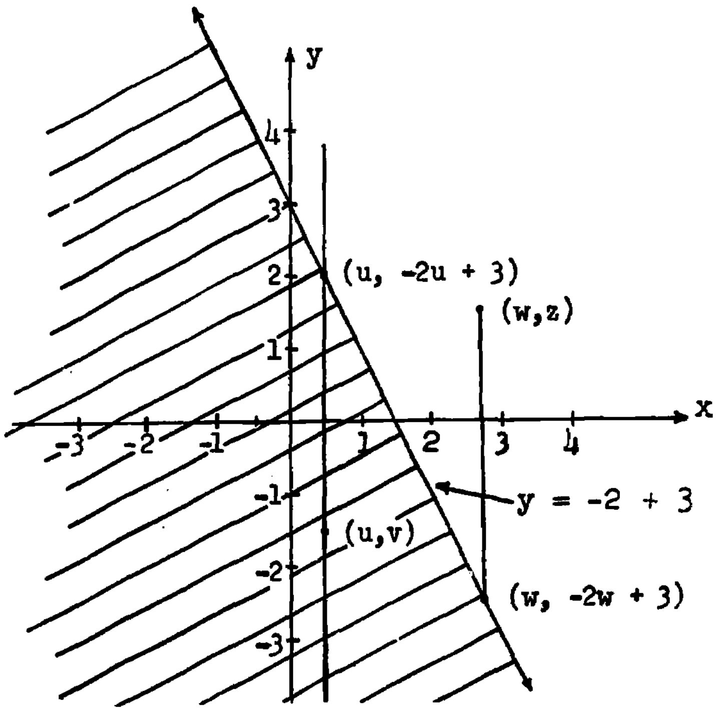


Figure 4.2

The point on the line with the same x-coordinate, u, has y-coordinate $-2u + 3$. Clearly, $v < -2u + 3$. That is, all points below the line must have coordinates satisfying the condition $y < -2x + 3$. The graph of our solution set for $y \leq -2x + 3$ is thus the line plus all points "below" the line. This is indicated by "shading in" the graph below the line. On the other hand, it is easy to see that points "above" the line must have y-coordinates satisfying $y > -2x + 3$, as is shown in the diagram for the point (w, z). In this chapter we will be studying <u>conditions</u>, which are open sentences in two variables, denoted $C(x, y)$.

<u>Example</u> 2. Construct the graph of the condition $C(x,y)$:

$$-x + 3y > 12.$$

We first solve the condition $-x + 3y > 12$ for y. Thus $-x + 3y > 12$ iff $3y > x + 12$ iff $y > \frac{1}{3}x + 4$. We then graph $y = \frac{1}{3}x + 4$. The graph of $y > \frac{1}{3}x + 4$ is the set of points above the line $y = \frac{1}{3}x + 4$ (the shaded region is Figure 4.3).

To indicate that the line is not part of the graph, it is "broken" or "dashed."

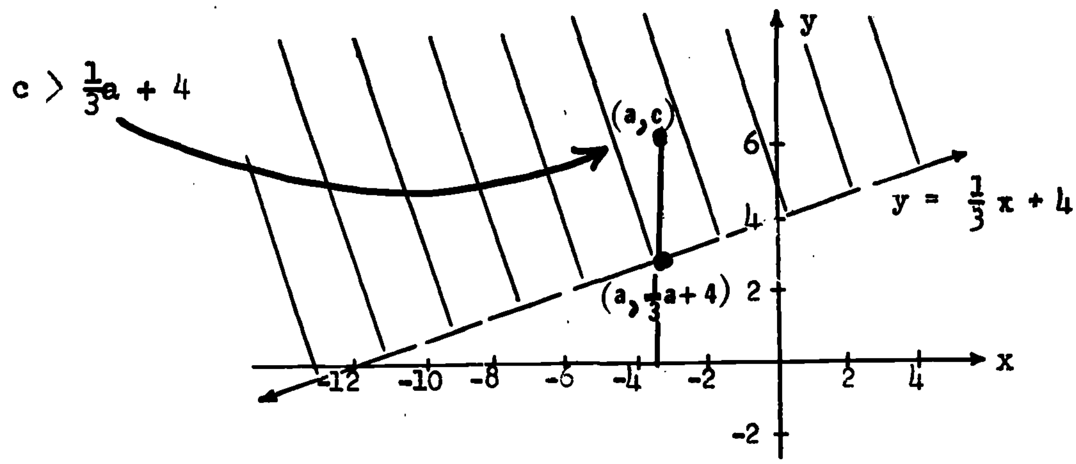


Figure 4.3

Thus, given the condition for a non-vertical line, $y = ax + b$, we can write the conditions for the halfplanes determined by the line:

(1) $y \geq ax + b$ is the condition for the halfplane above the line. $y > ax + b$ is the condition for the open halfplane above the line.

(2) $y \leq ax + b$ is the condition for the halfplane below the line. $y < ax + b$ is the condition for the open halfplane below the line.

Because of the correspondence between lines, halfplanes, and their conditions we often speak of the line $y = 3x + 2$ or the open halfplane $y > 3x + 7$, and so forth.

<u>Example</u> 3. Graph the condition $C(x,y)$: $y \leq |x|$.

Since

$$y \leq |x| \text{ iff } y = |x| \text{ or } y < |x|,$$

the solution set is

$$\{(x, y): y = |x|\} \cup \{(x, y): y < |x|\}.$$

To graph y , $|x|$, break the problem into two parts.

1. If $x \geq 0$, $|x| = x$; so that the graph of $y = |x|$ is the same as the graph of $y = x$ for $x \geq 0$.
2. If $x < 0$, $|x| = -x$ so that the graph of $y = |x|$ is the same as the graph of $y = -x$ for $x < 0$.

The graph of $y = |x|$ is shown in Figure 4.4(a). Notice that this graph partitions the plane into three sets of points - the points of the graph of $y = |x|$, those above this graph, and those below it. The coordinates of all points below the graph of $y = |x|$ satisfy the condition $y < |x|$. The graph of $y \leq |x|$ is shown in Figure 4.4(b).

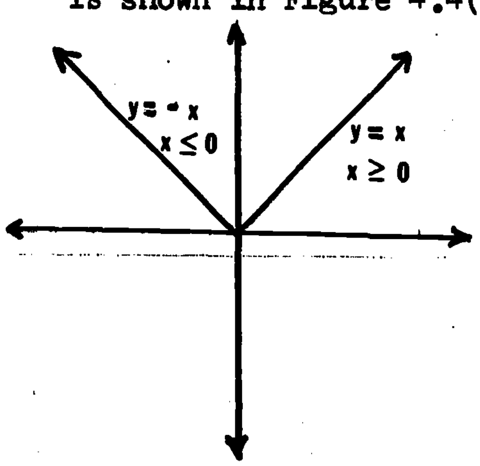


Figure 4.4 (a)

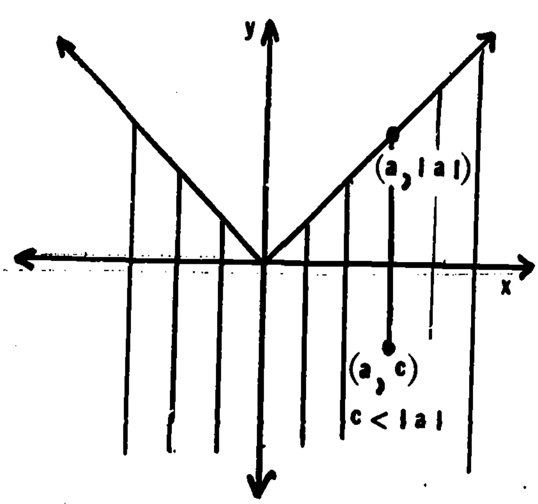
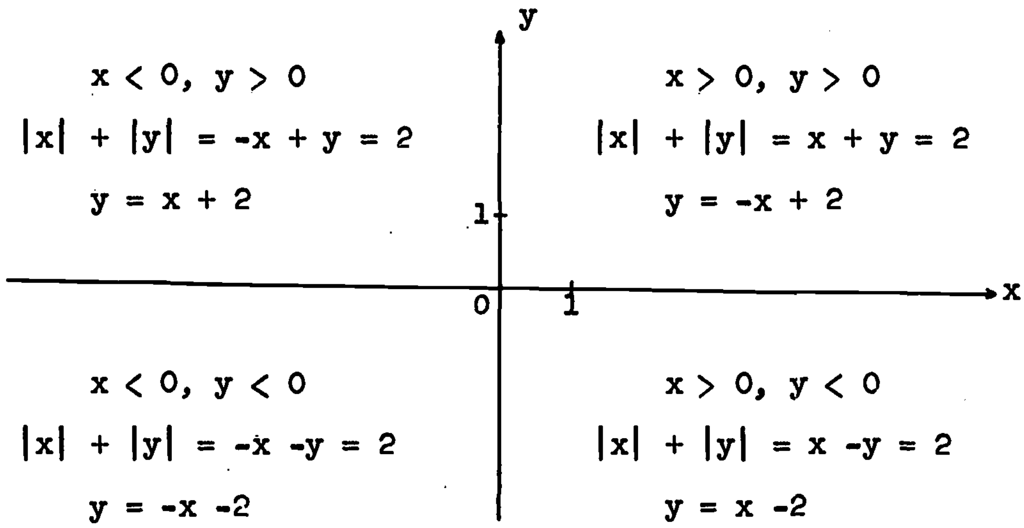


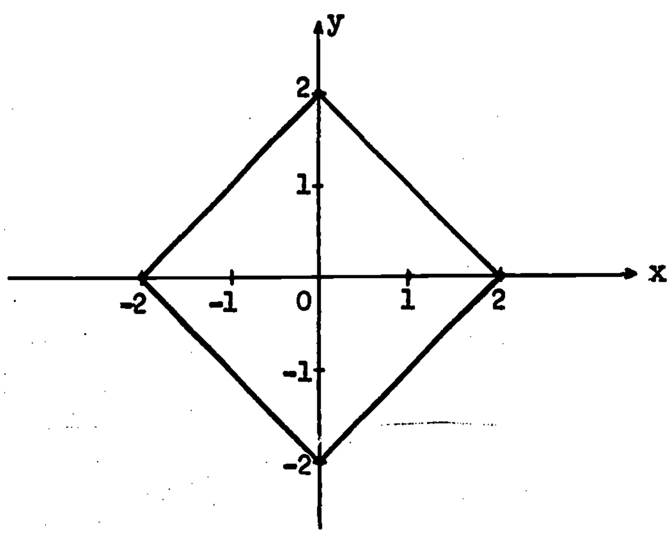
Figure 4.4 (b)

<u>Example</u> 4. Graph the condition $|x| + |y| = 2$.

It is best to do this problem by constructing the graph one quadrant at a time. Figure 4.5(a) shows the details of the analysis, and the graph is constructed in Figure 4.5(b).



(a)



(b)

Figure 4.5

Thus, the graph is constructed in "pieces," one for each quadrant.

<u>Questions</u>. (1) For what points is $|x| + |y| < 2$?

(2) For what points is $|x| + |y| > 2$?

The answer to (1) is the points inside the square, and for (2) the answer is the points outside the square. Check several points to see that this is a reasonable answer.

The graphs in Figures 4.4(a) and 4.5(b) both have symmetry with respect to the y-axis and the graph in Figure 4.5(b) has several other symmetries. Knowledge of these symmetries in advance is helpful in constructing graphs of conditions. For example, in graphing $y \leq |x|$ we could have plotted the points for $x \geq 0$ and drawn in the part for $x < 0$ so as to produce the required symmetry with respect to the y-axis.

A figure is symmetric with respect to a line if it is its own image under the reflection in the line. For the y-axis, the rule of the line reflection is $(x, y) \longrightarrow (-x, y)$. This means that for a graph to be symmetric with respect to the y-axis, (x, y) is in the graph if and only if $(-x, y)$ is in the graph (see Figure 4.4 (a)). In terms of the condition $y \leq |x|$, this means that (x, y) satisfies the condition if and only if $(-x, y)$ satisfies the condition. Since $|-x| = |x|$ for all $x \in R$, the desired property holds for $y \leq |x|$; that is

$$y \leq |x| \text{ iff } y \leq |-x|.$$

The graph of $C(x, y)$ is then symmetric with respect to the y-axis if and only if $C(x, y)$ and $C(-x, y)$ are equivalent (that is, have the same solution set).

Hence, the graph of $|x| + |y| = 2$ is symmetric with respect to the y-axis since

$$|-x| + |y| = 2 \text{ iff } |x| + |y| = 2.$$

Again, the reason is that $|-x| = |x|$ for all $x \in R$. What about the graph of $y = x^2$? Is it true that

$$y = x^2 \text{ iff } y = (-x)^2?$$

Yes, since $(-x)^2 = x^2$. Therefore, the graph of $y = x^2$ is symmetric with respect to the y-axis.

<u>Questions</u>. (1) Is the graph of $|x| + |y| = 2$ symmetric with respect to the x-axis?

(2) What is the test that you apply?

(3) How is the test stated for any condition $\mathcal{C}(x,y)$?

The graph of $|x| + |y| = 2$ is also symmetric with respect to the line $y = x$. A coordinate rule for the reflection in the line $y = x$ is $(x,y) \longrightarrow (y,x)$. As before, then, the graph of a condition $\mathcal{C}(x,y)$ has symmetry with respect to the line $y = x$ if and only if $\mathcal{C}(x,y)$ and $\mathcal{C}(y,x)$ are equivalent. It is clear that

$$|x| + |y| = 2 \text{ iff } |y| + |x| = 2.$$

4.2 Exercises

(All graphing is to be done in a rectangular coordinate system.)

1. Construct a graph for each of the following conditions on the same set of coordinate axes.

(a) $y = 3x$

(b) $y = 3x - 1$

(c) Do you see any pattern in your results?

(d) $y = 3x + 2$

(e) $y = 3x + 7$

(f) What does the number a in the equation $y = ax + b$ for a line tell you about the graph?

2. Construct the graph of each of the following conditions on the same set of coordinate axes.

(a) $y = 3x + 4$

(b) $y = -\frac{1}{3}x + 4$

(c) Do you see any pattern in your results?

(d) $y = -3x + 4$

(e) $y = 2x + 4$

(f) What does the number b in the equation $y = ax + b$ for a line tell you about the graph?

3. Construct the graph of each of the following conditions. Use symmetry as an aid in graphing whenever possible.

(a) $3x - 2y \leq 6$

(b) $y = |x| + 3$

(c) $y = 2|x|$

(d) $y = |x - 2|$

(e) $x = |y|$

(f) $x = |y| + 1$

(g) $x - 5y > 10$

(h) $y = |x| - 2$

(i) $y = -2|x|$

(j) $y = |x + 3|$

(k) $x = -|y|$

(l) $x = |y| - 2$

4. Construct the graph of each of the following conditions.

(a) $y < 3x$

(b) $y < -3x + 4$

(c) $y \leq -3x + 4$ and $x \leq 0$ and $y \geq 0$

(d) $y \geq 3x + 2$

(e) $y \leq -3x + 4$ and $x \geq 0$

5. Construct the graph of each of the following conditions:

(a) $|x| + |2y| = 5$ (e) $|x| - |y| = 3$

(b) $|x + 2| + y = 1$ *(f) $|x| - |y - 1| = -2$

(c) $|x + 2y| = 4$ (g) $2y = |x| + x$

(d) $|y| > |x|$ (h) $|x| + x < 2y$

6. The rule for the reflection in the origin (a point reflection) is $(x,y) \longrightarrow (-x, -y)$. A graph has symmetry with respect to the origin if and only if it is its own image under the reflection in the origin.

   (a) If a graph is symmetric with respect to the origin and $(-3, 4)$ is in the graph, must $(3, -4)$ be in the graph? Must $(4, -3)$ be in the graph?

   (b) What must be true of a condition $C(x,y)$ in order that its graph be symmetric with respect to the origin?

   (c) Is the graph of $|x| + |y| = 2$ symmetric with respect to the origin?

7. Construct the graph of each of the following conditions. Before graphing, determine whether or not the graph is symmetric with respect to (1) the y-axis, (2) the x-axis, (3) the origin, and (4) the line $y = x$.

   (a) $|x| + |y| = 5$ (c) $|x| + |2y| = 3$

   (b) $|x - 2| + |y| = 4$ *(d) $|x + y| = 1$

4.3 Regions of the Plane and Translations

In Section 4.1 the condition $|x| + |y| = 2$ was found to have a square as its graph. It was observed that points interior to this square have coordinates satisfying the condition $|x| + |y| < 2$. Similarly, points exterior to the square have coordinates satisfying the condition $|x| + |y| > 2$. Thus, the graph of $|x| + |y| = 2$ can be considered as dividing the plane into two parts of which it is the common boundary.

You also saw that a condition such as $y = -\frac{1}{3}x + 4$ divides the plane into two open halfplanes of which it is the common boundary. Sets of points in the plane such as the closed halfplanes determined by a line, or the union of a square and its interior, or the union of a square and its exterior, as in the examples above, are called <u>regions</u> of the plane. In these cases they are regions determined by conditions in x and y.

<u>Example</u> 1. Graph the solution set of the condition $y \leq -\frac{1}{3}x + 4$ and $y \leq 2x$ and $y \geq 0$. We first graph the boundary lines $y = -\frac{1}{3}x + 4$, $y = 2x$, and $y = 0$.

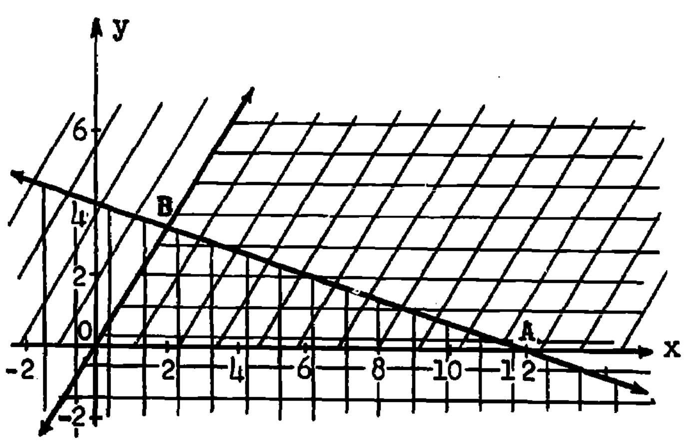


Figure 4.6

The shaded areas in Figure 4.6 show the graphs of the solution sets of $y \leq -\frac{1}{2}x + 4$ (||||), $y \leq 2x$ (≡), and $y \geq 0$ (////). The triply-hatched triangular region OAB (#) is (with the boundary lines) the graph of the given condition. The condition thus determines a triangular region; that is the union of a triangle and its interior.

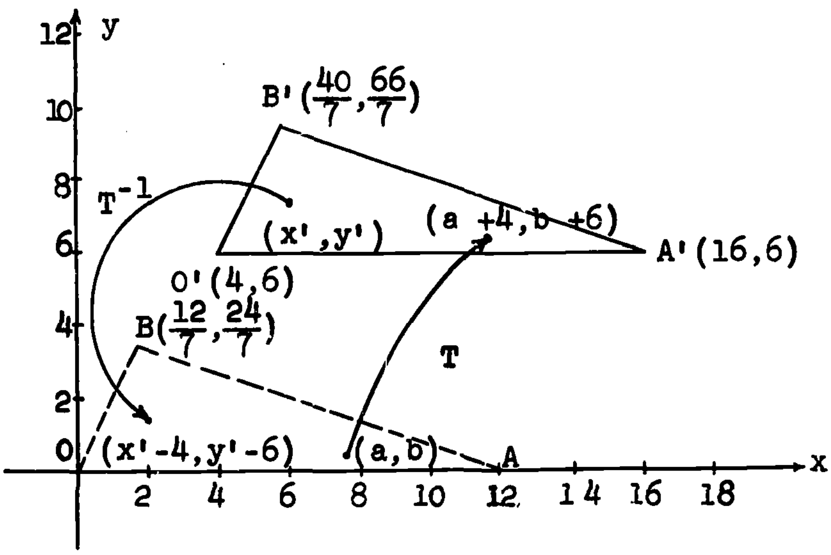


Figure 4.7

In Figure 4.7, $\Delta O'A'B'$ is given by the coordinates of its vertices. We refer to this triangle and its interior as the region $O'A'B'$.

<u>Question.</u> Region $O'A'B'$ is the intersection of three half-planes. What are they?

We know that any line in a coordinate plane is the graph of an equation $y = ax + b$ or an equation $x = c$ (if the line is vertical). Using the methods of coordinate geometry, we find that

(1) $\overleftrightarrow{O'A'}$ is the graph of $y = 6$

(2) $\overleftrightarrow{O'B'}$ is the graph of $y = 2x - 2$

(3) $\overleftrightarrow{A'B'}$ is the graph of $y = -\frac{1}{3}x + \frac{34}{3}$

(See Course II, Section 6.15, Exercise 6.)

Thus, region $O'A'B'$ is the graph of the compound condition $y \geq 6$ and $y \leq 2x - 2$ and $y \leq -\frac{1}{3}x + \frac{34}{3}$. Let us denote this condition by $\mathcal{C}'(x,y)$.

$\Delta\, O'A'B'$ was obviously chosen with malice aforethought for it is easy to see that $\Delta O'A'B'$ is the image of ΔOAB under the translation $T = T_{4,6}$. Let us now explore the relationship of this fact to the conditions $\mathcal{C}(x,y)$ and $\mathcal{C}'(x,y)$ which determine the triangular regions OAB and $O'A'B'$, respectively.

First, let (a,b) be any point in the triangular region OAB. Then its image point under T in the region $O'A'B'$ is the point $(a + 4,\ b + 6)$. Since T is a translation (and hence a one-to-one mapping of the plane onto the plane), it has an inverse $T^{-1} = T^{-1}_{4,6}$. Then we have, by coordinate rules,

$$(x,y) \xrightarrow{T} (x + 4,\ y + 6)$$

$$(x,y) \xrightarrow{T^{-1}} (x - 4,\ y - 6)$$

How is this related to the conditions? First, consider any point (a,b) in region OAB. Its image $(a + 4,\ b + 6)$ must satisfy the condition $\mathcal{C}'(x,y)$ for region $O'A'B'$. This is stated, and the equivalents worked out below.

(1) $(b + 6) \geq 6$ iff $b \geq 0$

(2) $(b + 6) \leq 2(a + 4) - 2$ iff $b \leq 2a$

(3) $(b + 6) \leq -\frac{1}{3}(a + 4) + \frac{34}{3}$

$$\text{iff } b + 6 \leq -\tfrac{1}{3}a - \tfrac{4}{3} + \tfrac{34}{3}$$
$$\text{iff } b + 6 \leq -\tfrac{1}{3}a + \tfrac{30}{3}$$
$$\text{iff } b + 6 \leq -\tfrac{1}{3}a + 10$$
$$\text{iff } \quad b \leq -\tfrac{1}{3}a + 4$$

The equivalents give us precisely the condition $C(x,y)$ for region OAB, stated in terms of a and b. What does this say? In particular, it says that knowledge of the condition $C'(x,y)$ for region O' A' B' enables us to find the condition $C(x,y)$ for region OAB, given that region O' A' B' is the image under a translation of region OAB.

Now take (x',y') any point in region O' A' B'. Its pre-image under T (its image under T^{-1}), $(x' - 4, y' - 6)$, is in region OAB and must satisfy $C(x,y)$. This is stated and the equivalents worked out below.

(1) $(y' - 6) \geq 0$ iff $y' \geq 6$

(2) $(y' - 6) \leq 2(x' - 4)$ iff $y' - 6 \leq 2x' - 8$ iff $y' \leq 2x' - 2$

(3) $(y' - 6) \leq -\frac{1}{3}(x' - 4) + 4$ iff $y' - 6 \leq -\frac{1}{3}x' + \frac{4}{3} + 4$ iff $y' \leq -\frac{1}{3}x' + \frac{34}{3}$. (Check the computations.)

Again, from the condition $C(x,y)$ for region OAB and knowledge of the translation T the condition $C'(x,y)$ for the image region O' A' B' is obtained. This is a general result concerning the graphs of conditions $C(x,y)$ and translations.

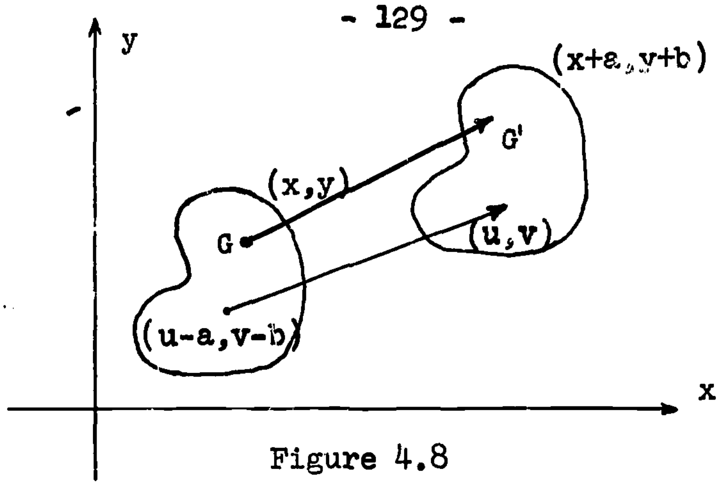


Figure 4.8

If G is the graph of a condition $C(x,y)$ and T is a translation such that

$$(x,y) \xrightarrow{T} (x + a,\ y + b) \quad \text{or} \quad (u - a,\ v - b) \xrightarrow{T} (u,v)$$

and G' is the image of G under T, then a condition $C'(u,v)$ whose graph is G' is given by

$$C'(u,v) = C(u - a,\ v - b)$$

(See Figure 4.8.)

To prove this, note that if (u, v) is in G' then it has a pre-image in G, since T is an "onto" mapping. Since T is one-to-one, that pre-image is precisely one point, $(u - a, v - b)$. But $(u - a, v - b)$ is in G if and only if it satisfies the condition $C(x, y)$. That is, $C(u - a, v - b)$ is true.

Example 2. Graph the condition $|x - 5| + |y - 3| = 2$ (See Figure 4.9.)

Using what we have observed about translations, this graph can be constructed easily from the graph constructed in Example 4 of Section 4.1.

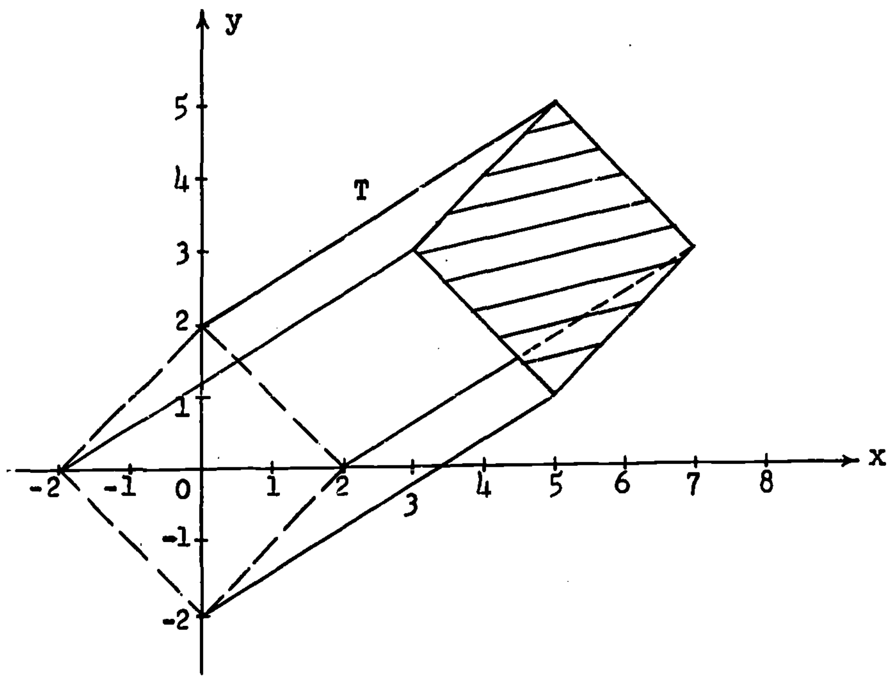


Figure 4.9

The condition is in the form $\mathcal{C}(x - 5, y - 3)$. This suggests that we graph the condition $\mathcal{C}(x,y)$, that is, $|x| + |y| = 2$ and then apply the translation T whose coordinate rule is

$$(x,y) \xrightarrow{T} (x + 5, y + 3).$$

The graph of the image should then satisfy the condition $\mathcal{C}(x - 5, y - 3)$ (i.e. $|x - 5| + |y - 3| = 2$). To accomplish this, as shown in Figure 4.9, we find the image of each vertex of the graph under T and connect them in the proper order.

<u>Question</u>. What is the graph of the condition $|x - 5| + |y - 3| \leq 2$? (See Figure 4.9.)

4.4 <u>Exercises</u>

1. Graph each of the following conditions.

(a) $x \leq 0$ and $y \geq 0$ and $y \leq -x$.

(b) $y \geq 0$ and $x \leq 4$ and $y \leq x$.

(c) $y \leq 0$ and $y \geq -6$ and $x \geq 0$ and $x \leq 3$.

(d) $y \geq 0$ and $y \leq 5$ and $y \leq 2x$ and $y \leq -3x + 18$.

(e) $x \leq 0$ and $x \geq -6$ and $y \leq 4$ and $y \geq 2x + 4$ and $y \leq 2x + 12$.

2. Graph each of the following conditions:

   (a) $x \geq 0$ and $3y - 2x \leq 6$ and $5y - 3x \geq -3$ and $4y + x \leq 20$ and $y \geq 0$.

   (b) $3y + 2x < 9$ and $y < 1$ and $y > x - 7$.
   (Use a "dashed" line to show that a boundary does not belong to the graph of a condition.)

   (c) Find a condition for the complement of the region graphed in (a). (The complement of a region is the set of points of the plane that are not in the region.)

   (d) Find a condition for the complement of the region graphed in (b).

3. (a) Graph the compound condition $y \leq -3x + 4$ and $x \geq 0$ and $y \geq 0$. Find the image of the graph under the translation T with coordinate rule

$$(x,y) \xrightarrow{T} (x + 5,\ y - 7)$$

   (b) Find a condition whos graph is the set of points found as the answer to 3(a).

4. (a) Graph the condition $|x| + |y| = 5$.

   (b) Find the image of the set of points in this graph under the translation T given by $(x,y) \xrightarrow{T} (x + 6,\ y + 6)$.

(c) Find a condition in x and y for the image set.

Answer: $|x - 6| + |y - 6| = 5$.

5. (a) Graph the condition $|x + 2| + |y + 3| \leq 3$.

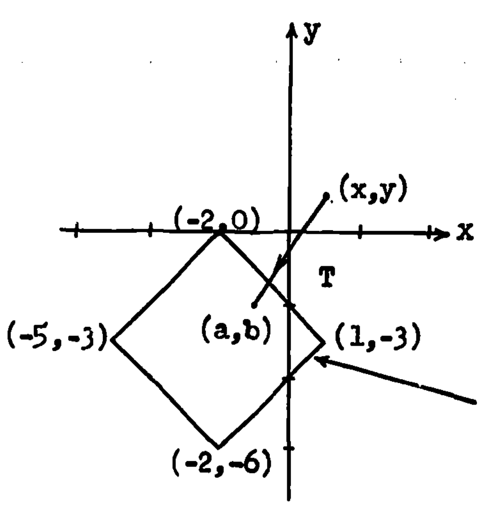
(b) Find the image set of this set of points under the translation T with coordinate rule $(x,y) \xrightarrow{T} (x - 7,\ y - 3)$.

(c) Find a condition in x and y for the image set.

6. (a) Graph the condition $|x| + |y| \leq 3$.

(b) Find the image of this set of points under the translation T with coordinate rule $(x,y) \xrightarrow{T} (x - 2,\ y - 3)$.

(c) Do you see how the translation T can be used to graph the condition $|x + 2| + |y + 3| \leq 3$ from the graph of the condition $|x| + |y| \leq 3$?



If $(a,b) \in G'$, and (x,y) is its pre-image under T, how may (a,b) be written in terms of x and y? What condition must hold for x and y in the pre-image set G?

Graph G' of $|x + 2| + |y + 3| \leq 3$ (the union of the square and its interior)

7. (a) Use the graph of $y = 3x$ and the translation $T_{4,5}$ to

graph the condition $y - 5 = 3(x - 4)$.

(b) Why must the image be a line?

(c) What is the slope of the image line?

(d) Do you think you can get the graph of any line with slope 3 by a translation of the line $y = 3x$? Why?

4.5 Functions and Conditions

In Course II you learned how to represent a real function $f: A \longrightarrow B$ by its graph in the coordinate plane. For example, consider the real function $g: R \longrightarrow R$ with rules $x \xrightarrow{g} |x|$. The graph of this function is shown in Figure 4.10.

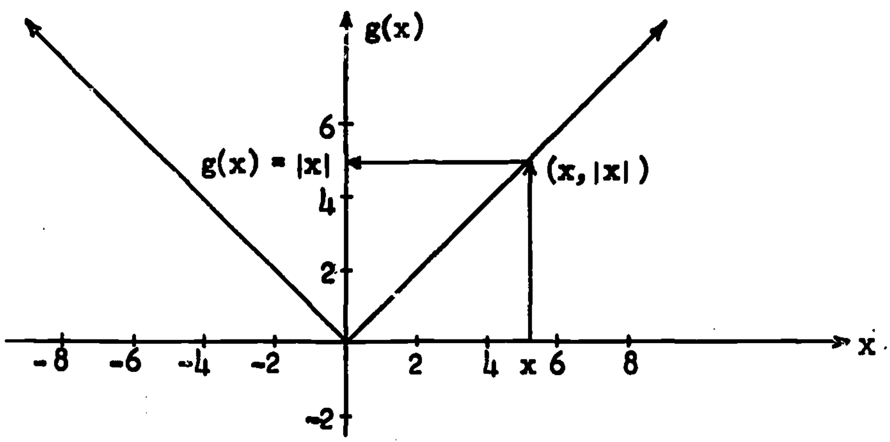


Figure 4.10

Thus, the function $g: R \longrightarrow R$ determines the set of ordered pairs $(x, |x|)$, for $x \in R$. Since $g(x)$ represents the image of x under g, we write; for any $x \in R$,

$$(x, g(x)) = (x, |x|).$$

Also, given domain R and codomain R, and the set of ordered pairs, $\{(x, g(x)) : g(x) = |x| \text{ and } x \in R\}$, the function $g: R \longrightarrow R$

is completely determined. This means that from the set of ordered pairs we can obtain the assignment of exactly one real number $g(x)$ to each real number $x \in R$. This is illustrated graphically in Figure 4.10. The process consists of locating the point (ordered pair) whose first coordinate is x and taking the second coordinate of the point as the image, $g(x)$, of x under g.

Now consider the condition, the equation, $y = |x|$. Here both y and x are variables whose allowable replacements are real numbers. The solution set of this equation is precisely the same set of ordered pairs as the set of ordered pairs determined by $g: R \longrightarrow R$. In this way there is associated with g the equation $y = g(x) = |x|$.

Now suppose we consider the function $g': [-3, 3] \longrightarrow R$, whose rule of assignment is $x \xrightarrow{g'} |x|$. The associated equation for this function is also $y = |x|$. But the solution set of $y = |x|$ is far larger than the set of ordered pairs determined by g'. This can be patched up by restricting the solution set of $y = |x|$ by adding the obvious restriction that x must be in $[-3, 3]$. Then the solution set, $\{(x,y): y = |x| \text{ and } x \in [-3, 3]\}$ is the set of ordered pairs of g'. But still, the solution set of the condition $y = |x|$ and $x \in [-3, 3]$ does not determine a function completely since it could be the set of ordered pairs for any function with rule $x \longrightarrow |x|$, domain $[-3, 3]$ and a codomain which contains the interval $[0, 3]$. However, all these functions would have the same range, $[0, 3]$, and would thus be equivalent. In this sense, the solution set of the condition $y = |x|$ and $x \in [-3, 3]$ determines a function.

In general, if f: $A \longrightarrow R$ is a real function with domain A, there is associated with this function the equation, called a <u>function equation</u>, $y = f(x)$ such that the solution set of the condition $y = f(x)$ and $x \in A$ is the set of ordered pairs determined by f: $A \longrightarrow R$. Thus, graphing the function f: $A \longrightarrow B$ means graphing the solution set of the associated function equation, with the restriction that $x \in A$. Graphing a function then becomes a special case of graphing a condition $\mathbf{C}(x,y)$.

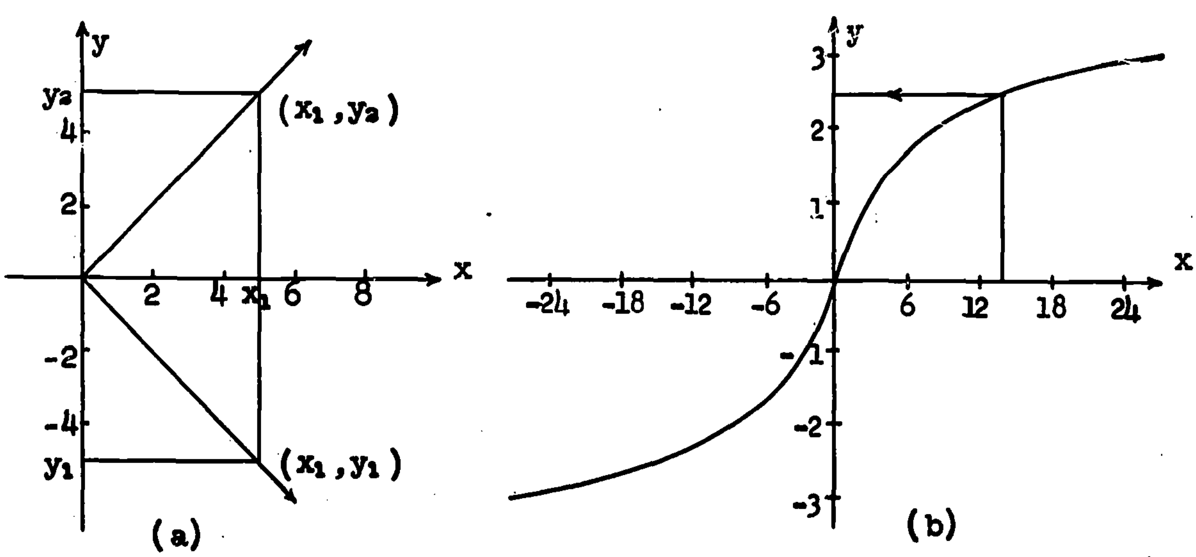


Figure 4.11

<u>Question</u>. Which of the graphs in Figure 4.11 can be the graph of a function?

In (a), if we pick a point x_1 on the x-axis whose x-coordinate is positive, we find that there are two ordered pairs (x_1, y_1) and (x_1, y_2) which have x_1 as a first element. This, then, cannot be the graph of a function with domain R^+ since a function must assign exactly one image to each element of its domain. In the set of ordered pairs determined by a function, no two distinct ordered pairs can have the same first element.

Geometrically, this means that any line perpendicular to the x-axis intersects the graph of a function in at most one point. You can readily see that Figure 4.11(b) can be the graph of a function.

A condition for the graph in Figure 4.11(a) is $|y| = x$. A condition for the graph in Figure 4.11(b) is $y^3 = x$ and $x \in [-27, 27]$. (Different scales are used on the axes of Figure 4.11(b) to make a reasonable display on the text page.) $y^3 = x$ is certainly not an equation in the form $y = f(x)$ but its solution set and graph satisfy the conditions for a function f with domain $[-27, 27]$.

<u>Questions</u>. (1) Can the condition $y^3 = x$ and $x \in [-27, 27]$ define a function with codomain less extensive than R?

(2) What is the range of any function determined by this condition?

As before, $y^3 = x$ and $x \in [-27, 27]$ determines a set of equivalent functions. Any function whose domain is $[-27, 27]$ and whose codomain contains $[-3, 3]$ would be a function determined by the given condition. $y^3 = x$ and $x \in [-27, 27]$ is called a <u>function condition</u>.

<u>Definition</u> 1. A condition $\mathcal{C}(x,y)$ with solution set S is a function condition if and only if no two distinct ordered pairs of S have the same first element. The condition $\mathcal{C}(x,y)$ is then said to determine a function with domain $A = \{x: (x,y) \in S\}$.

If the codomain of the functions considered is R, that is, we consider only functions $f: A \longrightarrow R$, then the function condition $y^3 = x$ and $x \in [-27, 27]$ defines a single function. Likewise, any function condition then determines a function with domain $A = \{x: (x,y) \in S\}$.

<u>Example</u> 1. Consider the conditions

(a) $|x| + |y| = -7$ (b) $|x| + |y| = 5$ and $y \geq 0$ and $x \in [-5, 5]$. What are the graphs of these conditions? Are they function conditions?

(a) The solution set of this condition is empty, since $|x| \geq 0$, $|y| \geq 0$ and hence $|x| + |y| \geq 0$ for all x, y.

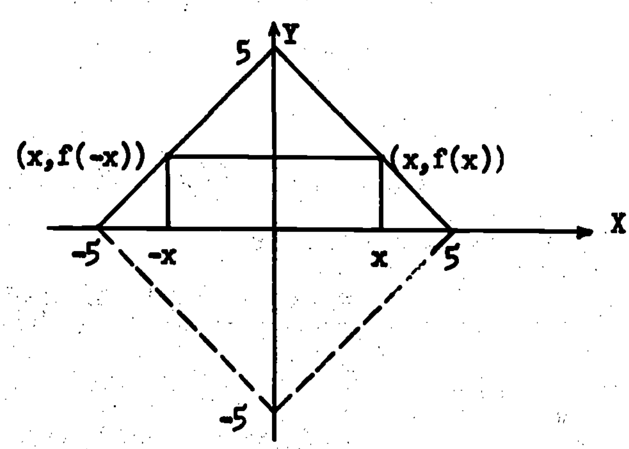


Figure 4.12

(b) The restriction $y \geq 0$ makes this a function condition. Otherwise, the dotted lines in Figure 4.12 would be part of the graph, in which case the condition would not be a function condition.

You may have noticed that function graphs as well as graphs of conditions may have symmetry with respect to the y-axis or with respect to the origin. For example, in Figure 4.12, the graph has symmetry with respect to the y-axis. That is $C(x,y)$ is equivalent to $C(-x,y)$ so that the graph of the condition is mapped onto itself by the line reflection $(x,y) \longrightarrow (-x,y)$. It is easy to see that a corresponding criterion for the graph of a function to be symmetric with respect to the y-axis is that for all x in the domain of f, $f(x) = f(-x)$. The graph of a function cannot have symmetry with respect to the x-axis. Do you see why?

Also, in Figure 4.11(b), the graph has symmetry with respect to the origin, since

$$(-y)^3 = (-x) \text{ iff } y^3 = x.$$

For a function graph the criterion for such symmetry is that for all x in the domain of f, -x is in the domain of f and $f(-x) = -f(x)$. Thus, note that for (x,y) in the graph of f,

$$(x,y) = (x,f(x))$$

so that,

$$(-x, -y) = (-x, -f(x)) = (-x, f(-x)).$$

If $f(x) = x^3$, the test is

$$f(-x) = (-x)^3 = -(x^3) = -f(x).$$

<u>Example</u> 2. Test the graph of (a) $f(x) = |x| + 2$
(b) $g(x) = \frac{1}{x}$, for symmetry with respect to the y-axis and with respect to the origin.

(a) $f(-x) = |-x| + 2 = |x| + 2 = f(x)$. Therefore the graph is symmetric with respect to the y-axis, but not symmetric with respect to the origin.

(b) $g(-x) = \frac{1}{-x} = -\frac{1}{x} = -g(x)$. Therefore the graph is symmetric with respect to the origin but not the y-axis.

In Course II, when you studied real functions, a special function called the postal function $p: R^+ \longrightarrow W$ was introduced. The rule for p was that if $b - 1 < x \leq b$, where $b - 1$ and b are consecutive natural numbers, then $x \xrightarrow{p} b$. Thus $p(.5) = 1$, $p(2\frac{2}{3}) = 3$, etc. This function is a variation on a special function that is a useful and interesting one to study in developing a deeper understanding of real functions, called the "greatest integer function." The greatest integer function assigns to each real number x the greatest integer that is smaller than or equal to x. It is usually denoted by the symbol [x], whence it is called the <u>bracket function</u>.

More formally, the "greatest integer function" ,[ ], is the function of R to R given by the rule $x \longrightarrow [x]$ where $[x] = a$ if a is an integer and $a \leq x < a + 1$. Hence $[.5] = 0$, $[2\frac{2}{3}] = 2$, $[\sqrt{2}] = 1$, $[-2.3] = -3$. To satisfy yourself that this last is true, locate -2.3 on a number line. The first integer to the left of -2.3 (less than -2.3) is -3

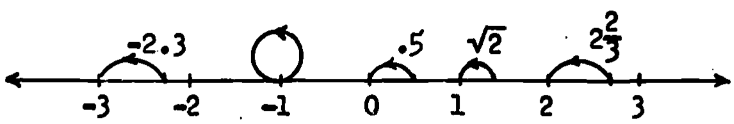


Figure 4.13

In terms of an arrow diagram, Figure 4.13, this function maps each integer onto itself; and every real number between two consecutive integers is mapped onto the immediately preceding integer.

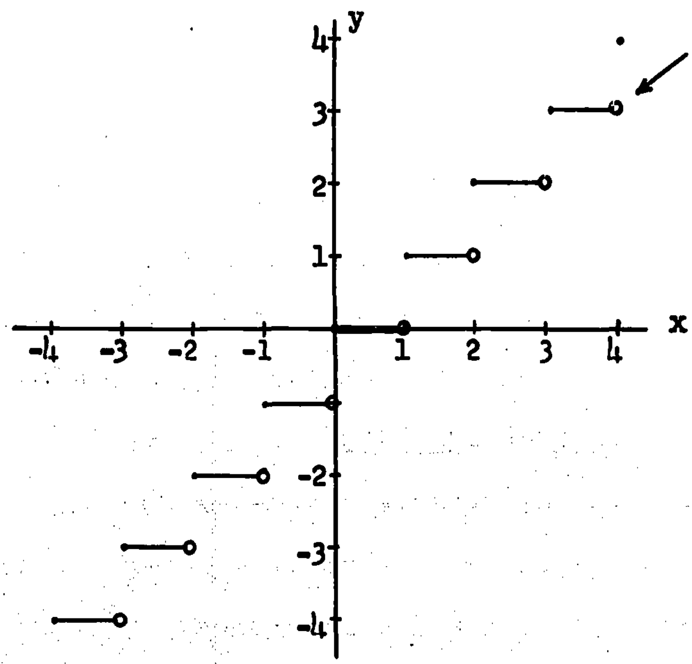


Figure 4.14

The graph of $y = [x]$, which is the graph of the greatest integer function, [ ], restricted to the interval [-4, 4] is shown in Figure 4.14. Because of the appearance of this graph, this function is sometimes called a <u>step function</u>. Notice that there is only one point of the graph with first coordinate 3, for example. The point (4, 3) is not in the graph. This is denoted by the little circle (see arrow).

4.6 Exercises

1. Write the function equation for each of the real functions of R^+ to R given as follows:

   (a) $x \xrightarrow{f} 1 + x^2$ (d) $x \xrightarrow{f} |x|$

   (b) $x \xrightarrow{f} \frac{1}{x}$ (e) $x \longrightarrow \frac{|x|}{1 + x^2}$

   (c) $x \xrightarrow{f} 3x + 5$

2. Consider each of the following equations carefully. Which of them are function equations for the domain specified? Explain why or why not in each case.

   (a) $y = 2x - 7$, $x \in R$ (g) $y = \frac{3}{x}$, $x \in R^+$

   (b) $x^2 + y = 7$, $x \in R$ (h) $|x| = y$, $x \in R$

   (c) $|y| = x$, $x \in R$ *(i) $|x + y| = 7$, $x \in R$

   (d) $|x| + |y| = 17$, $x \in R$ (j) $x^2 + |y| = 10$, $x \in R$

   (e) $\frac{y^2}{1 - x^2} = 12$, $x \in \{x: x \in R, x > 1\}$

   (f) $\frac{|y|}{1 + x} = 108$, $x \in R^+$

   Graph each condition.

3. Discuss the symmetry of the graphs of each of the following functions with domain as given and codomain R.

   (a) $f(x) = |x|$, $x \in R$ (c) $f(x) = [x]$, $x \in R$

   (b) $f(x) = x^3$, $x \in R$ (d) $f(x) = 3x$, $x \in R$

(e) $f(x) = 3x + 4,\ x \in R$ (f) $f(x) = |x| - x,\ x \in R$

(g) $f(x) = \begin{cases} -1 \text{ if } x < 0 \\ +1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \end{cases}$ (h) $f(x) = 6,\ x \in R$

*4 Graph each of the following conditions. Determine which of them are function conditions with codomain R. Give a reason for your answer.

(a) $|y| = |x|$ and $x \in R$.

(b) $[y] = x$ and $x \in Z^+$.

*(c) $[y][x] = 1$ and $x \in [0, 1]$.

(d) $|y| = |x|$ and $y \leq 0$ and $x \in R$.

(e) $|y| = x$ and $y \geq 0$ and $x \in R^+ \cup \{0\}$.

(f) $y^2 = x$ and $y \geq 0$ and $x \in R^+ \cup \{0\}$.

(g) $y = x^3$ and $x \in R$.

*(h) $[y] = [x]$ and $x \in R$.

4.7 Functions and Solution of Equations

There are many problems that can be solved using the graphs of functions. Some of these applications are not readily seen at first. Let us begin by examining a function given by its graph.

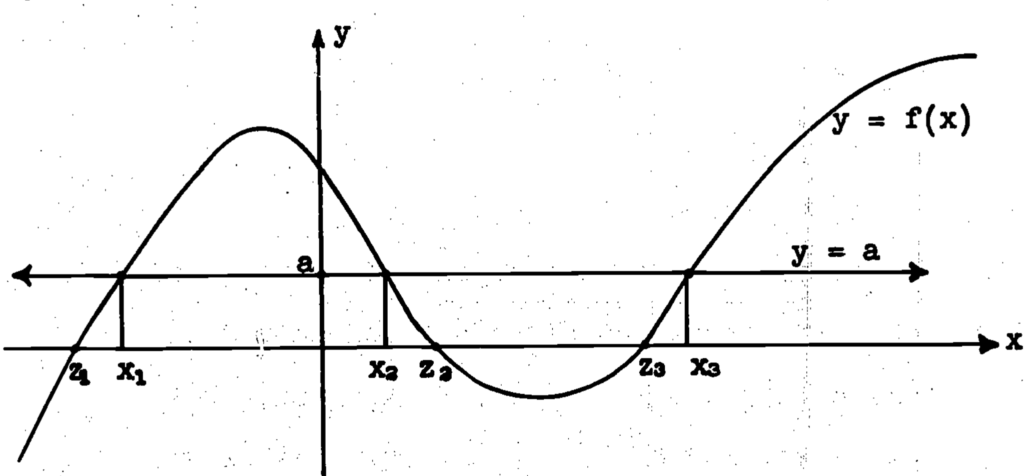


Figure 4.15

A basic problem that may be solved using a graph is finding all values x in the domain of f such that f(x) = a. The points that have these x-values are called <u>a-points</u> of f. The x-values of the a-points constitute the x-values of the solution set of the system of equations y = f(x) and y = a. This is illustrated in Figure 4.15, where x_1, x_2, and x_3 are the x-values of the a-points of f. An important special case is the set of zero-points of f. The x-values of these points represent the solutions of f(x) = 0, and these x-values are called the <u>zeros</u> of f. For our example, the zeros of f are z_1, z_2 and z_3, and are the x-coordinates of the intersection of the graphs of y = f(x) and y = 0 (the x-axis).

Given the graphs of two functions f and g (Figure 4.16) we may solve the equation f(x) = g(x) graphically. The solution set is {x: f(x) = g(x) and x is in the domain of f and in the domain of g}. Graphically, these are the x-coordinates of the points of intersection of the graphs of y = f(x) and y = g(x).

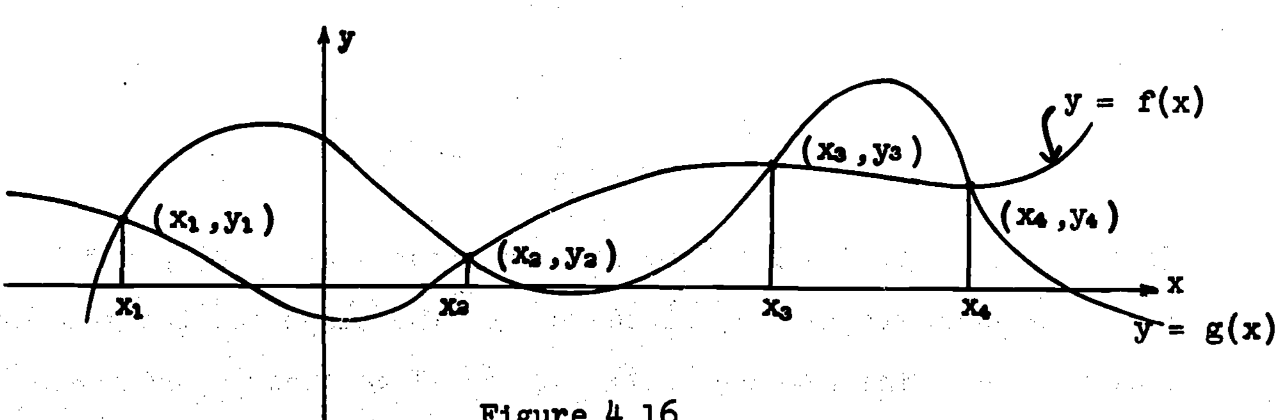


Figure 4.16

This process is illustrated in Figure 4.16. The solution set is

$\{x_1, x_2, x_3, x_4\}$. Of course we may also read the y-coordinates and obtain the solution set for the system $y = f(x)$ and $y = g(x)$. Now let us look at some examples.

In Chapter 2 of Course III we solved systems of linear equations such as:

$$3x + 2y = 12$$
$$5x - 3y = 27$$

Since the graphs of these equations are non-vertical lines, they are function conditions and the solution of such a system can be reconsidered from the point of view of functions. Solving each equation for y we obtain:

$$y = -\frac{3}{2}x + 6$$
$$y = \frac{5}{3}x - 9$$

Since these are function conditions, they define two functions f and g with domain R. Hence we write:

$$x \xrightarrow{f} -\frac{3}{2}x + 6$$
$$x \xrightarrow{g} \frac{5}{3}x - 9$$

We must, then, find all values of x such that $f(x) = g(x)$. This is easily done graphically since we know the slope and the y-intercept for each line (Figure 4.17). We shall get only approximate solutions from the graph. It appears that the single value of x is approximately 4.7 and the corresponding value of y is about -1.1. You should check this in the original equations and also solve the original equations algebraically as a check on this approximate solution.

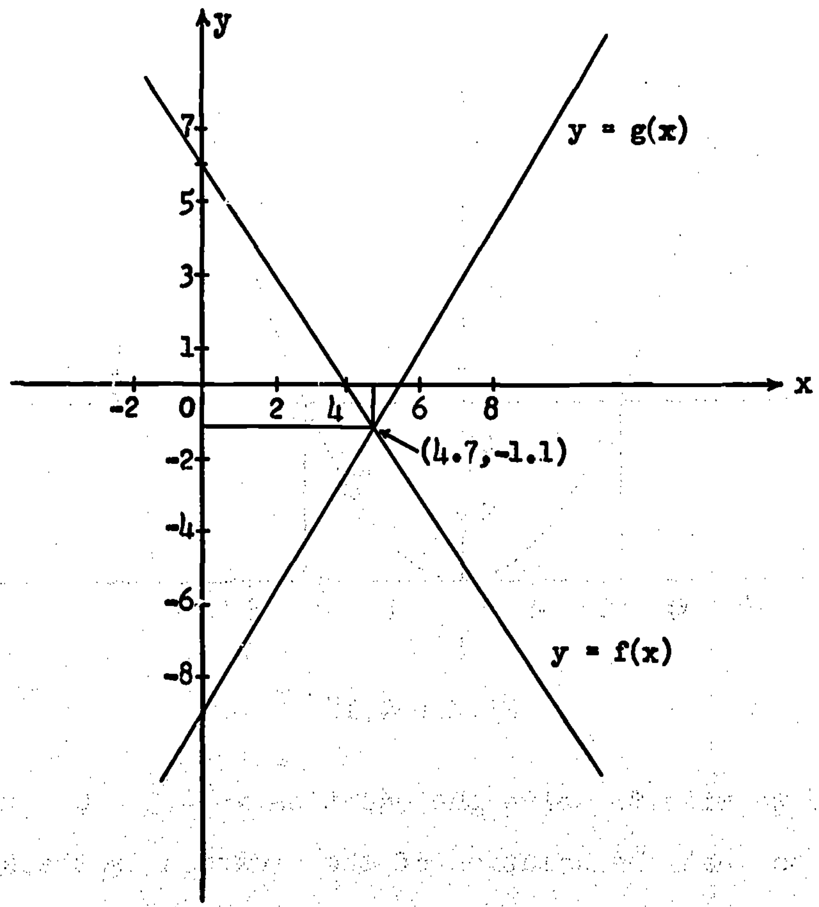


Figure 4.17

Another use of these methods is to solve several equations from the graph of a single function equation. For example, the graph of $y = f(x) = x^2$ is given in Figure 4.18, with the unit on the x-axis larger than that on the y-axis, for convenience.

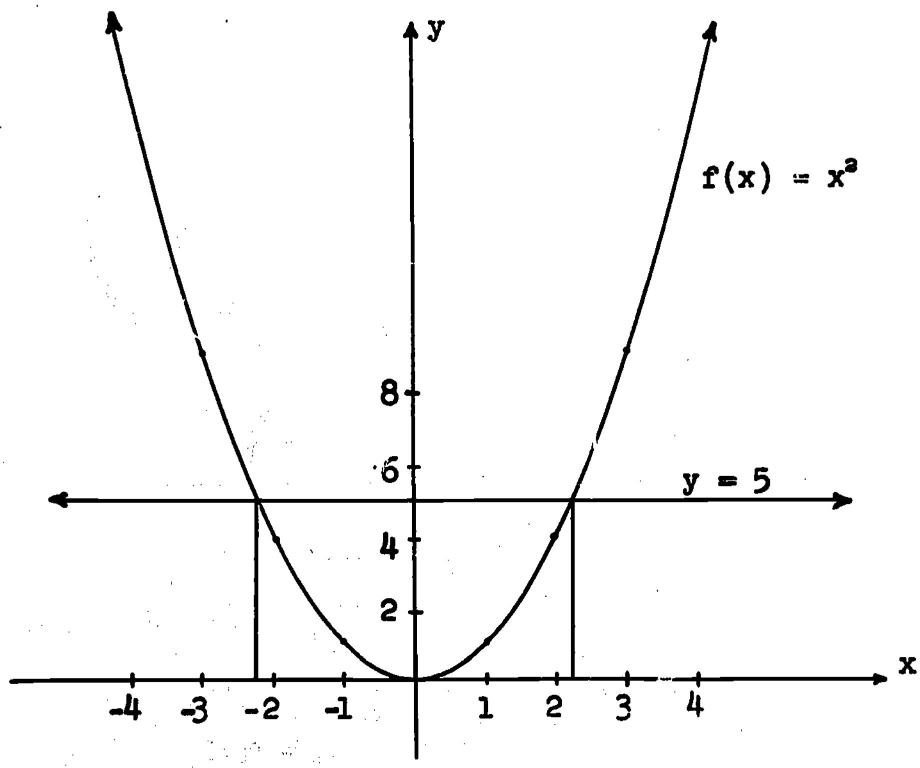


Figure 4.18

Now suppose we wish to solve the equation $x^2 - 5 = 0$. $x^2 - 5 = 0$ iff $x^2 = 5$ so that the solution of the equation is the set of all values x such that $f(x) = x^2 = 5$. Which, in turn, is the set of 5-points of f. Thus, we draw the line $y = 5$ on our graph and read the x-coordinates of the intersection with the graph of $f(x) = x^2$. These are approximately $x = 2.2$ and $x = -2.2$.

<u>Question</u>. Does f have any -3 points? What geometrical reason can you give for your answer?

An interesting application of these methods is the graphical solution of space-time problems.

<u>Example</u> 1. A car travelling in a straight line at a uniform speed of 50 miles per hour passes point A at 2:00 P.M. Point A is ten miles from the starting point O. How far is the car from A at 4:00 P.M.?

To set the stage with this simple example, we denote the path along which the car is travelling by a vertical axis, the s-axis. To denote the passage of time, we use a horizontal t-axis, t being the time elapsed since the car passed point A (Figure 4.19). The equation relating the s-coordinates and the t-coordinates is $s = 50t + 10$, since the car ten miles from O when time begins, and the speed, v, is uniformly 50 miles per hour. This is the equation for a line in the s,t-system with slope 50 which intersects the vertical axis at $s = 10$.

This line has been drawn in Figure 4.19. (It is customary to refer to this as an s,t-coordinate system, even though the first coordinate of an ordered pair that designates a point is always the t-coordinate.)

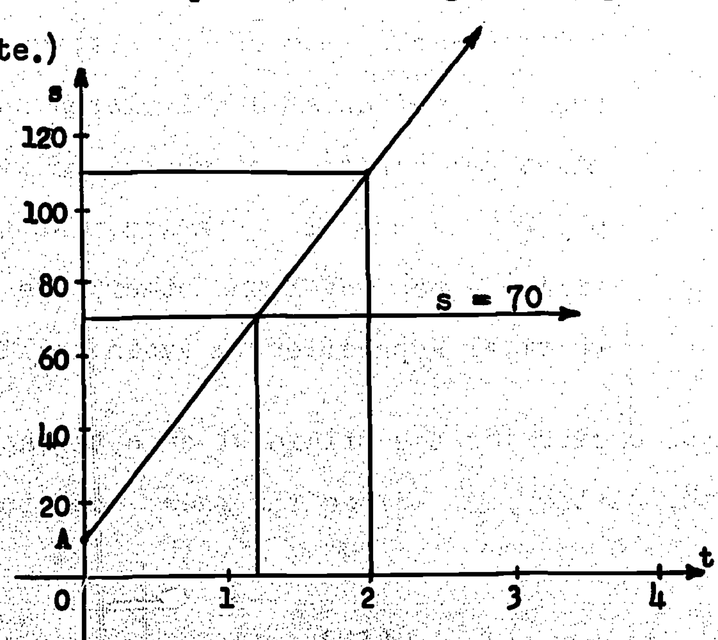


Figure 4.19

This is a function graph. We denote the function by its equation $s = f(t) = 50t + 10$. To answer the question asked in the example, we must find $f(2)$. This is done graphically. $f(2) = 110$. But this is the distance from 0. Hence the car is 100 miles from A at 4:00 P.M.

At what time is the car 70 miles from 0? This is a "70-point" of f. Draw the line $s = 70$ and read the t-coordinate, 1.25, approximately. Checking algebraically,

$$70 = 50t + 10$$

$$50t = 60 \text{ so that } t = 1.20.$$

We have used this very simple example to introduce the ideas. Now a more challenging application.

Example 2. A radar station located at point 0 picks up airplane A 150 miles due east of 0 at 1:00 A.M. The plane is approaching the station at a calculated speed of 6 miles per minute. At 1:16 A.M. a second airplane is picked up 240 miles due west of the station and approaching the station at a calculated speed of 8 miles per minute. At what time will each plane pass over the station? At what time will one pass over the other? We assume some vertical separation to avoid collision!

A natural choice for the origin of an s,t-graph for this problem is the station, 0. Since the radar operator becomes concerned about the problem at 1:16 A.M. when the second plane appears, we have made the zero point for elapsed time at 1:16 A.M. Thus, 1:00 A.M. is -16 minutes elapsed time. On this s,t-system the function graphs for the progress of the two planes are drawn.

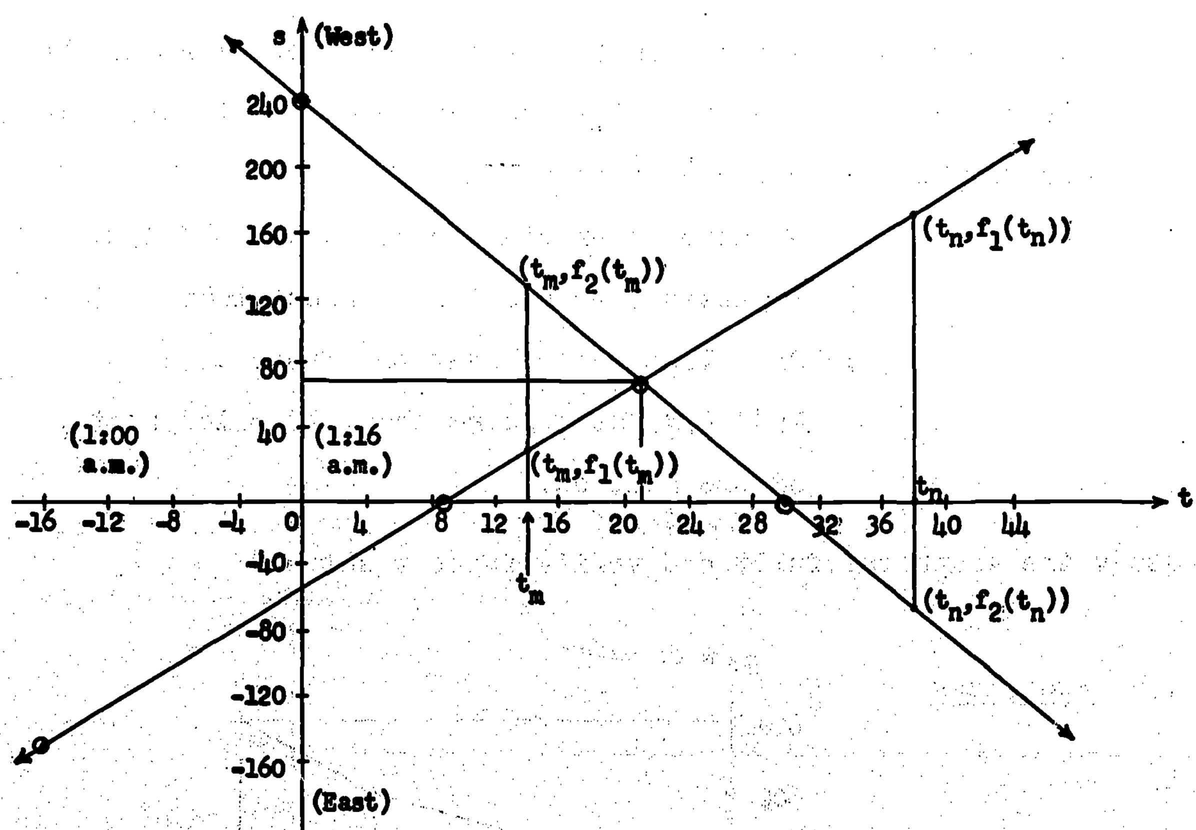


Figure 4.20

The s,t-coordinates of plane A when sighted are then (-16, -150). (We have chosen west to be the positive direction from O. See Figure 4.20.) Since its velocity is 6 miles per minute, the function graph for $s = f_1(t)$ is a line through (-16,-150) with slope 6. For the plane B, approaching from the west, the s,t-coordinates are (0, 240) and the function graph for $s = f_2(t)$ is a line through (0, 240) with slope -8.

The distance between the planes can be determined for any time. For time t_m it is $f_2(t_m) - f_1(t_m)$ (See Figure 4.20). For time t_n it is $f_1(t_n) - f_2(t_n)$. In general, it is $|f_1(t) - f_2(t)|$.

To answer the questions in the problem one merely reads the 0-points of f_1 and f_2 and the time coordinate of the intersection of their graphs. From these we obtain:

(1) A passes over station at about 1:25 A.M.

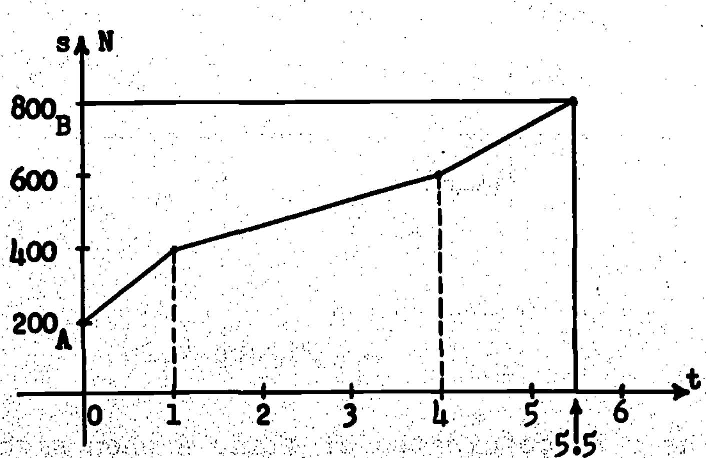
(2) B passes over station at about 1:46 A.M.

(3) One plane passes over the other at about 1:37 A.M. at a point about 70 miles west of the station.

Study the graph carefully and verify these results.

4.8 Exercises

1.



An airplane's position relative to a base station at 0 is located at time 0 hours as 200 miles due north of 0. The plane is flying due north. The graph represents the plot of distance travelled against time as recorded on the instruments at the base station.

(a) How far did the airplane travel in the first hour?

(b) How far did the airplane travel in the next three hours?

(c) In approximately how many hours was the plane 500 miles from the base station?

(d) Is there a functional relationship between elapsed time and the distance the plane travels?

(e) What do you know about the plane's ground speed from time $t = 0$ to $t = 1$? From time $t = 1$ to $t = 4$? From $t = 4$ to $t = 5.5$?

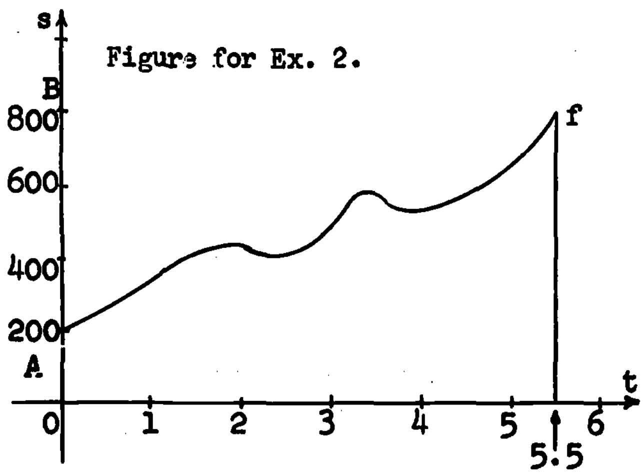
(f) Can you make up a reasonable explanation for the fluctuation in ground speed from interval to interval?

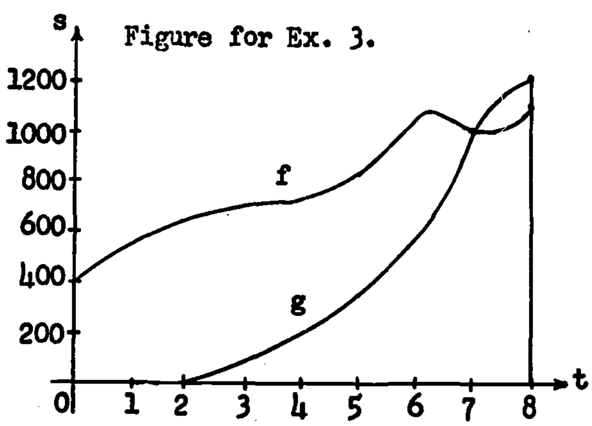
2. Because of the influence of wind, slight course changes, gaining altitude, losing altitude, and so forth, it is likely that an actual plot of the plane's progress would appear as in the graph below:

(a) Answer questions (a)--(c) of Exercise 1. We have here a function $s = f(t)$ where a graph is known but no equation is given.

(b) Find a 600-point of f. Is there more than one 600-point of f?

(c) How can the result of (b) be explained? Is it possible?





3. Consider the graph (above right) showing the tracks of two planes. Make up a story to explain what the graph shows.

4. Given $f(x) = \frac{1}{4}x^2 - 2$, $g(x) = x + 1$.

   (a) Construct the graph of f and the graph of g on the same set of axes.

   (b) Find graphically the 0-points of f.

   (c) Find graphically the 0-points of g.

   (d) Use the graph of f to solve the equation $\frac{1}{4}x^2 = 6$.

   (e) Use the graphs of f and g to solve the system of equations

$$y = \frac{1}{4}x^2 - 2$$
$$y = x + 1$$

   Explain your work.

5. Given the system of equations:

$$3x + 2y = 7$$
$$x - 2y = 18$$

   (a) Write the rules for the functions f and g determined by the given equations.

   (b) Find graphically the set of values of x such that $f(x) = g(x)$.

(c) Use the values of x found in part (b) to obtain the solution set of the system given.

6. Repeat Exercise 5 for each of the following systems:

(a) $x + 2y = 4$
$2x - 2y = 3$

(b) $5x + y = 10$
$2x + 2y = 8$

(c) $2x + 3y = 10$
$\frac{1}{2}x + \frac{3}{4}y = 5$

(d) $x - 3y = 17$
$\frac{1}{2}x - \frac{3}{2}y = 8\frac{1}{2}$

(e) $y = \frac{1}{2}x + 2$
$x = \frac{1}{4}y + 3$

(f) $3x + y = 7$
$x + 3y = 9$

4.9 Operations on Functions

In the study of real functions in Course II Chapter 7, various operations on functions of R to R were defined. We summarize the definition of these operations on functions below.

Definition 2. If $f : R \longrightarrow R$ and $g : R \longrightarrow R$ then $[f + g]$, $[f - g]$, and $[f \cdot g]$ are functions of R to R with rules $[f + g](x) = f(x) + g(x)$, $[f - g](x) = f(x) - g(x)$ and $[f \cdot g](x) = f(x) \cdot g(x)$. Furthermore, $[\frac{f}{g}] : A \longrightarrow R$ is the function with the rule $[\frac{f}{g}](x) = \frac{f(x)}{g(x)}$, where the domain $A = \{x: x \in R \text{ and } g(x) \neq 0\}$.

Now each of these functions has an associated function equation. For $[f + g]$ it is $y = f(x) + g(x)$. For $[f \cdot g]$ it is $y = f(x) \cdot g(x)$. For example, if $f(x) = 2x^2 - 3$ and $g(x) = x + 2$, the function equation of $[f + g]$ is

$$y = (2x^2 - 3) + (x + 2) = 2x^2 + x - 1.$$

The function equation for $[f \cdot g]$ is then

$$y = (2x^2 - 3)(x + 2) = (2x^2 - 3)x + (2x^2 - 3)2$$
$$= 2x^3 - 3x + 4x^2 - 6$$
$$= 2x^3 + 4x^2 - 3x - 6.$$

The function equation for $[\frac{f}{g}]$ is

$y = \dfrac{2x^2 - 3}{x + 2}$ and the domain A of $[\frac{f}{g}]$ is $\{x: x \in R \text{ and } x \neq -2\}$, since $x + 2 = 0$ if and only if $x = -2$.

There are also some other ways that new functions can be constructed from given functions. You may recall that if $f : R \longrightarrow R$, then $[af]: R \longrightarrow R$ is the function defined by $[af](x) = a \cdot f(x)$. For example, if f has the rule $f(x) = \frac{1}{2}x + 2$, and $a = 2$, then $[2f]$ has the rules $[2f](x) = 2(\frac{1}{2}x + 2) = x + 4$. This notion can also be considered graphically as can the operations of addition and subtraction of functions (which was done in Course II, Chapter 7).

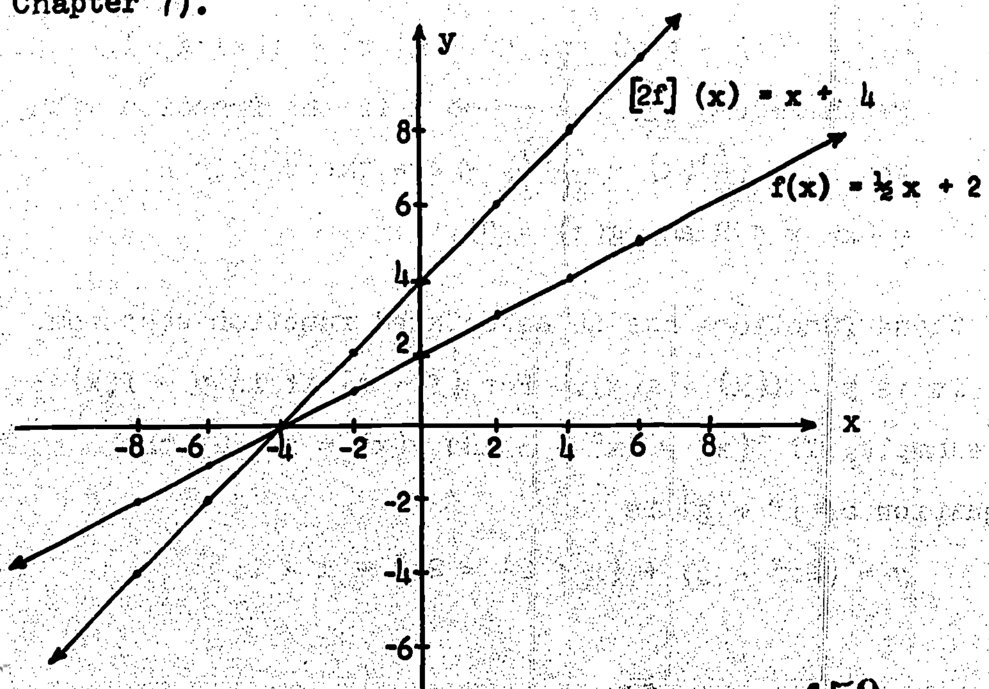


Figure 4.21

In Figure 4.21, to draw the graph of $[2f]$ from the graph of f, each function value is doubled. That is, the distance of each point of the graph from the x-axis is doubled.

Questions. (1) What is the relationship between the y-intercept of $y = [2f](x)$ and of $y = f(x)$?

(2) What is the relationship between the x-intercept of $y = [2f](x)$ and of $y = f(x)$?

In Course II, Chapter 7, a function of R to R with rule $x \longrightarrow 0$ was denoted by c. However, since there are many such functions, c is inadequate to name all of them. The notation c_0 is used to name the particular constant function in our example.

Definition 3. For any real number a, $y = a$ is a function equation for the function $c_a : R \longrightarrow R$. The functions c_a are called constant functions.

If $f : R \longrightarrow R$ and $c_a : R \longrightarrow R$ are given, $[c_a \cdot f] : R \longrightarrow R$ has the function equation $y = c_a(x) \cdot f(x) = a \cdot f(x)$. Hence, $[c_a \cdot f]$ and $[af]$ make the same assignments, have the same domain and the same codomain. They are, therefore, two names for the same function.

Another way to obtain a new function in F, the set of all functions of R to R, is to add a constant function to f forming the function $[f + c_a]$. Since

$$[f + c_a](x) = f(x) + c_a(x) = f(x) + a,$$

$[f + c_a]$ is also denoted by $[f + a]$. The graph of $[f + c_a]$ can be obtained from the graph of f as seen in Figure 4.22.

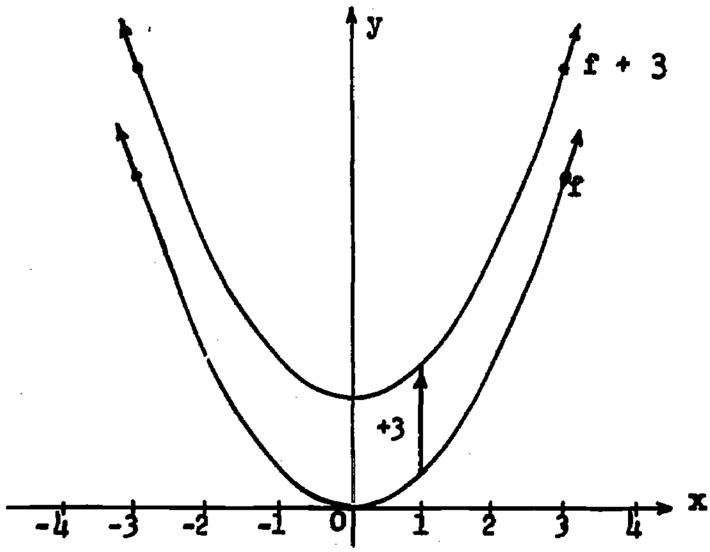


Figure 4.22

Here $f(x) = x^2$ and $a = 3$. Thus, $[f + c_3](x) = f(x) + c_3(x) = x^2 + 3$. To obtain the graph of $[f + c_3] = [f + 3]$ from the graph of f, each point of the graph is moved upwards 3 units. Thus, the graph of $[f + 3]$ is the image under the translation with rule $(x,y) \longrightarrow (x, y + 3)$ of the graph of f.

Another operation defined on real functions is composition of functions. If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are real functions, $g \circ f : A \longrightarrow C$ is the real function defined by $g \circ f\ (x) = g(f(x))$.

Thus, if $f : R \longrightarrow R$, $g : R \longrightarrow R$, $f(x) = 2x + 3$ and $g(x) = x^2$,

(1) $f \circ g\ (x) = f(g(x)) = f(x^2) = 2(x^2) + 3 = 2x^2 + 3$ and

(2) $g \circ f\ (x) = g(f(x)) = g(2x + 3) = (2x + 3)^2 = 4x^2 + 12x + 9$.

Note that in (1), "x^2" replaces "x" in the rule for f. In (2) "$2x + 3$" replaces "x" in the rule for g.

There are two compositions which are very special in terms of their effects in graphing. Let $g(x) = 2x$ be a rule for a function from R to R. A partial arrow diagram for this function is shown in Figure 4.23.

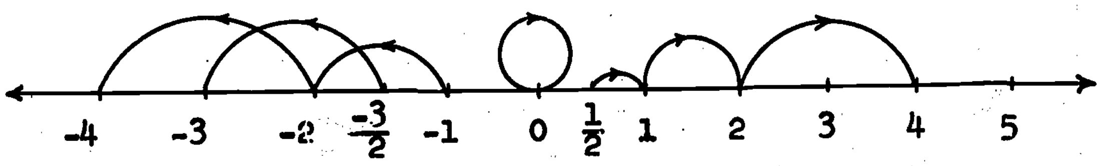


Figure 4.23

g determines, of course, a dilation of the line (Course I, Section 6.9). Now let f be any function of R to R. Then f o g has the rule

$$f \circ g\ (x) = f(g(x)) = f(2x).$$

In general, if g has the rule $g(x) = ax$, f o g has the rule $f \circ g\ (x) = f(ax)$. If $a = 0$, $f \circ g\ (x) = f(0)$ for all x, so we exclude $a = 0$ from our discussion.

There is a strong relationship between the graph of f and the graph of f o g, where $g(x) = ax$ and $a \neq 0$. For example, let $f : R \longrightarrow R$ have the rule $f(x) = |x|$ and $g : R \longrightarrow R$ have the rule $g(x) = 2x$. Then $f \circ g\ (x) = f(g(x)) = f(2x) = |2x|$. This is the algebraic story. In Figure 4.24, we show what happens on the graph.

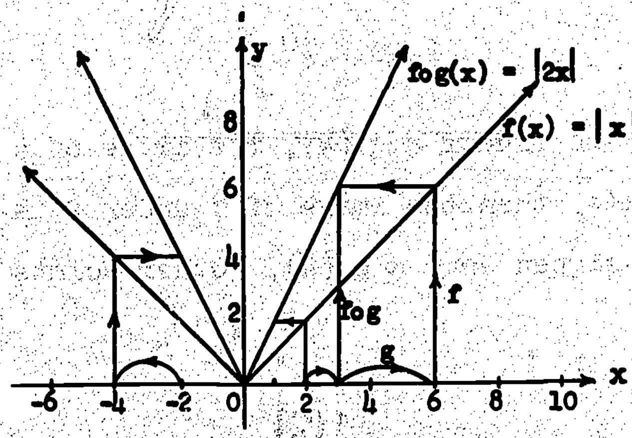


Figure 4.24

For example, to find $f \circ g\ (3)$, we find the image of 3 under g on the x-axis, then find $f(g(3)) = f(6)$ by going up to the graph of f and then go back to locate the point (3, 6) in the graph of $f \circ g$. That is $f \circ g\ (3) = f(6) = 6$. Also, $f \circ g\ (-2) = f(g(-2)) = f(-4) = 4$, so that (-2, 4) is in the graph of $f \circ g$. Follow the arrows! The effect of g in the composition in this case is to "accelerate" the effect of f.

Next, consider what happens if $g(x) = x + a$. Then $f \circ g\ (x) = f(g(x)) = f(x + a)$. Again, the only interesting cases are for $a \neq 0$. Suppose $a = 3$. Then $f \circ g\ (x) = f(g(x)) = f(x + 3)$. $f : R \longrightarrow R$ is again given by $f(x) = |x|$. Thus,

$$f \circ g\ (x) = f(g(x)) = f(x + 3) = |x + 3|.$$

The graphic process is illustrated in Figure 4.25.

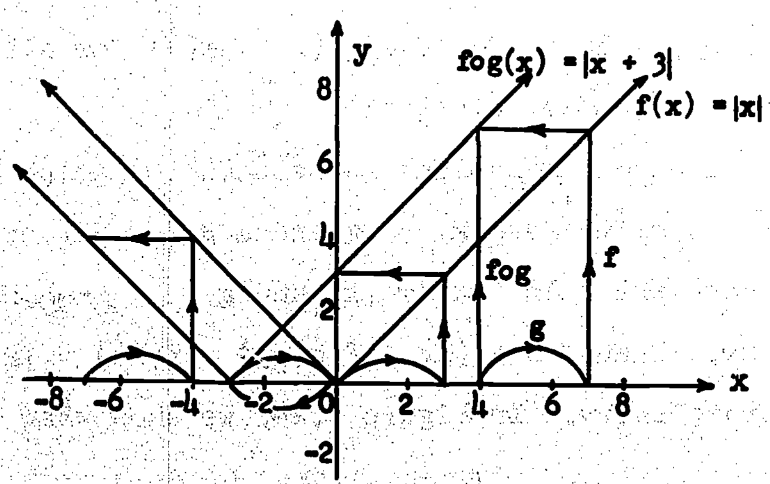


Figure 4.25

For example, to find $f \circ g\ (4)$, the image of 4 under g, 7, is found on the x-axis, then the image of 7 under f is located on the graph of f and assigned as $f \circ g\ (4)$. Other points of the graph of f o g (x) = f(x + 3) are located in the same manner. It is easy to see that the graph of f o g is the image of the graph of f under the translation with rule $(x,y) \longrightarrow (x - 3, y)$. In general, the graph of f o g (x) = f(x + a) is the image of the graph of f under the translation with rule $(x,y) \longrightarrow (x - a, y)$. The function equation for f is y = f(x) and the function equation for f o g is y = f(x + a). You should compare this result with the work in Section 4.3 of this chapter.

4.10 Exercises

1. Given the following functions of R to R and rules as shown:

$x \xrightarrow{f} |x|$ $\qquad$ $x \xrightarrow{j} x$

$x \xrightarrow{g} [x]$ $\qquad$ $x \xrightarrow{c_1} 1$

$x \xrightarrow{h} x + 5$ $\qquad$ $x \xrightarrow{q} x^2$

$x \xrightarrow{k} 4x$

(a) Write the function equation for each function.

(b) Write the function equation for:

(i) [f + g]	(vi) [f · k]	(xi) [q + k + c_1]
(ii) [f + h]	(vii) [h · c_1]	(xii) [j · j]
(iii) [h + k]	(viii) [q + h]	(xiii) [j · j + 4j + 1]
(iv) [f · g]	(ix) [q + k]	(xiv) [h · k]
(v) g o h	(x) g o k	(xv) g o q

(c) Graph q and h on the same set of axes and then find the graph of [q + h] by the graphic method.

(d) Graph q, k, and c_1 on the same set of axes. Find the graph of $[q + k + c_1]$ by the graphic method.

*(e) Graph g and j on the same set of axes. Then find the graph of $[j - g]$ by the graphic method.

(f) Use the graph of g to construct the graph of $[-2g]$, where $[-2g](x) = -2[x]$.

(g) Use the graph of q to construct the graph of $[-\frac{1}{2} q]$, where $[-\frac{1}{2} q](x) = -\frac{1}{2}x^2$.

(h) Use the graph of h to construct the graph of $[h + c_3]$, where $[h + c_3](x) = (x + 5) + 3 = x + 8$.

(i) Use the graph of g to construct the graph of $[g + c_2]$, where $[g + c_2](x) = [x] + 2$.

2. Use the graph of (see Exercise 1):

(a) f to construct the graph of f o k. Note that f o k (x) $= f(k(x)) = f(4x) = |4x|$.

(b) f to construct the graph of f o h (x) $= f(h(x)) = f(x + 5) = |x + 5|$.

(c) g to construct the graph of g o k.

(d) g to construct the graph of g o h.

(e) q to construct the graph of q o k.

(f) q to construct the graph of q o h.

3. Set $\ell(x) = \frac{1}{2}x$ and $m(x) = x - 3$. f, k, and q are defined in Exercise 1.

(a) Use the graph of f to construct the graph of f o ℓ, of f o m.

(b) Use the graph of q to construct the graph of q o ℓ, of q o m.

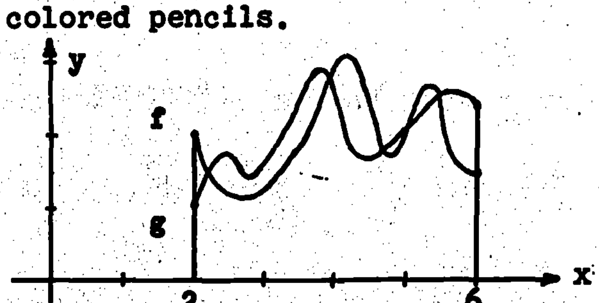
(c) What do you observe about the effect of ℓ and m in composition, as contrasted with k and h? (See Exercise 2.)

4. The function $|u| : A \longrightarrow R$ can be defined as follows: $|u| = f \circ u$, where f is the absolute value function, $x \xrightarrow{f} |x|$, of R to R. Thus $|u|\ (x) = f \circ u\ (x) = f\ (u(x)) = |u(x)|$.

   (a) Graph $u : R \longrightarrow R$ where $u(x) = x^2 - 3$.

   (b) Graph $|u| : R \longrightarrow R$. Hint: For each point of the graph below the x-axis, $(x, |u(x)|)$ is the image of the point by a reflection in the x-axis.

5. Given two real functions f and g of A to R, a function $\max(f,g) : A \longrightarrow R$ is defined by $\max(f,g)(x) = \max(f(x),g(x))$. That is, for each x, the new function chooses the larger of f(x) and g(x). If $f(x) = g(x)$, $\max(f,g)(x) = f(x) = g(x)$. Copy the following graph and construct from it the graph of max(f,g). Use colored pencils.



4.11 Bounded Functions and Asymptotes

If f is any real function of A to R then $[\frac{c_1}{f}]$ is by definition the real function of B to R with rule $[\frac{c_1}{f}](x) = \frac{1}{f(x)}$, for all $x \in B$, where $B = \{x : x \in A \text{ and } f(x) \neq 0\}$. Normally this function is written simply $[\frac{1}{f}]$ or $\frac{1}{f}$ and is called the <u>reciprocal</u> of f. The use of reciprocal is restricted since B is often a proper subset of A. But for all $x \in B$, $f(x) \cdot [\frac{1}{f}](x) = 1$.

The simplest reciprocal function is $\frac{1}{j}$ where j is the identity function on R. Then $\frac{1}{j} : B \longrightarrow R$ has the rule $x \longrightarrow \frac{1}{x}$ and $B = \{x : x \in R \text{ and } x \neq 0\}$. A good picture of the action of $\frac{1}{j}$ may be obtained by looking at an arrow diagram on a line for $\frac{1}{j}$.

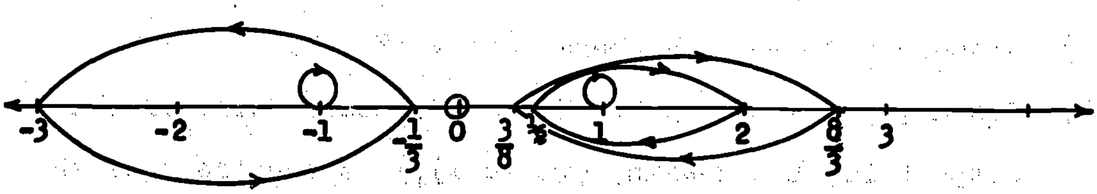


Figure 4.26

Study of Figure 4.26 shows that 1 and -1 are fixed points in the mapping and that if $|x| > 1$, j maps x onto the point $\frac{1}{x}$ such that $0 < |\frac{1}{x}| < 1$, and conversely. Also, as $|x|$ gets close to 0, $|\frac{1}{j}(x)|$ gets very large. For example:

$$.0001 \xrightarrow{\frac{1}{j}} 10{,}000 \text{ and } .00000001 \xrightarrow{\frac{1}{j}} 100{,}000{,}000.$$

Conversely, as $|x|$ gets very large, $|\frac{1}{j}(x)|$ gets very close to zero.

The graph of $y = \frac{1}{x}$, the function equation of $\frac{1}{j}$, shows these relationships in another way. To construct the graph of $\frac{1}{j}$ we shall use a technique that can then be applied to sketching the graph of $\frac{1}{f}$ from the given graph of f. We begin with the graph of $y = x$, the function equation of j. (See Figure 4.27.)

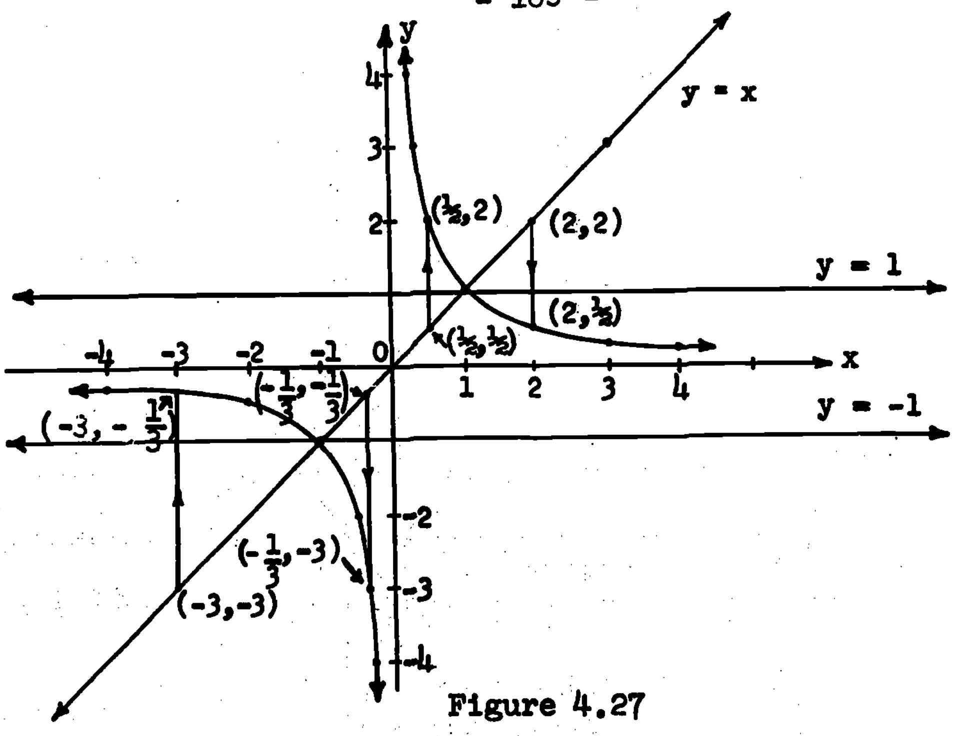


Figure 4.27

The points $(-1, -1)$ and $(1, 1)$ are in the graph of $y = \frac{1}{x}$.

For each value of x between -1 and 1, the point on the graph of $y = x$ lies between the lines $y = -1$ and $y = 1$. Corresponding to each of these points (except $(0,0)$) we will get a point outside these lines whose y-coordinate is reciprocal to that of the given point. For example, from $(\frac{1}{2}, \frac{1}{2})$ we get $(\frac{1}{2}, 2)$ and from $(-\frac{1}{3}, -\frac{1}{3})$ we get $(-\frac{1}{3}, -3)$ in the graph of $y = \frac{1}{x}$. Likewise, for points of the line $y = x$ outside the given region we get points inside the region between the lines $y = 1$ and $y = -1$. From $(2, 2)$, we get $(2, \frac{1}{2})$ in the graph of $y = \frac{1}{x}$. From $(-3, -3)$ we get $(-3, -\frac{1}{3})$. Continuing in this way, enough points may be located to sketch the graph.

A similar procedure can be used to sketch the graph of $\frac{1}{f}$ for any function, given its graph. The result of this procedure for a function f given by its graph is shown in Figure 4.28 and following are the steps in the procedure.

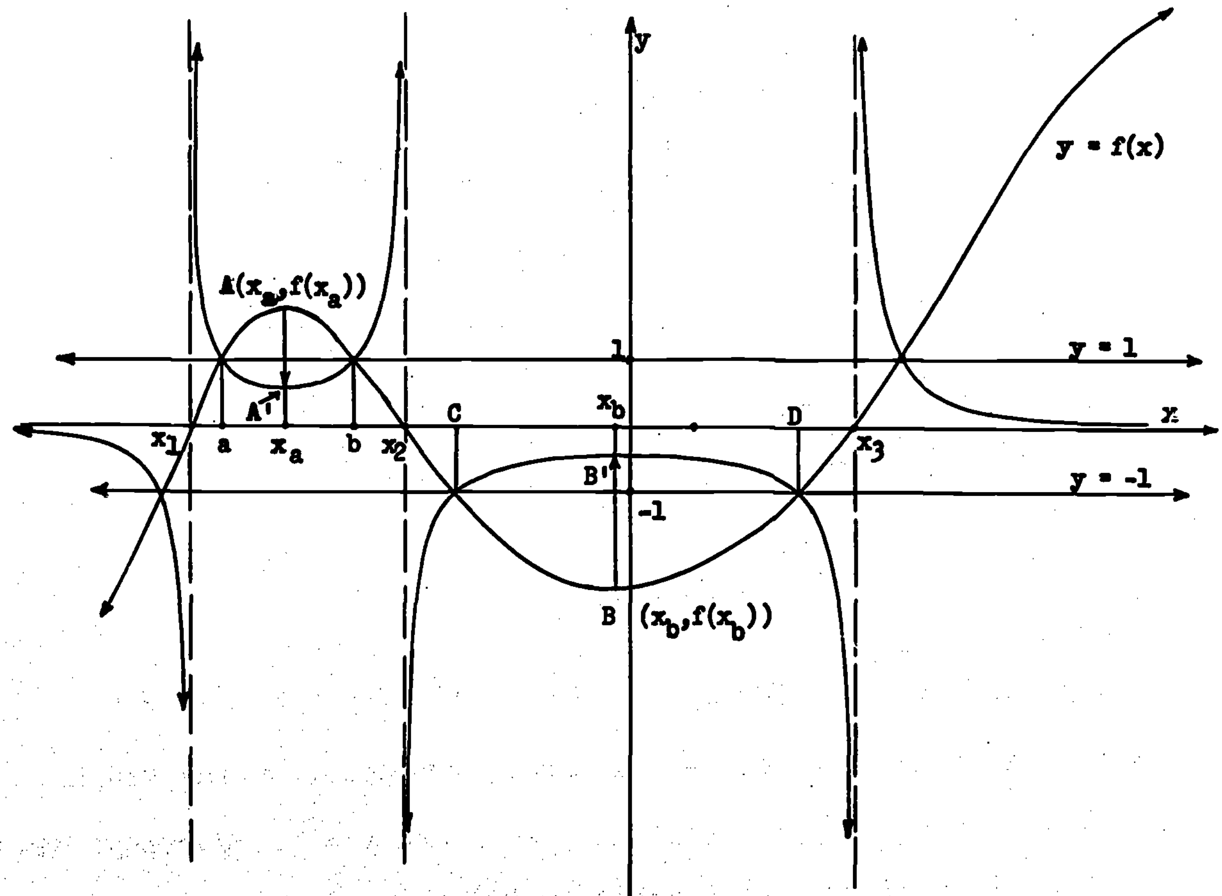


Figure 4.28

(1) The lines $y = 1$ and $y = -1$ are drawn and the intersections with the graph of f marked. These intersection points are also in the graph of $\frac{1}{f}$. Why?

(2) Vertical lines (dashed) are drawn through the points of the graph of f where $f(x) = 0$. $\frac{1}{f}$ will have no values for these values of x. Why?

(3) Points such as A and B, where the graph of f has a "peak" or a "valley" are noted. f is said to have a <u>local</u> <u>maximum</u> at A and a <u>local</u> <u>minimum</u> at B.

(4) A' and B' are located by estimation of $\frac{1}{f(x_a)}$ and $\frac{1}{f(x_b)}$.

(5) A' is a local minimum for $\frac{1}{f}$ and B' is a local maximum for $\frac{1}{f}$. The curve can be then sketched in between the fixed points either side of A' and either side of B'.

(6) Everywhere that $|f(x)|$ becomes close to 0, $|\frac{1}{f(x)}|$ becomes larger and larger. Thus to the right of x_1, as $f(x)$ gets close to $f(x_1) = 0$, $\frac{1}{f(x)}$ gets very large so the graph of $\frac{1}{f}$ extends upward and ever closer to the line $x = x_1$. Note also what happens near x_1, but to the left of x_1; also, near x_2 and near x_3.

(7) Where $|f(x)|$ becomes large, $|\frac{1}{f(x)}|$ becomes close to 0. Thus for x going to the right of x_3, the graph of $\frac{1}{f}$ comes ever closer to the x-axis. For x going to the left of x_1, $\frac{1}{f(x)}$ is negative but the graph also comes ever closer to the x-axis.

The vertical dashed lines drawn in the construction of the graph of $\frac{1}{f}$ are of special interest. If we carefully examine the graph of $\frac{1}{f}$ in the vicinity of one of these, say $x = x_3$, we see that as x gets closer to x_3 the graph of $\frac{1}{f}$ gets closer to the vertical line. However, the graph of $\frac{1}{f}$ never actually touches the line. A line which is approached arbitrarily closely (but never intersected) by a graph is called an asymptote of the graph. The three vertical dashed lines are thus vertical asymptotes of the graph of $\frac{1}{f}$. If we assume that f is increasing for $x > x_3$, decreasing for $x < x_1$, and unbounded in both cases, then the x-axis is a horizontal asymptote of the graph of $\frac{1}{f}$.

We say that the function $\frac{1}{f}$ is <u>not</u> <u>bounded</u> because, given any number $k > 0$ we can find an x in the domain of $\frac{1}{f}$ such that $|\frac{1}{f}(x)| > k$. For example, if k is 1,000,000,000, we take a value of x so close to x_3, on either side, that $|\frac{1}{f}(x)| > 1,000,000,000$. Do you believe this is possible? Think about it.

A function g is said to be <u>bounded</u> iff there is a real $k > 0$ such that $|g(x)| \leq k$ for all x in the domain of g.

A function, even if not bounded on its entire domain, may be bounded on some interval of its domain. For example, f is bounded on the interval $[x_1, x_3]$. A suitable value of k here is $k = 3$. Then $|f(x)| \leq k$ for all $x \in [x_1, x_3]$. Geometrically, for f to be bounded, the graph of f must lie entirely between the lines $y = k$ and $y = -k$ for some $k > 0$.

The following are examples of bounded functions of R to R:

(1) any constant function (trivial)

(2) $x \xrightarrow{g} \frac{|x|}{x}$ if $x \neq 0$, $0 \xrightarrow{g} 0$

(3) $x \xrightarrow{h} \frac{x}{|x| + 1}$

(4) $x \xrightarrow{f} x - [x]$.

In (2), $g(x) = -1$ if $x < 0$, $g(x) = 0$ if $x = 0$, and $g(x) = 1$ if $x > 0$. The graph of (3) lies entirely between or on $y = 1$ and $y = -1$.

4.12 <u>Exercises</u>

1. Construct the graph of the reciprocal function from the graph of the function for the functions of R to R given by:

(a) $x \xrightarrow{f} |x|$

(b) $x \xrightarrow{g} [x]$

(c) $x \xrightarrow{h} x + 5$

(d) $x \xrightarrow{h} 4x$

(e) $x \xrightarrow{c_1} 1$

(f) $x \xrightarrow{c_3} 3$

(g) $x \xrightarrow{c_{.4}} .4$

(h) $x \xrightarrow{q} x^2$

2. For each graph in Exercise 1:

   (a) Give the horizontal asymptotes, if any.

   (b) Give the vertical asymptotes, if any.

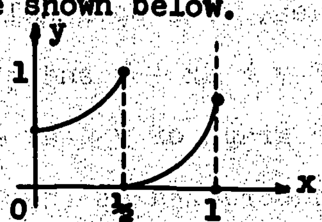
   (c) Locate on the graph the points where the reciprocal function has a local maximum or a local minimum, if any.

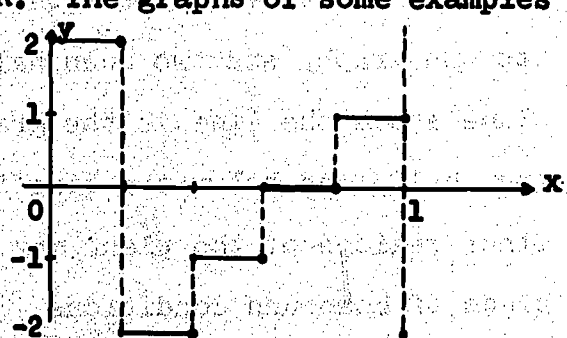
   (d) Determine whether the given function is bounded on its domain.

   (e) Determine whether each reciprocal function is bounded on its domain.

   (f) For each function and for each reciprocal function find an interval on which the function is bounded.

*3. Let B represent the set of all bounded functions with domain [0, 1] and codomain R. The graphs of some examples are shown below.





   (a) Is (B, +) an operational system? That is, for $f \in B$ and $g \in B$, is $[f + g]$ a unique bounded function of [0, 1] to R?

   (b) Since addition of real functions is generally commutative and associative, addition of functions in (B, +)

has these properties. Is (b, +) an abelian group? What is the identity? If $f \in B$, is the additive inverse $-f$, defined by $[-f](x) = -f(x)$ for all $x \in [0, 1]$, also in B?

(c) For every real number m and every $f \in B$, is $[mf]$ in B; that is, is $[mf]$ bounded?

4.13 Summary

1. A condition $\mathcal{C}(x, y)$ on $R \times R$ has a solution set $S \subset R \times R$ and a graph G, which is the set of points determined by S in a rectangular coordinate plane. The "graph of $\mathcal{C}(x, y)$" is the graph of its solution set, S. The graph of $\mathcal{C}(x, y)$ has symmetry in the y-axis if $\mathcal{C}(x, y)$ and $\mathcal{C}(-x, -y)$ are equivalent; symmetry in the origin if $\mathcal{C}(x, y)$ and $\mathcal{C}(-x, -y)$ are equivalent.
2. The line $y = ax + b$ partitions the plane into three sets: its own graph, and two open halfplanes. The open half-plane above the line is the graph of $y > ax + b$ and the open halfplane below the line is the graph of $y < ax + b$. Other regions of the plane may be constructed as the graph of compound conditions.
3. If G is the graph of a condition $\mathcal{C}(x, y)$, $(x, y) \xrightarrow{T} (x + a, y + b)$ is a translation, and if G′ is the image of G under T,

then $\mathcal{C}'(x,y)$, the condition for G', is $\mathcal{C}(x - a, y - b)$.

4. If f is a function of A to R with rule $x \longrightarrow f(x)$, then $y = f(x)$ is the function equation of f. The condition $y = f(x)$ and $x \in A$ determines a function equivalent to f. (The graph of f is the same as the graph of $y = f(x)$ and $x \in A$). A condition $\mathcal{C}(x,y)$ is a function condition if and only if no two distinct ordered pairs in the solution set of $\mathcal{C}(x,y)$ have the same first element. The condition $\mathcal{C}(x,y)$ is then said to determine a function with domain $A = \{x : (x,y)$ is in the solution set of $\mathcal{C}(x,y)\}$.

5. The graph of a real function $f: A \longrightarrow R$ is symmetric with respect to the y-axis iff $f(-x) = f(x)$. It is symmetric with respect to the origin iff $f(-x) = -f(x)$.

6. The special functions of R to R given by $x \longrightarrow |x|$, $x \longrightarrow [x]$, and $x \longrightarrow \frac{1}{x}$, $x \longrightarrow x$, $x \longrightarrow x^2$ were used to construct many other functions of interest.

7. The a-points of f are the solutions for x of the system $y = f(x)$ and $y = a$. If f and g are any two functions, the graphs of f and g may be used to find the solution of $y = f(x)$ and $y = g(x)$ by finding x such that $f(x) = g(x)$.

8. If f is any function, $a \in R$ and $a \neq 0$, $g(x) = ax$, and $h(x) = x + a$, $f \circ g$ and $f \circ h$ have special properties relative to f. The graph of $f \circ g$ is similar to the graph of f in that $f \circ g\ (x) = f(ax)$. The graph of $f \circ h$ is a translation to the right or to the left of f, depending on whether $a < 0$ or $a > 0$.

9. In investigating the graph of $\frac{1}{f}$, where the graph of f is given it is found that
   (a) if $f(a) = 0$, $\frac{1}{f}(a)$ is not defined, but the line $x = a$ may be an asymptote of the graph of $\frac{1}{f}$.
   (b) if f has a local maximum (minimum) at $x = b$, then $\frac{1}{f}$ has a local minimum (maximum) at $x = b$.

4.14 Review Exercises

1. Construct the graph of the condition $|x| + |y - 2| = 6$. Use the methods of this chapter to obtain the graph efficiently. Determine the symmetries of the graph.
2. For each of the following symmetries, draw a graph having the given symmetry:
   (a) Symmetry in the y-axis.
   (b) Symmetry in the x-axis.
   (c) Symmetry in the origin.
   (d) Symmetry in the x-axis and in the y-axis.
   (e) Symmetry in the origin and in the x-axis.
3. (a) Graph the compound condition $y - 2 \geq |x + 3|$ and $y \leq 6$.
   (b) Find the image of this region under the translation $(x,y) \xrightarrow{T} (x + 3,\ y - 2)$.
   (c) What is the condition for the image region?
4. Is $y^2 = x^2 + 2$ and $x \in R^+$ a function condition?
5. (a) Use the methods of this chapter to construct the graph of the function g of R to R with rule $x \xrightarrow{g} -\frac{1}{2}x^2 + 3$ from the graph of $f : R \longrightarrow R$ with rule $x \xrightarrow{f} x^2$.
   (b) Use the graph of g to solve the equations

(i) $-\frac{1}{2}x^2 + 3 = 0$ (ii) $-\frac{1}{2}x^2 + 2 = 0$ (iii) $-\frac{1}{2}x^2 + 5 =$

(c) Use the graph of g and the graph of $y = 2x - 7$ to solve the system:

$$y = -\tfrac{1}{2}x^2 + 3$$
$$y = 2x - 7$$

6. Given real functions having the domains specified and codomain R:

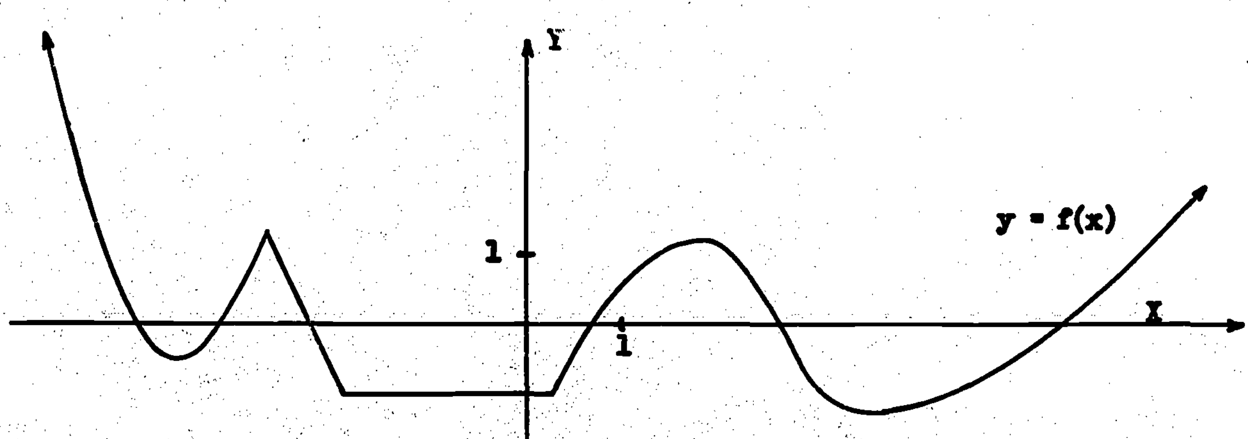
$$x \xrightarrow{f} \tfrac{1}{2}x^2,\ x \in R \qquad x \xrightarrow{h} \frac{1}{x},\ x \in R\setminus\{0\}$$
$$x \xrightarrow{g} 2x + 3,\ x \in R \qquad x \xrightarrow{\ell} \tfrac{1}{2}x,\ x \in R$$
$$x \xrightarrow{m} x - 2,\ x \in R$$

(a) Construct the graphs of f, g, and h.

(b) Construct the graph of $f \circ \ell$, of $g \circ \ell$, and $h \circ \ell$.

(c) Construct the graph of $f \circ m$, $g \circ m$, and $h \circ m$.

7.



(a) Trace the graph of f above on your paper and sketch from it the graph of $\frac{1}{f}$.

(b) Find the local maxima and minima of f and of $\frac{1}{f}$.

(c) Write the equations $x = a$ and $y = b$ of the vertical and horizontal asymptotes of $\frac{1}{f}$ (estimating a and b from your graph).

(d) Give an interval in which f is bounded. Give an interval in which $\frac{1}{f}$ is bounded.

Chapter 5

COMBINATORICS

5.1 Introduction

The study of combinatorics had its origin in problems involving counting. These problems may have involved, for example, finding the number of one-to-one mappings of a set onto itself or finding the total number of subsets of a given set that have some specified number of members.

The above mentioned types of problems come from a class of mathematical ideas known generally as <u>combinatorial counting</u>. Although combinatorics today encompasses a much wider range of ideas and overlaps such studies as group theory, graph theory, and topology, as well as others, we will restrict our interest in this chapter to combinatorial counting. Sometimes combinatorial counting is referred to as <u>sophisticated counting</u>. This means that instead of counting each member of a set individually when trying to determine its total number of members, it is sometimes possible to find this number more efficiently.

5.2 Counting Principle and Permutations

<u>Example</u> 1. Suppose that A, B and C are three cities, and you wish to travel from City A to City C by passing through City B. There are exactly three roads from City A to City B--the red

road, the blue road and the yellow road. There are exactly two roads from City B to City C-- the green road and the orange road. How many ways are there to make the trip from A to C? (See Figure 5.1.)

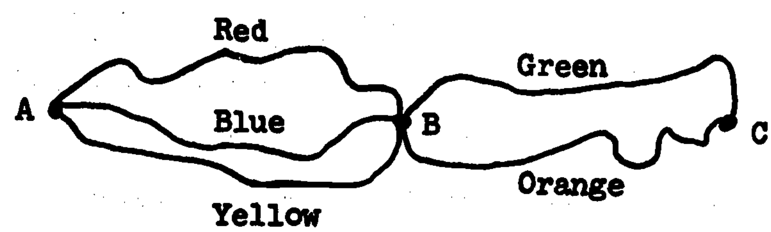


Figure 5.1

One way is to take the red road from A to B, and then the green road from B to C; we shall call this route the red-green route. All the possible routes are shown in Table 5.1.

Roads from A to B	Roads from B to C	Routes from A to C
red	green	red-green
blue	orange	red-orange
yellow		blue-green
		blue-orange
		yellow-green
		yellow-orange

Table 5.1

The total number of routes is 6. Notice that $6 = 3 \cdot 2$, where 3 is the number of ways to make the first part of the trip, and 2 is the number of ways to make the second part of the trip.

<u>Example</u> 2. Let S be the set {a,b,c,d} consisting of four different letters of the alphabet. How many two-letter "words" can you make using the letters in this set? Before answering the question, we must agree to certain rules. One rule is that a "word" does not necessarily have any meaning; another rule is that a letter may not be used more than once in the same "word." Thus, while we accept "bd" as a "word", we do not accept "dd".

All possible "words" follow:

ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc.

There is a total of 12 "words." As in Example 1, there are two choices to be made in forming a "word." First, choose the first letter of the "word." There are 4 choices, since you may use any one of the four letters in the set. Next, choose the second letter of the "word." How many choices are there in this case? Not 4, since the second letter cannot be the same as the first. Therefore, there are just 3 choices for the second letter, once the first letter has been selected. Do you see from Table 5.2 that we have the same sort of situation as we had in Example 1?

Number of Choices for First Letters	Number of Choices for Second Letter	Total Number of Words
4	3	$12 = 4 \cdot 3$

Table 5.2

Specifically, in this case we have $12 = 4 \cdot 3$ "words." The "tree" diagram, Figure 5.2, is another way to make this clear.

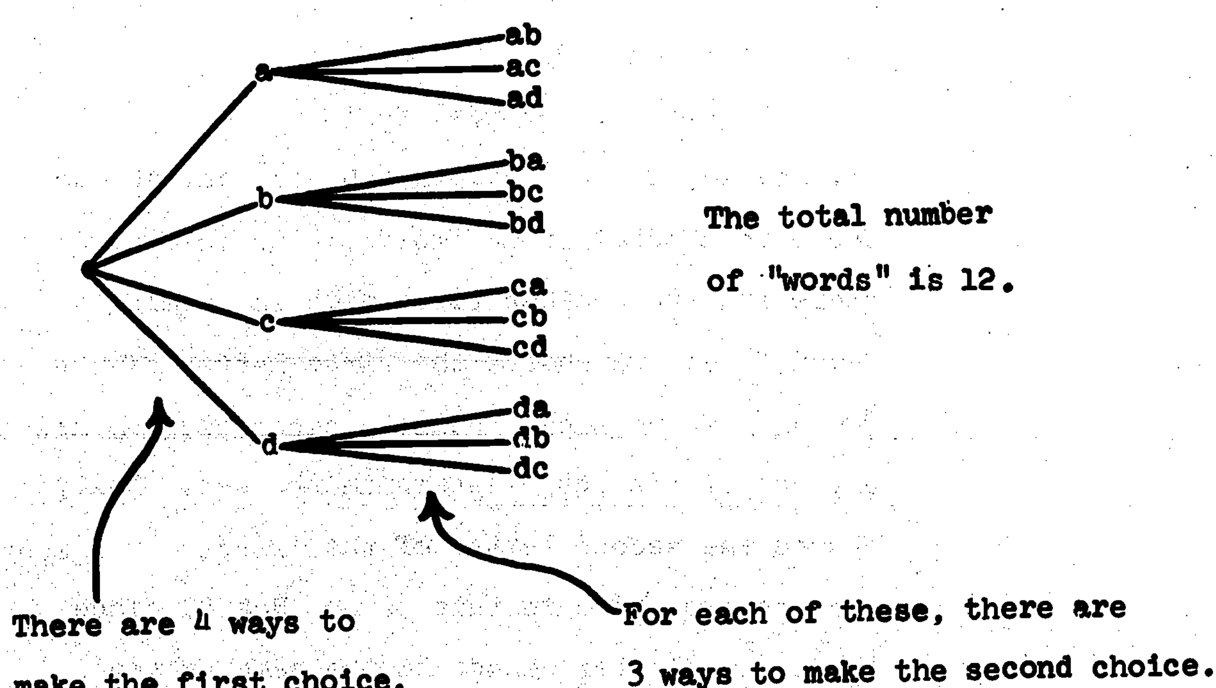


Figure 5.2

The two examples just discussed illustrate a principle called the <u>counting principle</u>. It may be stated as follows:

<u>CP</u> If an activity can be accomplished in <u>r</u> ways, and after it is accomplished, a second activity can be accomplished, in <u>s</u> ways, then the two activities can be accomplished, one after the other, in $r \cdot s$ ways.

<u>Example</u> 3. Suppose in Example 2 we lift the restriction that no letter can be selected twice. If we do so we will have four ways to select the first letter, and then four ways to select the second letter. Therefore we will have $4 \cdot 4 = 16$ distinct possible words. This result is illustrated in the tree diagram of Figure 5.3 and suggests that we may state a more general counting principle, "CP′".

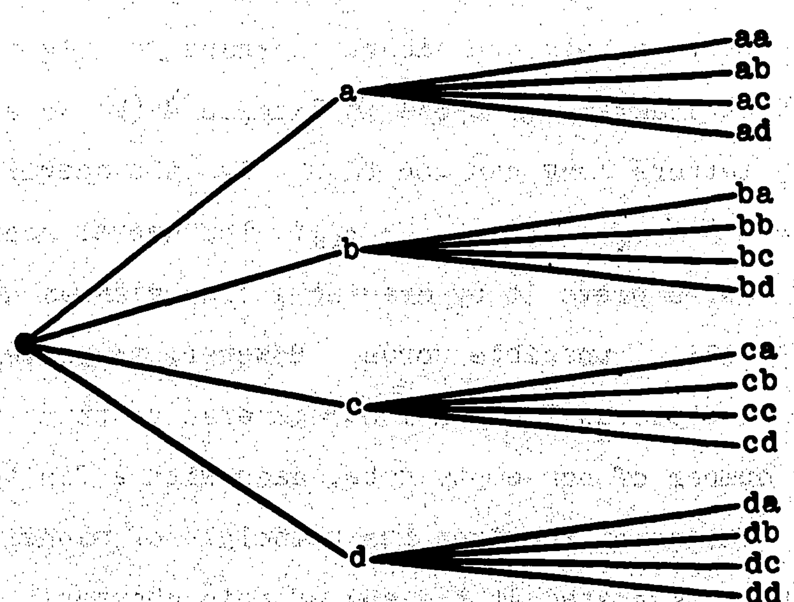


Figure 5.3

CP′ Let A_1 and A_2 be sets with r_1 and r_2 elements respectively, where $r_1, r_2 \in Z^+$. Then $A_1 \times A_2 = \{(a_1, a_2): a_1 \in A_1 \text{ and } a_2 \in A_2\}$ contains $r_1 \cdot r_2$ elements.

Example 4. Given the set of letters $\{a,e,i,o,u\}$, how many two letter "words" can be formed, using the same rules as in (a) Example 2? The first letter may be chosen in 5 ways ($r_1 = 5$). The second letter may then be chosen in 4 ways ($r_2 = 4$). The total number of "words" is $5 \cdot 4 = r_1 \cdot r_2 = 20$. (b) Example 3? Here $r_1 = r_2 = 5$ and thus the total is $25 = 5 \cdot 5$.

One might well wonder if the counting principle CP and its generalization CP′ can be extended to more than two sets A_1 and A_2. For instance suppose, in Example 4 (a) we wanted to form 3 letter "words." Is the number of such "words" $5 \cdot 4 \cdot 3 = r_1 \cdot r_2 \cdot r_3 = 60$? Would the number in Example 4 (b) be $5 \cdot 5 \cdot 5 = r_1 \cdot r_2 \cdot r_3 = 125$? The answer is yes, to both questions. Perhaps you might confirm this with a tree diagram. Suppose in Example 4 (b) we ask how many words 15 letters long can you form? Is the answer $5 \cdot 5 \cdot \ldots \cdot 5 = 5^{15} = r_1 \cdot r_2 \cdot \ldots \cdot r_{15}$? The answer again is yes. You could of course prove it by drawing a tree diagram and counting the 30,517,578,125 possible words. However, to prevent you from tiring, we state as Theorem 1 our general counting principle for a finite number of non-empty sets, each with a finite number of elements. The proof requires the principle of mathematical induction, which is stated at the end of this chapter. After you have gained some facility with this principle, you will be asked to write a proof of Theorem 1. To simplify the writing of the theorem and subsequent statements we adopt the following notation. If a set S contains r elements we will write $n(S) = r$.

<u>Theorem</u> 1. <u>CP</u> Let $A_1, A_2, \ldots, A_k$ be non-empty sets and let $n(A_i) = r_i$ for $i = 1, 2, \ldots, k$. where each $r_i \in Z^+$. Let $A_1 \times A_2 \times \ldots A_k = \{(a_1, a_2, \ldots, a_k): a_i \in A_i, i = 1, 2, \ldots, k\}$ Then $n(A_1 \times A_2 \ldots \times A_k) = r_1 \cdot r_2 \cdot \ldots \cdot r_k$.

<u>Example</u> 5. A direct mail firm plans to send out a letter to an assortment of people. Each letter is to contain four pieces of literature, one piece from each of the four companies this firm represents. Company A_1 has made available six different pieces of literature, Company A_2 three pieces, Company A_3 two pieces and Company A_4 eight pieces. How many different mailings are possible? We have
$n(A_1) = r_1 = 6$, $n(A_2) = r_2 = 3$,
$n(A_3) = r_3 = 2$, $n(A_4) = r_4 = 8$. Therefore
$n(A_1 \times A_2 \times A_3 \times A_4) = r_1 \cdot r_2 \cdot r_3 \cdot r_4 = 6 \cdot 3 \cdot 2 \cdot 8 = 288$.

<u>Example</u> 6. In a certain school, the student council decides to give each student an ID number consisting of a letter of the alphabet followed by two digits. What is the maximum number of students that can

be accommodated by this procedure?

Let A_1 = {all letters in the alphabet}, $A_2 = A_3$ = {all digits}. Therefore $n(A_1) = 26$, $n(A_2) = n(A_3) = 10$. Therefore the number of ID numbers is $n(A_1 \times A_2 \times A_3) = 26 \cdot 10 \cdot 10 = 2600$.

In Course II Chapter 2, Section 2.3 we defined a <u>permutation</u> of a set S as a one-to-one mapping of the set onto itself; and saw that if the set contains n elements, then there are $n! = n(n - 1) \cdot \ldots \cdot 2 \cdot 1$ such permutations. In Example 7, we shall see that the counting principle may be used to get the same result.

<u>Example</u> 7. How many permutations are there of the set $S = \{a,b,c\}$?

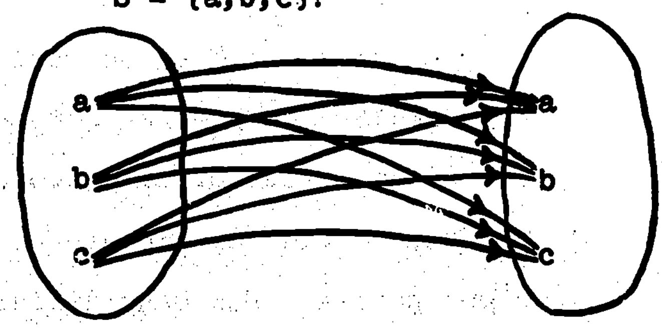


Figure 5.4

As illustrated in Figure 5.4 we may choose any one of the 3 arrows starting at a; that is, there are 3 choices. Next, we move to b. We do not have 3 choices, since we cannot assign the same image to b as we did to a, if we want a one-to-one mapping. So, the number of choices here is 2. Next, we move to c. Two of the images have now been used. So here we have only 1 choice.

To summarize: At a we have 3 choices; at b we have 2 choices; at c we have 1 choice. The total number of one-to-one mappings is $3 \cdot 2 \cdot 1 = 6 = 3!$ In the language of our theorem, $n(A_1) = 3$, $n(A_2) = 2$, $n(A_3) = 1$, and therefore $n(A_1 \times A_2 \times A_3) = 3 \cdot 2 \cdot 1 = 3!$

Example 8. Given the sets in Figure 5.5 how many ways are there to make a one-to-one mapping from set A to set B?

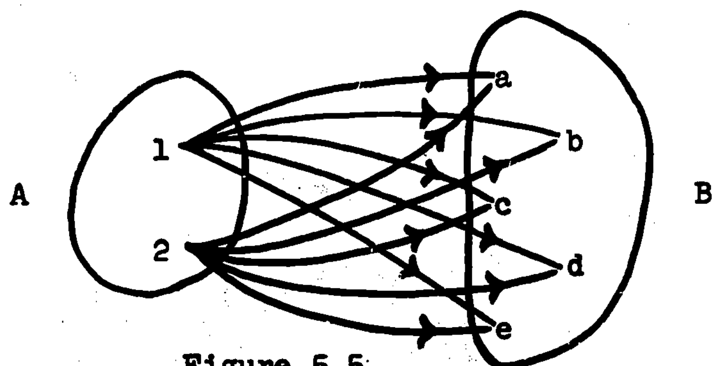


Figure 5.5

We may choose any one of the 5 arrows starting at 1; there are 5 choices. Then we may choose any one of 4 arrows, starting at 2; we cannot choose the arrow which goes to the same image as our first arrow. Therefore, the total number of one-to-one mappings from A to B is $5 \cdot 4 = 20$.

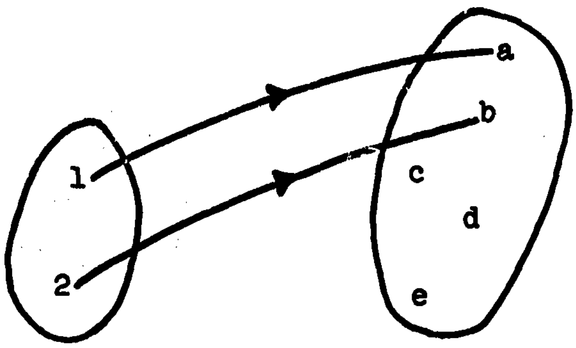
The word permutation is also used to describe a situation such as that in Example 8. Specifically, we would say that the number of permutations of 5 elements taken 2 at a time is 20. In Example 8, the 5 elements are a, b, c, d, and e. And the 20 permutations of these elements taken 2 at a time are listed in

Table 5.3.

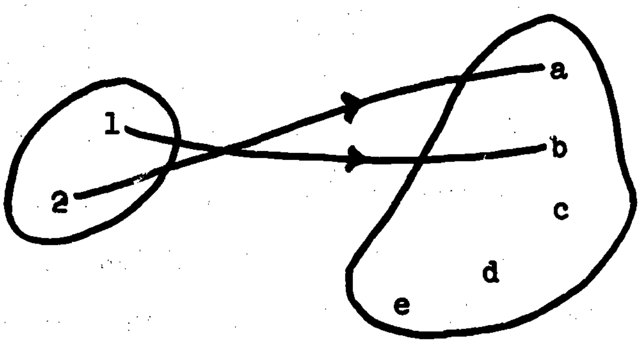
ab	ac	ad	ae
ba	bc	bd	be
ca	cb	cd	ce
da	db	dc	de
ea	eb	ec	ed

Table 5.3

Each of these, of course, corresponds to one of the 20 mappings mentioned in Example 8. For instance, "ab" refers to the mapping in Figure 5.6 (a).



(a)



(b)

Figure 5.6

On the other hand "ba" refers to the mapping in Figure 5.6 (b). Thus, "ab" and "ba" are different permutations (i.e., they are different mappings).

<u>Example</u> 9. What is the number of 4-letter "words" that can be formed from the set {a,b,c,d,e,f,g}? The number is $7 \cdot 6 \cdot 5 \cdot 4$. (Express in the language of Theorem 1.) This is the number of permutations of 7 elements taken 4 at a time.

<u>Example</u> 10. What is the number of permutations of 10 elements taken 3 at a time?

$$10 \cdot 9 \cdot 8 = 720$$

This is the number of one-to-one mappings from a set containing 3 elements to a set containing 10 elements.

<u>Example</u> 11. What is the number of permutations of 5 elements taken 5 at a time?

This is the number of one-to-one mappings from set A to set B, where both A and B have 5 elements. (See Figure 5.7.)

Figure 5.7

But the number of such mappings is the same as the number of mappings of A onto itself.

Therefore, the answer is 5! or 120.

Example 12. Suppose you had five colored flags, one in each of the following colors: red, white, blue, green, yellow. If you agree that a given signal is to be represented by a particular arrangement of three colored flags, how many different signals could you devise using the five flags? For example, the arrangement

RED YELLOW BLUE

might mean "Help". This problem really asks for the number of one-to-one mappings from a set containing 3 elements to a set containing 5 elements. This number is:

$$5 \cdot 4 \cdot 3 = 60$$

In Examples 8 to 12 we have been considering the number of one-to-one mappings from a set A, with r members, to a set B, with n members, where $r \leq n$. Another way to describe the number of one-to-one mappings from a set with r members to a set with n members ($r \leq n$) is the number of permutations of n elements taken r at a time.

Figure 5.8 indicates that there are n ways of finding an image in B for the first selection from A, (n - 1) ways to find the image for the second selection from A, and so on.

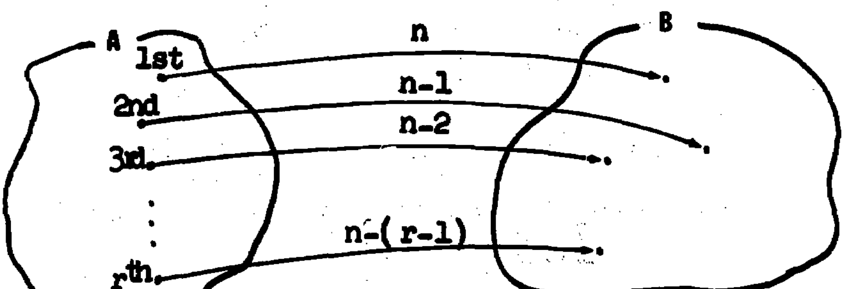


Figure 5.8

This is expressed by Table 5.4.

selection from A	1st	2nd	3rd	...	10th	...	rth
ways to find image in B	n	n-1	n-2	...	n-9	...	n-(r-1)

Table 5.4

The symbol $(n)_r$ is used to represent the number of permutations of n elements taken r at a time. Referring to the preceding table and applying the counting principle, we can conclude that:

$$(n)_r = n \cdot (n-1) \cdot (n-2) \cdot \ldots (n-(r-1))$$

Since $n - (r-1) = n - r + 1$, we could express the above formula as follows:

$$(n)_r = n(n-1)(n-2)\cdots(n-r+1)$$

Example 13. (a) $(8)_5 = 8\cdot7\cdot6\cdot5\cdot4 = 6720$

(b) $(4)_4 = 4\cdot3\cdot2\cdot1 = 4! = 24$

The exercises in Section 5.3 will contain specific examples of permutations of n elements taken r at a time. An alternative form of the general formula for $(n)_r$ will be developed in Exercise 17 of Section 5.3.

5.3 Exercises

1. Given the set of letters $\{r,s,t,u,v,w,x\}$, how many "words" without repeated letters can be formed having:
   - (a) one letter
   - (e) two letters
   - (b) three letters
   - (f) four letters
   - (c) five letters
   - (g) six letters
   - (d) seven letters
2. If set B contains seven elements, how many one-to-one mappings are there from set A to set B if set A contains:

(a) one element (b) two elements

(c) three elements (d) four elements

(e) five elements (f) six elements

(g) seven elements

3. Use the results of Exercises 1 or 2 to answer the following:

   (a) What is $(7)_1$? (b) What is $(7)_2$? (c) What is $(7)_3$?

   (d) What is $(7)_4$? (e) What is $(7)_5$? (f) What is $(7)_6$?

   (g) What is $(7)_7$?

4. How many permutations are there of the set $\{a,b,c,d,e,f,g,h\}$ taken 5 at a time?

5. Suppose you have 5 books to put on a shelf. In how many orders can the 5 books be arranged?

6. In Exercise 5, suppose there is room for only 3 of the books on the shelf, but you may use any 3. How many arrangements are possible? That is, what is the number of permutations of 5 elements taken 3 at a time?

7. In a certain state, the license tags consist of two letters of the alphabet followed by three digits.

   (a) How many different license "numbers" are possible?

   (b) How many are possible if the letters O and I are not used?

8. A telephone number consists of 10 digits.

   (a) How many numbers are possible if there are no restrictions?

   (b) How many are possible if the digit "0" cannot be used as the first digit?

(c) How many are possible if the digit "0" cannot be used as the first digit and also cannot be used as the fourth digit?

9. If a baseball team has 10 pitchers and 4 catchers, how many batteries (pitcher-catcher pairs) are possible?

10. If a girl has 5 blouses and 4 skirts, how many blouse-skirt combinations can she arrange?

11. If you toss one die for a first number, then toss a second die for a second number, how many results (ordered number pairs) are possible?

12. Find:

(a) $(5)_4$ (b) $(8)_3$ (c) $(8)_5$ (d) $(20)_2$ (e) $(9)_5$

13. (a) What is $(8)_3$? (b) What is $8!$?

(c) What is $(8 - 3)!$ (d) What is $\frac{8!}{(8-3)!}$?

14. What is: (a) $(6)_4$? (b) $6!$ (c) $(6 - 4)!$?

(d) $\frac{6!}{(6-4)!}$

15. What is: (a) $(10)_3$ (b) $10!$ (c) $(10 - 3)!$

(d) $\frac{10!}{(10-3)!}$

16. Let n and r be positive integers and $r \leq n$. Give an argument to justify:

$n! = n(n - 1)(n - 2) \ldots (n - r + 1)\cdot[(n - r)!]$

17. Using the formula $(n)_r = n(n - 1)\cdots(n - r + 1)$ and Exercise 16, gi an argument to justify this new formula for $(n)_r$: $(n)_r = \frac{n!}{(n-r)!}$

18. Use the formula in Exercise 17 to find:

(a) $(11)_3$ (b) $(7)_5$ (c) $(15)_3$ (d) $(100)_2$

19. Make up permutation problems for each of the following answers:

(a) $\frac{8!}{(8-2)!}$ (b) $\frac{9!}{(9-3)!}$ (c) $\frac{9!}{5!}$

20. Use the formula in Exercise 17 to find the number of permutations of 5 elements taken 5 at a time. Do you see that the denominator is 0!? 0! has no meaning. We define $0! = 1$ so that the formula in Exercise 17 holds for all whole numbers n, r with $r \leq n$ without exception.

21. Find a standard name for each of the following:

(a) $\frac{8!}{(8-8)!}$ (b) $\frac{12!}{(12-12)!}$ (c) $3! + 2! + 1! + 0!$

(d) Express as a product in powers of 1, 2, 3, and 4: $(4!)\cdot(3!)\cdot(2!)\cdot(1!)$ (e) Evaluate $\sum_{i=1}^{4} i!$

22. Computers use binary numbers where only 0 and 1 are used as digits. How many 2-digit binary numbers are there? 3-digit? 4-digit?

5.4 The Power Set of a Set

Given a set S with n elements, we know that there are various subsets of S that may be formed. The empty set, S itself, as well as one-member subsets, two-member subsets, and so on, are examples that might be considered. The set of all subsets of S is called the power set of S.

<u>Definition</u>. The <u>power set</u> of a set S, denoted $\mathcal{P}(S)$, is the set whose elements are the subsets of S. (Thus, $A \in \mathcal{P}(S)$ if and only if $A \subset S$.)

Table 5.5 lists the power set of S, for several different sets S. Copy and complete the table in order to test your understanding of the notion of a power set. Perhaps you will see a pattern that indicates how the number of elements in the power set of S is related to the number of elements in S.

S	n(S)	$\mathcal{P}$(S)	n($\mathcal{P}$(S))
{} = ∅	0	{∅}	1
{a}	1	{∅, {a}}	2
{a,b}	2	{∅, {a},{b},{a,b}}	4
{a,b,c}	3		
{a,b,c,d}	4		

Table 5.5

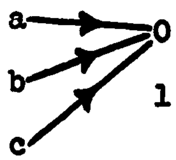
Once again, we may apply the counting principle to help us determine the total number of subsets of a given set S. Suppose S contains k elements; that is n(S) = k. We are interested in forming every possible subset of S. Selecting any one of these subsets may be thought of as a sequence of k tasks. A task is a decision for each member of S; either you select the first member or reject it, and likewise for the second member, third member, and so on. In other words, there are two possibilities for each member of S. Then, since S has k members, the counting principle tells us that the product of k factors, each equal to 2, is the number of ways of performing these tasks one after the other. Each subset of S is the result of exactly one performance of the tasks, and each performance of the tasks results in exactly one subset of S. Accordingly the number of subsets of a set S with k elements is:

$$\underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{k \text{ factors}} = 2^k$$

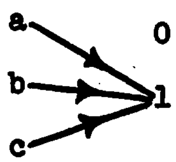
Is this the conclusion you drew when you completed Table 5.5? In the language of Theorem 1, for each $i \in S$, $i = 1, \ldots, k$ let $A_i = \{\text{select, reject}\}$. Therefore $r_1 = r_2 = \ldots = r_k = 2 = n(A_i)$. Thus $n(A_1 \times A_2 \times \ldots \times A_k) = r_1 \cdot r_2 \ldots \cdot r_k = 2^k$. If we replace the word select by the digit 1 and the word reject by the digit 0 then $A_i = \{1,0\}$ and we can reason as follows:

The number of elements in the power set of S is equal to the number of mappings with domain S and codomain {0,1}. The elements in S that map onto 1 are selected and those that map onto 0 are rejected for the subset generated by that particular mapping. Here we do not require that the mappings be one-to-one, nor do we require that they be onto. For example, each member of S may be mapped onto 1 and the set S itself would be the generated subset. Likewise each member of S may be mapped onto 0 and then the empty set would be selected.

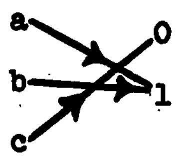
Example 1. Figure 5.9 exhibits some mappings from {a,b,c} to {0,1} and the sets they generate.



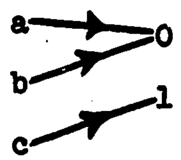
generates $\{\ \} = \emptyset$



generates $\{a,b,c\}$



generates $\{a,b\}$



generates $\{c\}$

Figure 5.9

Complete the rest of the mapping diagrams from $\{a,b,c\}$ to $\{0,1\}$ as an exercise.

5.5 Number of Subsets of a Given Size

We will now turn our attention to the number of subsets of S that have some given number of elements; for example the number of subsets of $\{a,b,c\}$ that have exactly two elements. From your mapping diagrams you can see that this number is 3. In general we will be concerned with the number of r-member subsets of a set S with n members.

Example 2. Suppose that {a,b,c,d,e} is a set of club members. How many committees can be formed which have exactly two members? The committees are listed below:

{a,b}, {a,c}, {a,d}, {a,e}, {b,c}, {b,d}, {b,e}, {c,d}, {c,e}, {d,e}

The number in this case is 10. This question is the same as asking how many subsets of 2 elements can be formed from a set of 5 elements.

In general, questions such as this may be phrased as follows: Given a set containing n elements, how many of its subsets contain exactly r elements? The word combination is also used to describe this situation. Specifically, we would ask, how many combinations are there of n elements, taken r at a time?

In order to answer the general question, let us first look again at the question raised in Example 2, a question whose answer we already know. Given the set {a,b,c,d,e}, how many different subsets of 2 elements can be formed? We introduce the symbol

$$\binom{5}{2}$$

to represent this number. That is, $\binom{5}{2}$ is the number of subsets of 2 elements that can be formed from a set of 5 elements.

Figure 5.10 shows a one-to-one onto mapping from the set {1,2} to the subset {a,b}. The set {1,2} is used since we want a subset having two elements. However, the diagram shows only one such mapping. How many one-to-one onto mappings are there from {1,2} to the subset {a,b}? Since {a,b} has the same number

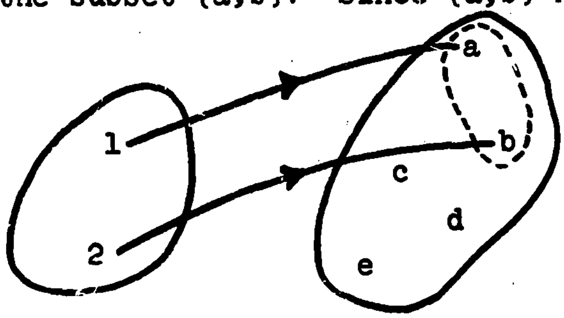


Figure 5.10

of elements as {1,2}, this is the same as the number of permutations of a set of 2 elements--that is 2!. So there are 2 different one-to-one onto mappings from {1,2} to {a,b}. (Be sure that you can draw a diagram for each.)

Also there are 2! different one-to-one onto mappings from {1,2} to the subset {a,c}, to the subset {a,d}, etc. In fact, there are 2! different one-to-one onto mappings from {1,2} to every subset of S containing two elements. Now how many such subsets are there? We have agreed to let $\binom{5}{2}$ represent this number. Thus if we form the product

$$2!\binom{5}{2}$$

we should get the total number of ways to form a one-to-one mapping from {1,2} to the set S. However, from CP we know this number is: $(5)_2$. Therefore we have: $2!\binom{5}{2} = (5)_2$.

Then dividing by 2! we get:

$$\binom{5}{2} = \frac{(5)_2}{2!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

Of course this agrees with our earlier observation that there are 10 possible subsets, each with 2 persons, that can be formed from a club of 5 persons. We may also say that the number of combinations of 5 persons, taken 2 at a time, is 10.

Example 3. Consider the problem of finding how many subsets of 3 elements can be formed from a set of 7 elements. Again, let $\binom{7}{3}$ represent this number. To find the standard name for $\binom{7}{3}$ we begin by examining the mapping of Figure 5.11.

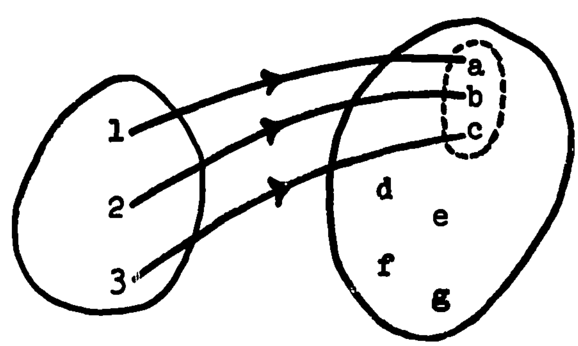


Figure 5.11

The diagram shows a one-to-one mapping from {1,2,3} to the subset {a,b,c}. The diagram shows only one such mapping, but there are 3! of them. (Why?) Furthermore, there are 3! different one-to-one onto mappings from {1,2,3} to every one of the $\binom{7}{3}$ subsets having 3 elements. Therefore,

$$3!\binom{7}{3} = (7)_3$$

where $(7)_3$ is obtained from the counting principle. Dividing by 3! gives:

$$\binom{7}{3} = \frac{(7)_3}{3!} = \frac{7\cdot6\cdot5}{3\cdot2\cdot1} = 35$$

Therefore, a set of 7 elements has 35 different 3-element subsets.

The two preceding examples suggest a perfectly general argument for finding the number of subsets having r elements that can be formed from a set having n elements, where $r \leq n$. Using $\binom{n}{r}$ to represent this number, we have,

<u>Theorem</u> 2.

$$r!\binom{n}{r} = (n)_r$$

<u>Proof</u>. Exercise 24, Section 5.6.

From Theorem 2, dividing by r! we obtain

$$\binom{n}{r} = \frac{(n)_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!\,(n-r)!}$$

<u>Example</u> 4. In a club with 12 members, how many 5 member subsets are there?

$$\binom{12}{5} = \frac{(12)_5}{5!} = \frac{\frac{12!}{(12-5)!}}{5!}$$

$$= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot (7!)}{7!\ 5!}$$

$$= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= 792$$

Notice that in Example 4 each time you selected a subset of 5 elements from the set of 12 elements, there were 7 elements remaining that were not selected. In general, whenever you select a subset of r elements from a set of n elements there are n - r elements remaining that are not selected. This means that there are just as many subsets with n - r elements as there are subsets with r elements.

Example 5. (a) Compute $\binom{7}{5}$ and $\binom{7}{2}$.

(b) Did you get the same number for each of the computations in part (a)?

(c) If the answer to (b) is yes explain why. If not, do your computations again.

(d) Which of the two computations in (a) was easier? Why?

These results may be expressed more generally as:

Theorem 3.

$$\binom{n}{r} = \binom{n}{n-r}$$

The proof is left as an exercise.

5.6 Exercises

1. In a voting body of 7 members, how many 3-man subsets are there?
2. In a voting body of 12 persons, how many 5-man subsets are there?
3. If set S has 6 elements, how many elements are in $\mathcal{P}(S)$? How many of these subsets have exactly 3 elements?
4. Find a standard name for each of the following:

   (a) $\binom{7}{3}$ (b) $\binom{12}{5}$ (c) $\binom{6}{3}$
5. There are 8 books lying on the table, and you are to choose 3 of them.

(a) How many ways are there to choose 3 books from 8?

(b) How many ways are there to choose the 3 books and arrange them on a shelf?

6. (a) Verify the following formula for special cases of n and m (e.g. $n = 5$ and $m = 3$):

$$\binom{n}{m-1} + \binom{n}{m} = \binom{n+1}{m}$$

*(b) Now show by using the formula,

$$\binom{n}{r} = \frac{(n)_r}{r!}$$

that formula in 6(a) is true when $m \leq n$.

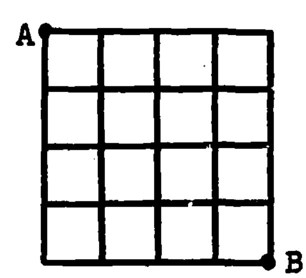
7. Use the fact that the formula in Exercise 6 is true for all natural number replacements for m and n, $m \leq n$, to complete the following:

$$\binom{x-1}{y} + \binom{x-1}{y+1} = \left(\qquad \right)$$

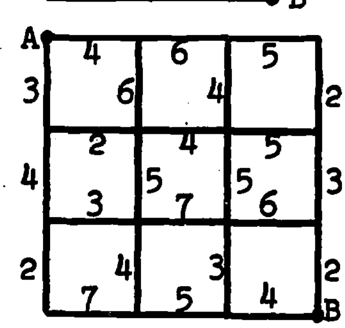
What relation must hold between x and y?

8. If n is a non-negative integer, then $\binom{n}{0} = \boxed{}$

9. If you can move only along the drawn segments down and to the right, how many paths are there from A to B? (Do this by figuring the number of paths to each point.)



10. If the numerals recorded at right indicate the length of the segments, find the shortest distance from A to B. (Travel rules are those of Exercise 9.)



11. If n is a positive integer, then $\binom{n}{1} = \boxed{}$.

12. For n = 4, expand:

$$\sum_{k=0}^{n} \binom{n}{k}$$

into a sum where each term makes use of the formula for $\binom{n}{r}$; then evaluate the sum and express the results in standard form. (Hint: The first two terms of the summation are $\binom{4}{0}$ and $\binom{4}{1}$.)

*13. Prove:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

for any positive integral n. (Hint: For a set with n elements count the number of subsets in two different ways.)

14. If n is a non-negative integer, then $\binom{n}{n} = \boxed{}$.

15. What meaning can we give to $\binom{3}{5}$? From a set of 3 elements, how many 5 element sets can be formed? Obviously there are none. Therefore, we shall define $\binom{3}{5} = 0$. What standard name would you suggest for each of the following?

(a) $\binom{2}{8}$ (b) $\binom{7}{8}$ (c) $\binom{3}{9}$ (d) $\binom{0}{4}$ (e) $\binom{1}{3}$

*16. In a deck of 52 playing cards, how many 13-card hands are possible?

17. Draw diagrams for each of the possible mappings from a set of 3 elements to a set of 2 elements. Don't restrict the mappings to one-to-one or onto.

18. Use the counting principle to suggest a way of expressing the number of mappings in Exercise 17 in exponential form.

19. Use the counting principle to construct an argument that justifies the following:

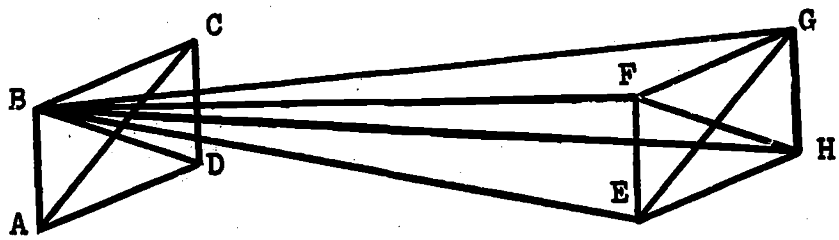
    The number of mappings from a set of b elements to a set of a elements is a^b.

20. In the diagram below there are two graphs each consisting of four nodes (points) and paths connecting the nodes by pairs:

Graph I   Graph II

(a) Explain why each graph has $\binom{4}{2}$ paths, and the total number of paths for the two graphs is $2\binom{4}{2}$.

In the next diagram node B is connected with each node in Graph II to illustrate how each node of Graph I may be connected with a path to each node in Graph II.



Graph I   Graph II

(b) Use the counting principle to explain why there are 16 or 4^2 paths required to connect each node of Graph 1 with each node of Graph II (that is to complete it).

(c) Assuming that the above graph is completed, explain why the number of paths is $\binom{8}{2}$ or $\binom{2\cdot 4}{2}$.

(d) Use an argument concerning the above graphs to justify the statement: $2\binom{4}{2} + 4^2 = \binom{2\cdot 4}{2}$.

Use computation to justify the statement.

*21. Use the graphs and explanations in Exercise 20 for this exercise.

(a) Suppose that you repeated the procedures in Exercise 20 using 5 nodes in each graph. Write the statement in Exercise 20(d) for the case of 5 nodes.

(b) Revise the statement in Exercise 20(d) for n nodes.

(c) Revise the statement in Exercise 20(d) for the case where Graph I has 6 nodes and Graph II has 4 nodes.

(d) Repeat part (c) where Graph I has n nodes and Graph II has m nodes.

*22. Show that the following statements (a) and (b) are equivalent:

(a) $2\binom{n}{2} + n^2 = \binom{2n}{2}$

(b) $n(n - 1) + n^2 = n(2n - 1)$

23. Use what you have learned in this chapter on combinatorics in addition to what you learned in the chapter on affine geometry to justify the following:

(a) If each line in the affine plane π contains k points, then π contains k^2 points.

(b) If the affine plane π contains k^2 points, then it contains $k\cdot(k + 1)$ lines.

*24. Prove Theorem 2.

25. Prove Theorem 3. (Hint: Use the formula developed in Theorem 2.)

5.7 The Binomial Theorem

Example 1. Suppose that you were given the problem of expanding the following power of a binomial:

$(a + b)^5 = (a + b)(a + b)(a + b)(a + b)(a + b)$.

After some labor you would find that the expansion of the above expression is:

$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

The symmetry of the coefficients in the above terms (1,5,10,10,5,1), and the decreasing powers of a (5,4,3,2,1,0) with the corresponding increasing powers of b (0,1,2,3,4,5) leads us to suspect that there might be a more efficient way to get the result without resorting to brute force multiplication of binomials. Note also that the sum of the exponents of a and b in each term is 5.

In this section, we are going to develop a theorem, known as the Binomial Theorem, which will be useful in expanding powers of binomials. It also has other applications in mathematics, for example, to probability theory. The development of the Binomial Theorem will make use of many ideas which you have learned such as the power set of a given set, the number of r-member subsets of a set with n elements, and the use of the symbol Σ to indicate summation.

Example 2. To illustrate the general theorem we expand $(a + b)^3$ by using the distributive property:

(1) $(a+b)(a+b)(a+b) = a(a+b)(a+b) + b(a+b)(a+b)$

(2) $= a[a(a+b) + b(a+b)] + b[a(a + b) + b(a + b)]$

(3) $= a(aa + ab + ba + bb) + b(aa + ab + ba + bb)$

(4) $= aaa + aab + aba + abb + baa + bab + bba + bbb$

(5) $= a^3 + a^2b + a^2b + ab^2 + a^2b + ab^2 + ab^2 + b^3.$

(6) $= a^3 + 3a^2b + 3ab^2 + b^3$

(7) $= \binom{3}{0} a^3 + \binom{3}{1} a^2b + \binom{3}{2} ab^2 + \binom{3}{3} b^3$

(8) $= \sum_{r=0}^{3} \binom{3}{r} a^{3-r}b^r$

We can get the same result using the following combinatorial argument. We could get the terms in (4) directly from the left side of (1) by selecting just one of a or b from each of the binomial factors and recording them in the order of the factors from which they were chosen. The mapping diagrams in Figure 5.12 show all the ways that this selection can be made, where 1,2 and 3 stand for the 1st, 2nd and 3rd factors respectively and the mapping is from $\{1,2,3\}$ to $\{a,b\}$.

Note that the total number of mappings is $2^3 = 8$. (CP)

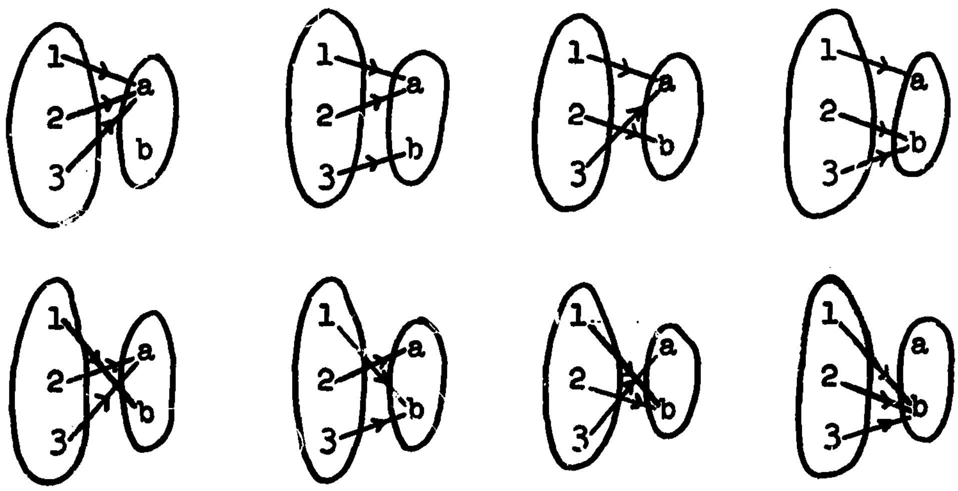


Figure 5.12

The number of times that b is selected as an image in a mapping determines the number of times that a is selected. If b is chosen r times, then a is chosen (3-r) times. Check this in the diagrams. Each mapping then is determined by the assignments of b.

The number of mappings in which

b is the image of 0 elements is 1.	$\binom{3}{0} = 1.$
b is the image of 1 element is 3.	$\binom{3}{1} = 3.$
b is the image of 2 elements is 3.	$\binom{3}{2} = 3.$
b is the image of 3 elements is 1.	$\binom{3}{3} = 1.$
	Total $8 = 2^3$

If b is the image of zero elements then a is the image of three elements, and thus the term which has coefficient $\binom{3}{0}$ is a^3.

If b is the image of one element then a is the image of two elements, and thus the term with coefficient $\binom{3}{1}$ is a^2b.

If b is the image of two elements then we deduce as above that the term with coefficient $\binom{3}{2}$ is ab^2.

Similarly if b is the image of three elements then the term with coefficient $\binom{3}{3}$ is b^3. Multiplying each term by its coefficient and adding again yields

$$\sum_{r=0}^{3} \binom{3}{r} a^{3-r}b^r = (a+b)^3.$$

You should recognize the above as a special case of ideas presented in this chapter:

(a) The number of subsets of a set with n elements is 2^n.

(b) The number of r-member subsets of a set with n elements is $\binom{n}{r}$. The binomial theorem can now be expressed.

<u>Theorem</u> 4. For any pair of real numbers, a and b, and any whole number n:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r =$$

$$\binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \dots + \binom{n}{n} b^n$$

Example 3. Expand $(a+b)^5$.

$$(a+b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{5}{5}b^5$$

$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Example 4. Expand $(p+q)^1$

$$(p+q)^1 = \sum_{r=0}^{1}\binom{1}{r}p^{1-r}q^r = \binom{1}{0}p^1 + \binom{1}{1}q^1 = p+q$$

Example 5. Expand $(1+k)^3$.

$$(1+k)^3 = \binom{3}{0}1^3 + \binom{3}{1}1^2k + \binom{3}{2}1k^2 + \binom{3}{3}k^3$$

$$= 1 + 3k + 3k^2 + k^3$$

Example 6. Expand $(1.03)^4$.

$$(1+.03)^4 = \binom{4}{0}1^4 + \binom{4}{1}1^3(.03) + \binom{4}{2}1^2(.03)^2 + \binom{4}{3}1(.03)^3 + \binom{4}{4}(.03)^4$$

$$= 1 + .12 + .0054 + .000108 + .00000081$$

$$= 1.12550881$$

Example 7. Expand $(a-b)^5$.

$(a-b)^5 = (a+(-b))^5$. Then apply Example 3.

5.8 Exercises

1. Show that $(a+b)^2 = a^2 + 2ab + b^2$ is correct when $a = 3$ and $b = 2$.
2. Show that $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ is correct when $x = 1$ and $y = 2$.

3. Expand the following:

   (a) $(a+b)^4$ (b) $(x+y)^6$

   (c) $(c+d)^7$ (d) $(a+b)^{10}$

4. $(a-b)^2 = (a + (-b))^2 = a^2 + 2a(-b) + (-b)^2 = a^2 - 2ab + b^2$

   Using a similar approach, expand the following:

   (a) $(a-b)^3$ (b) $(x-y)^4$

   (c) $(a-b)^5$ (d) $(x-y)^6$

5. The coefficients in the expansion of $(a+b)^n$ are as follows:

   1 11 55 165 330 462 462 330 165 55 11 1

   What is n?

6. Expand (a) $(x + 1)^3$ (b) $(x - 1)^3$

7. Expand (a) $(x + 2)^4$ (b) $(x - 2)^4$ (c) $(x - \frac{1}{2})^4$.

8. Expand (a) $(2x + 1)^5$ (b) $(2x - 1)^5$.

9. Find the first 3 terms of: (a) $(x - 1)^{20}$ (b) $(x + \frac{1}{2})^8$ (c) $(-2x -1)^7$.

10. Expand $(1+1)^n$ to show that it equals $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

11. Use the binomial expansion to find $(1.01)^5$; also $(.99)^5$.

*12. Show that $(1+x)^n \geq 1 + nx$, for $x > 0$ and $n \in Z^+$.

*13. Use the combinatorial argument to prove $(a+b)^6 =$

$$\sum_{r=0}^{6} \binom{6}{r} a^{6-r} b^r.$$

5.9 Mathematical Induction

An interesting unsolved problem in mathematics is to find a function f with the property that whenever n is a natural number then f(n) is a prime number. The story is told of the student who presented the following proposition as a solution to the problem:

> If n is a natural number, then $n^2 - n + 41$ is a prime number.

The student had much cause to think he was correct. If n is replaced by 1, then $n^2 - n + 41 = 41$, which is prime. If n is replaced by 5, then $n^2 - n + 41 = 61$, which is also prime. In fact, if n is replaced by any natural number up to and including 40, a prime number is produced. Not until n is replaced by 41 does the expression $n^2 - n + 41$ produce a number that is not prime. This student had generalized his argument to all natural numbers based upon his successful experience with some of them.

We observe that we must be careful before stating such generalizations. A statement may be true for many natural numbers and yet not be true for all of them. How do we prove that a given formula or statement is true for all natural numbers?

Suppose you were asked to find the sum of all natural numbers, beginning with 1 and ending with 8; that is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8.$$

After all the challenging work that you have been exposed to in previous chapters, this problem probably appears trivial and routine. The sum is 36. You may suspect that instead of finding the sum by adding up the numbers, one by one, there is a much shorter way to arrive at the same answer. One way is as follows:

(1) find the average of the first and last numbers in your sum;

(2) multiply this average by the number of terms in your sum.

Thus, we could have found the average of 1 and 8

$$\frac{1 + 8}{2},$$

and then multiplied by 8:

$$8 \cdot \frac{(1 + 8)}{2} = \frac{8}{1} \cdot \frac{9}{2} = \frac{72}{2} = 36.$$

If you now repeat the problem, ending the sum with 11 instead of 8, you get

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 = 11 \cdot \frac{(1+11)}{2}$$
$$= \frac{132}{2} = 66.$$

Suppose we wish to establish a general formula for adding all the natural numbers, beginning with 1 and ending with n. The process described above indicates that

$$1 + 2 + 3 + 4 + \ldots + n = n \cdot \frac{(1+n)}{2}.$$

We now know that this formula is true when $n = 8$ and $n = 11$; we could even verify that it is true for all natural numbers up to and including 40, as in the first problem. However, our experience tells us that this amount of evidence is not conclusive--we still would not be sure that the formula is true for all natural numbers.

Visualize a string of upright dominoes, equally spaced and close enough together so that any falling domino would hit its neighbor. (Assume that if a domino is hit from one side it falls to the other side.) Now, if the first domino is pushed over towards the second, the second will fall and push over the third; the third will then push over the fourth which will then push over the fifth. This would continue until all the dominoes are down. The situation appears to fit the following pattern:

(1) The first domino falls down:

(2) Whenever a particular domino falls, the next one falls too.

Thus, all the dominoes fall down.

Let us see how the "domino effect" can help us to formulate a procedure for showing that

$$1 + 2 + 3 + \ldots + n = n \cdot \frac{(n+1)}{2}$$

is true for all natural numbers n. Suppose we consider the set of all natural numbers for which the above statement is true; call this set S.

$$S = \{x : x \in Z^+ \text{ and } 1 + 2 + 3 + \ldots + x = x \cdot \frac{(x+1)}{2}\}$$

We already know that $8 \in S$ and $11 \in S$. We would like to show that S contains every natural number; that is $S = Z^+$.

Just as the dominoes were equally spaced, the difference between consecutive natural numbers is always the same. Just as the first domino had to fall to start the process, the natural number 1 must be in S. But the dominoes had to be arranged so that whenever a particular domino fell, the next one would fall too. The analogous requirement in our problem is that whenever a particular natural number k is in S, then the next larger natural number k + 1 must also be in S. Just as we could visualize that all the dominoes would fall, it seems plausible to conclude here that all the natural numbers would be in S.

These thoughts are summarized and expressed in a postulate about the natural numbers called the <u>Principle of Mathematical Induction</u>. (We shall denote this by "PMI.") It is important to note that the foregoing discussion does not constitute a mathematical proof; it was designed simply to indicate the plausibility of postulating PMI.

<u>Axiom</u> PMI. Let T be a set of natural numbers having the following two properties:

(1) $1 \in T$.

(2) Whenever the natural number $k \in T$, then $(k + 1) \in T$.

Then, T is the set of natural numbers. $(T = Z^+)$

We are now in a position to show that set S defined in our original problem is the set of natural numbers. We already know that $1 \in S$. Let us now show that whenever the natural number $k \in S$, then $(k + 1) \in S$.

<u>Proof</u>. $S = \{x : x \in Z^+ \text{ and } 1 + 2 + 3 + \ldots + x = x \cdot \frac{(1 + x)}{2}\}$

Let $k \in S$. This means that

I. $1 + 2 + 3 + 4 + \ldots + k = k \cdot \frac{(1 + k)}{2}$

We would like to show that $(k + 1) \in S$, that is:

II. $1 + 2 + 3 + \ldots + k + (k + 1) = \frac{(k + 1)[1 + (k + 1)]}{2}$

We notice that the left sides of I and II differ only by the term $(k + 1)$. Let us add $(k + 1)$ to both sides of I and see what we obtain.

$$\underline{1 + 2 + 3 + 4 + \ldots + k} + (k + 1) = k\left(\frac{1 + k}{2}\right) + (k + 1)$$

$$= \frac{k(k + 1)}{2} + 2\left(\frac{k + 1}{2}\right)$$

$$= \frac{(k + 1)(k + 2)}{2}$$

$$= (k + 1)\left[\frac{1 + (k + 1)}{2}\right]$$

Thus, $(k + 1) \in S$.

By PMI, we may conclude that $S = Z^+$. In other words, the statement $1 + 2 + 3 + \ldots + n = n\left(\frac{1 + n}{2}\right)$ is true for all natural numbers.

Example 1. In Section 5.8, Exercise 12, you were asked to show that $(1 + x)^n \geq 1 + nx$, for $x > 0$ and $n \in Z^+$. At that time, you probably had to rely on the Binomial Theorem for your proof. Let us now apply PMI.

Let $T = \{x : x \in Z^+ \text{ and } (1 + x)^n \geq 1 + nx\}$

Since $(1 + x)^1 = 1 + 1x$, we see that $1 \in T$.

Assume $k \in T$. This means that

$$(1 + x)^k \geq 1 + kx.$$

But, $(1 + x)^{k+1} = (1 + x)^k \cdot (1 + x) \geq (1 + kx)(1 + x)$.

$(1 + kx)(1 + x) = 1 + kx + x + kx^2 > 1 + kx + x =$
$1 + (k + 1)x$

Thus, $(1 + x)^{k+1} \geq 1 + (k + 1)x$ which means that

$$(k + 1) \in T.$$

By PMI, $T = Z^+$.

In Section 5.4 we discussed the power set of a set S, denoted $\mathcal{P}(S)$. By means of the counting principle, it was shown that if $n(S) = p$, then $n(\mathcal{P}(S)) = 2^p$. Let us now analyze this same problem in a different way. First, we define a function f whose domain is Z^+ and whose codomain is a set of statements. If $x \in Z^+$, then $f(x)$ or simply f_x is the statement:

$$\text{If } n(S) = x, \text{ then } n(\mathcal{P}(S)) = 2^x.$$

For example, f_1 is the statement: If $n(S) = 1$, then $n(\mathcal{P}(S)) = 2^1$, and f_9 is the statement: If $n(S) = 9$, then $n(\mathcal{P}(S)) = 2^9$.

Recall from Course I that a function whose domain is the set of natural numbers is called a sequence. Thus f is a sequence. Since the codomain of f is a set of statements, we refer to the images $f_1, f_2, f_3, f_4, \ldots f_n, \ldots$ as a <u>sequence of statements</u>. We are interested in showing that every statement in the sequence of statements is true. Paralleling the discussion in the earlier part of this section, it seems reasonable to expect the Principle of Mathematical Induction to apply equally well to a sequence of statements. An equivalent[1] form of this principle, which we shall denote by "PMI'" is stated as follows:

<u>Axiom</u> PMI'. Let $F_1, F_2, F_3, \ldots, F_x, \ldots$ be a sequence of statements having the following two properties:

(1) F_1 is true

(2) Whenever F_k is true, then F_{k+1} is true.

Then, for each natural number x, F_x is true.

[1] Two statements, A and B, are said to be equivalent if A implies B and B implies A. In this context, it is possible to prove that PMI implies PMI' and PMI' implies PMI.

We now have the mathematical machinery available for showing that the sequence of statements f_1, f_2, f_3, ..., f_x, ... where f_x was defined as the statement

$$\text{If } n(S) = x, \text{ then } n(\mathcal{P}(S)) = 2^x,$$

is true for all natural numbers x.

<u>Proof</u>. f_1 is the statement:

$$\text{If } n(S) = 1, \text{ then } (\mathcal{P}(S)) = 2^1.$$

If S consists of a single element, say S = {a}, then the only subsets of S are {a} and $\emptyset$. Thus, $n(\mathcal{P}(S)) = 2 = 2^1$. We see that f_1 is true. Let us assume that f_k is true and show that f_{k+1} must then be true too. Assume that when n(S) = k, then $n(\mathcal{P}(S)) = 2^k$. If a new element, say b, is added to set S, the resulting set S' will have (k + 1) elements. We are interested in determining the total number of subsets of S'. Observe that each subset of S is also a subset of S'. Thus, we obtain the subsets of S', first by taking every subset of S. In addition, the element b may be adjoined to each of these subsets in succession to form new subsets of S'. It is clear that if two subsets of S are distinct, then the adjunction of b to each set produces two distinct subsets of S'. Thus, the number of subsets of S' is twice the number of subsets of S. Since S contains 2^k subsets, S' must contain $2 \cdot 2^k = 2^{k+1}$ subsets. Consequently, if $n(S) = k + 1$, then $n(\mathcal{P}(S)) = 2^{k+1}$. This means that f_{k+1} is true.

By PMI', we conclude that every statement in the sequence is true.

You have now seen two versions of the Principle of Mathematical Induction, PMI and PMI'. The one to use depends upon your interpretation of a problem. For example, the theorem just proved in the preceding discussion -- for every natural number x, if set S contains x elements, then its power set contains 2^x elements -- was interpreted as a sequence of statements f_1, f_2, f_3,..., one statement for each natural number x. Thus, the proof depended upon PMI'. An alternate interpretation and analysis of the same problem might have been as follows:

Let $T = \{x : x \in Z^+ \text{ and } n(\mathcal{P}(S)) = 2^x \text{ whenever } n(S) = x\}$.

By showing that

(a) $1 \in T$ and

(b) whenever $k \in T$, then $(k + 1) \in T$

we could have concluded, by PMI, that set T contains every natural number.

Example 2. Show that for all natural numbers n, $\frac{5^n - 2^n}{3}$ is a natural number.

Proof. Consider the sequence of statements f_1, f_2, f_3, ... f_n,... where f_n is the statement:

$$\frac{5^n - 2^n}{3} \text{ is a natural number.}$$

Since $\frac{5^1 - 2^1}{3} = \frac{3}{3} = 1$, we see that f_1 is true.

Assume f_k is true; that is

$\frac{5^k - 2^k}{3}$ is a natural number, say p.

$$\frac{5^k - 2^k}{3} = p.$$

Thus, $5^k = 3p + 2^k$. (Why?)

We want to show that f_{k+1} must also be true.

$$\text{Now, } \frac{5^{k+1} - 2^{k+1}}{3} = \frac{5^k \cdot 5 - 2^k \cdot 2}{3}$$

$$= \frac{(3p + 2^k)5 - 2^k \cdot 2}{3}$$

$$= \frac{15p + 2^k \cdot 5 - 2^k \cdot 2}{3}$$

$$= \frac{15p + 2^k(5 - 2)}{3}$$

$$= \frac{15p + 2^k \cdot 3}{3}$$

$$= 3\left(\frac{5p + 2^k}{3}\right)$$

$$= 5p + 2^k$$

$5p + 2^k$ is a natural number. (Recall that $(Z^+,+)$ and $(Z^+,\cdot)$ are operational systems.)

Thus f_{k+1} is true. By PMI', all of the statements in the sequence $f_1, f_2, f_3, \ldots$ are true. Thus, for every natural number n, $\frac{5^n - 2^n}{3}$ is a natural number.

The notation "..." within a mathematical statement about the natural numbers is often a signal that the Principle of Mathematical Induction may be used to prove the statement.

<u>Example</u> 3. $2 + 4 + 6 + \ldots + 2n = n(n + 1)$ is the assertion that for all natural numbers n, the sum of the even numbers, beginning with 2 and concluding with 2n, is equal to the product of n and the next larger natural number, n + 1. If n = 1, the sum begins with 2 and ends with 2; consequently, there is just one term to be considered. If n = 5, the sum becomes 2 + 4 + 6 + 8 + 10. A simple check of both the sum and the product for n = 5 gives an answer of 30. Let us apply PMI to prove that Example 3 is true for all natural numbers.

<u>Proof</u>. Let $T = \{x : x \in Z^+ \text{ and } 2 + 4 + 6 + \ldots + 2x = x(x + 1)\}$.
Since $2 \cdot 1 = 1(1 + 1)$, we see that $1 \in T$.
Suppose $k \in T$. This means that

$$2 + 4 + 6 + \ldots + 2k = k(k + 1).$$

Add $2(k + 1)$ to both sides. (Why?)

$$\begin{aligned} 2 + 4 + 6 + \ldots + 2k + 2(k + 1) &= k(k + 1) + 2(k + 1) \\ &= (k + 1)(k + 2) \end{aligned}$$

Thus, $(k + 1) \in T$. By PMI, $T = Z^+$.

Can you see why it is essential to show that $1 \in T$? Why is it not sufficient to show that

whenever $k \in T$, then $(k + 1) \in T$?

Consider the following assertion:

For all natural numbers x, $x! > 2^x$.

Let $T = \{x : x \in Z^+ \text{ and } x! > 2^x\}$ and suppose that $k \in T$. This means that $k! > 2^k$. Then,

$$(k + 1)! = (k + 1) \cdot k! > (k + 1)2^k \geq 2 \cdot 2^k = 2^{k+1}. \text{ (Why)?}$$

Thus, $(k + 1) \in T$. Could we conclude, at this point, that for every natural number x, $x! > 2^x$? Table 5.6 compares the value of $x!$ and 2^x for x = 1, 2, 3, 4, and 5.

x	$x!$	2^x
1	1	2
2	2	4
3	6	8
4	24	16
5	120	32

Table 5.6

It is not true that $x! > 2^x$ for every natural number x. Not until $x = 4$ do we get a true statement. What we were able to prove in the preceding discussion is that if $k! > 2^k$, for some natural number k, then a similar statement is true for the next larger natural number $(k + 1)$. However, we had not shown that $x! > 2^x$ is ever true for any particular natural number k. Do you now see why the Principle of Mathematical Induction includes the requirement that $1 \in T$?

Before we leave the statement $x! > 2^x$, let us review what has been established. We know that if $T = \{x : x \in Z^+ \text{ and } x! > 2^x\}$, then

(a) $1 \notin T$; $2 \notin T$; $3 \notin T$

(b) $4 \in T$

(c) whenever $k \in T$, then $(k + 1) \in T$.

How do we know that $5 \in T$? $6 \in T$? in fact, any natural number $x \geq 4$? This thinking leads to a modification of the Principle of Mathematical Induction, allowing us to apply it to a greater variety of situations involving the natural numbers.

Axiom General PMI. Let T be a set of natural numbers having the following two properties:

(1) the natural number $\underline{a} \in T$

(2) Whenever the natural number $k \in T$, then $(k + 1) \in T$.

Then, T consists of all natural numbers greater than or equal to a.

5.10 Exercises

1. Use PMI or PMI' to prove each of the following:

(a) $\sum_{i=1}^{n} 7a_i = 7 \sum_{i=1}^{n} a_i$ for every $n \in Z^+$.

(b) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$ for every $n \in Z^+$.

* (c) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^n} < n$ for every $n \in Z^+$.

(d) $\frac{6^n - 2^n}{4}$ is a natural number for every $n \in Z^+$.

(e) $\frac{n(n + 1)}{2}$ is a natural number for every $n \in Z^+$.

(f) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$ for every $n \in Z^+$.

(g) $1 + 2 + 4 + 8 + \ldots + 2^{n-1} = 2^n - 1$ for every $n \in Z^+$.

2. Prove Theorem 1 in Section 5.2, using mathematical induction.

3. How does the principle of mathematical induction differ from the usual meaning of the word "induction"?

4. Let $T = \{n : n \in Z^{+} \text{ and } n = n + 1\}$.
Assume $k \in T$ and show that $(k + 1) \in T$. Does this mean that $T = Z^{+}$? Defend your answer.

5. Consider a polygon of n sides, where $n \geq 3$. Prove that the total number of diagonals that can be drawn is $\frac{n(n - 3)}{2}$.
(A diagonal is a line segment that joins two non-consecutive vertices.)

6. Prove or disprove the following assertion:

For every natural number n, $2^n > 3n$.

If this statement is not true, modify it so that a true statement results.

7. Let a and r be real numbers, $r \neq 1$. Prove that for every $n \in Z^{+}$,

$$a + ar + ar^2 + \ldots + ar^n = a\left(\frac{1 - r^{n+1}}{1 - r}\right).$$

8. Each of the following statements is false. Attempt to prove each one, using some form of the principle of mathematical induction. In each case, tell where the principle fails.

(a) $x^2 - x = 0$ for every $x \in Z^{+}$.

(b) The statement

If $10 \mid n$, the $10 \mid (n + 10)$

is true for every $n \in Z^{+}$. Therefore, we may say that every natural number is divisible by 10.

(c) $3 + 5 + 7 + \ldots + (2n + 1) = n^2 + 2$ for every $n \in Z^+$.

(d) $100n \geq n^2$ for every $n \in Z^+$.

9. Given the statement

$$\text{For every } n \in Z^+,\ 2n \leq 2^n,$$

present two different proofs, using PMI in one case and PMI' in the other. If the expression "$2n \leq 2^n$" is replaced by "$2n < 2^n$", is the result still true? If not, what changes would you make so that a true statement emerges?

10. Prove that for every $n \in Z^+$,

$$1 + 2 + 3 + 4 + \ldots + n = n + (n - 1) + (n - 2) + (n - 3) + \ldots + 1$$

5.11 Summary

1. The counting principle was illustrated for two and three finite sets and stated as a theorem for any finite number of sets.
2. If a set A contains a elements and set B contains b elements ($a \leq b$), the number of different one-to-one mappings from A to B is called the number of permutations of b elements taken a at a time, (a and b are whole numbers) written $(b)_a$.
   If $a = b$, then the number of permutations is $b!$.
   If $a < b$, then the number of permutations is $b(b - 1)(b - 2)\ldots(b - a + 1)$.
3. $0!$ is defined to be 1.
4. $\binom{n}{r}$ represents the number of subsets with r elements which can be formed from a set of n elements, where n and r are whole numbers.

If $n < r$, then $\binom{n}{r} = 0$.

If $n = r$, then $\binom{n}{r} = 1$.

If $r = 0$, then for any n, $\binom{n}{r} = 1$. In general $\binom{n}{r} = \frac{(n)_r}{r!}$, for $n \geq r$.

5. (The Binomial Theorem). If a and b are real numbers and n is a whole number then

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n}b^n = \sum_{r=0}^{n}\binom{n}{r} a^{n-r}b^r.$$

6. Principle of Mathematical Induction

   (a) Let T be a set of natural numbers having the following two properties:

      (1) $1 \in T$.

      (2) Whenever the natural number $k \in T$, then $(k + 1) \in T$.

   Then, T contains all the natural numbers.

   (b) Let $F_1, F_2, F_3, \ldots, F_x, \ldots$ be a sequence of statements having the following two properties:

      (1) F_1 is true.

      (2) Whenever F_k is true, then F_{k+1} is true.

   Then, for each natural number x, F_x is true.

5.12 Review Exercises

1. How many six-letter "words" can be formed from the set $\{t,h,e,o,r,y\}$ if

   (a) letters may not be repeated?

   (b) letters may be repeated?

2. A man conducts a probability experiment in which he does the following three things: he draws a marble from a bag of five differently colored marbles and records its color; then he tosses a die, recording the number the die shows; then he tosses a coin, recording the result "head" or "tail". How many possible outcomes are there in this experiment?

*3. In Exercise 2, what is the probability he will get an even number and a head?

4. If the call letters of a radio station must begin with "W" and contains three other letters (repetitions allowed) how many such arrangements of letters are there?

5. What is the answer to Exercise 4 if the call letters may begin with either "W" or "K"?

6. A person wishes to select 2 books from a set of 6 books. How many possible selections are there?

7. There are 5 points in a plane, no three of them in a line. How many lines can be drawn, with each line passing through exactly 2 of the points?

8. How many ways are there to arrange 3 books on a shelf if you have 8 books to choose from?

9. How many possible committees of 3 are there in a class of 8 persons?

10. Draw a "tree" diagram showing all the 2-letters words(no repetition) which can be formed from the set {a,e,i,o,u}.

11. If, from a set of 7 mathematics books and 5 history books, you must choose 1 mathematics book and 1 history book, in how many ways can you make your choice?

12. How many fractions can be formed having a numerator greater than 0 and less than 10, and a denominator greater than 0 and less than 15?

13. How many 3-digit numbers are there? (There are 10 digits to choose from, but the first digit cannot be 0.)

14. Referring to Exercise 13:

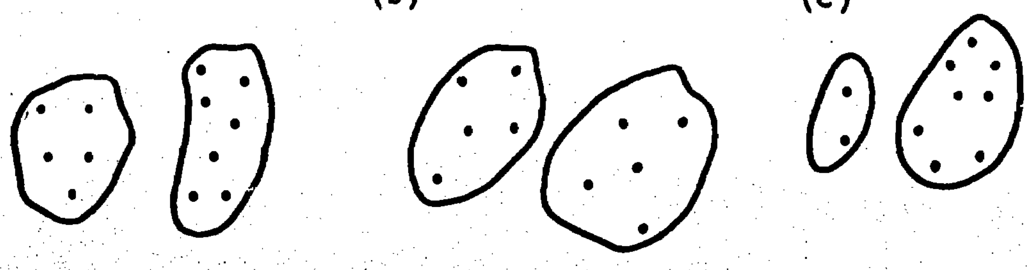
    (a) How many 3-digit numbers have no two digits alike?

    (b) How many 3-digit numbers have 3 digits alike?

    *(c) How many 3-digit numbers have exactly 2 digits alike?

15. For each of the following, tell how many one-to-one mappings are possible from set A to set B.

(a) (b) (c)



16. What is the number of permutations of 8 elements taken 2 at a time? of 10 elements taken 6 at a time?

17. A set has 10 elements.

    (a) How many of its subsets have exactly 3 elements?

    (b) How many of its subsets have exactly 7 elements?

    (c) How many of its subsets have exactly 10 elements?

    (d) How many of its subsets have exactly 1 element?

    (e) How many of its subsets have exactly 0 elements?

18. Barbara would like to take 5 books, 2 mathematics puzzle books and 3 novels with her on her vaction. Her library contains 5 puzzle books and 10 novels. In how many ways could she select her 5 books?

19. Find a standard name for each of the following:

(a) $\binom{9}{2}$ (b) $\binom{11}{8}$ (c) $\binom{7}{6}$ (d) $\binom{6}{7}$ (e) $\binom{16}{0}$

20. A student is instructed to answer any 8 of 10 questions on a test. How many different ways are there for him to choose the questions he answers?

21. A basketball squad consists of four centers, five forwards, and six guards. How many different teams may the coach form if players can be used only at their one position? (A basketball team consists of 1 center, 2 forwards and two guards.)

22. A sample of five light bulbs is to be taken from a set of 100 bulbs. How many different samples may be formed?

23. Complete the following: $\binom{8}{6} + \binom{8}{7} = \binom{\ }{\ }$

24. Expand $(a + b)^4$.

25. Expand $(a - b)^4$.

26. Write the first 6 terms in the expansion of $(a + b)^n$, where n is a positive integer greater than 6.

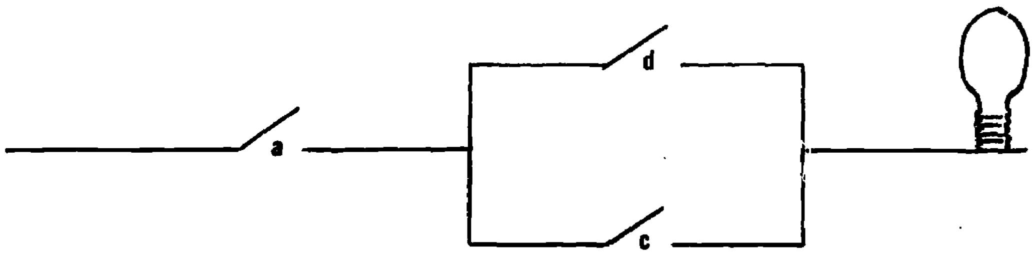
*27. Expand $(2u + v)^8$.

28. Prove by induction:

$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

*29. If $n(A) = x$ and $n(B) = y$ and there are exactly 720 1 to 1 mappings from set A into set B, find values for x and y.

30. If $\binom{n}{2} = 15$ and set S has n elements, how many subsets does S have?

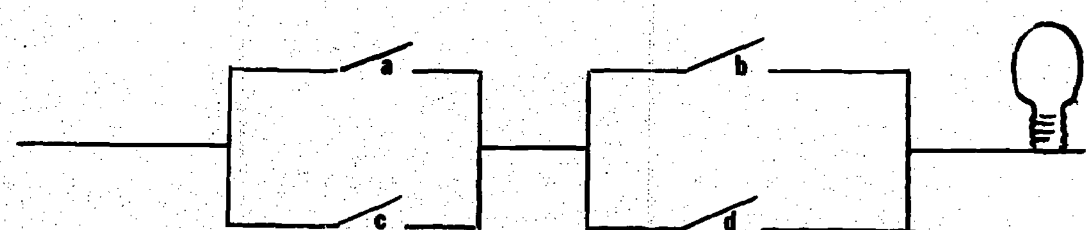


*31. The figure above represents a circuit diagram which consists of three switches a, d, c. If switch a and switch d are closed, the light will go on. If a, d, and c are closed, the light will also go on. However, if only d and c are closed, the light will not go on.

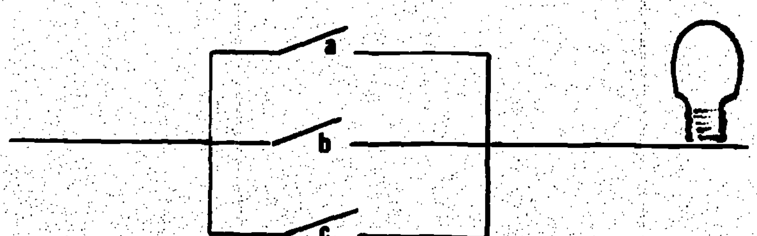
(a) In how many different ways can the switches be closed to turn on the light?

(b) In how many different ways can the switches be closed to turn on the light in the diagrams below?

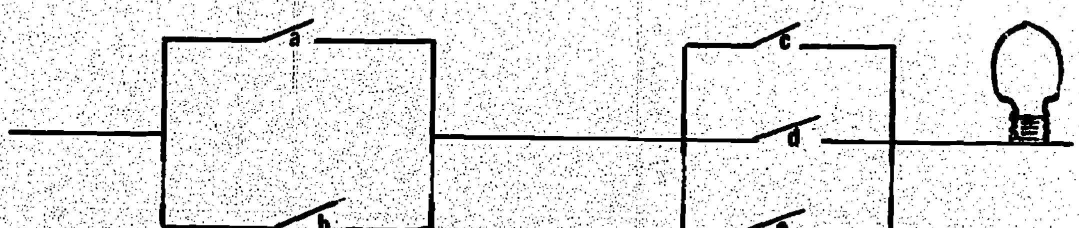
I.



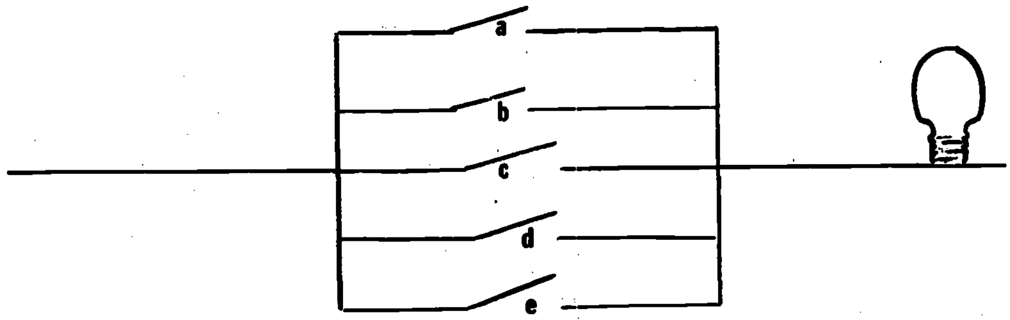
II.



III.



IV.



(c) Diagram IV has 5 switches in parallel connections. If a parallel circuit had 8 switches in it, how many ways could the light be turned on?

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