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ABSTRACT

The second part of Course III includes a study of probability, polynomial, rational and circular functions, and informal space geometry. The chapter on probability presents such topics as probability measure, outcome sets and events, and overview of topics studied in Courses I and II. Chapters on functions include polynomial algebra concepts and basic trigonometry. The space geometry chapter generalizes the notions of incidence, parallelism, perpendicularity, and coordinate systems to three dimensions. (FL)

Secondary School Mathematics
Curriculum Improvement Study

UNIFIED MODERN MATHEMATICS

COURSE III

PART II

TEACHERS COLLEGE COLUMBIA UNIVERSITY

Secondary School Mathematics

Curriculum Improvement Study

UNIFIED MODERN MATHEMATICS

COURSE III

PART II

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Chapter 6

PROBABILITY

6.1 Introduction

Probability theory is a field of mathematics that can be said to have been born in the year 1654 when the French mathematicians Blaise Pascal and Pierre de Fermat started to correspond on some problems related to games of chance. Since then many mathematicians have contributed to the development of probability theory; for instance, Christian Huygens, Jacob Bernoulli, Abraham de Moivre and Pierre Simon De Laplace. (Consult an encyclopedia to find out more about these men and their contributions to probability theory.)

It is interesting to note that probability theory did not get a well-organized mathematical foundation until 1933 when the Russian mathematician A. N. Kolmogorov published a famous book on probability in which he showed that probability theory could be based on set theory. Currently there is great interest in this field since probability is of fundamental importance in such areas as statistics, physical science, technology, social science, administration, predicting election results, life insurance, genetics, and in fact wherever analysis of data is used.

We have already had some contact with probability theory. In this chapter we shall build upon this, and extend this theory. In later courses we shall frequently return to continue our study of probability.

6.2 Outcome Set and Events

In Section 6.1 it was stated that Kolmogorov showed that probability may be based on set theory. We will use ideas and notation from set theory in describing situations where we can record and analyze the result of some action or observation. For example, we might observe a basketball player taking shots from the foul-line. This activity may be called an experiment with {basket, no basket} as the outcome set. Each shot is called a trial and each member of the outcome set is called an outcome. In this experiment basket and no basket are outcomes. Some texts use sample space and sample point instead of outcome set and outcome.

In this chapter we will consider only experiments with a finite number of outcomes, but subsequent study of probability requires the idea of infinite outcome sets. We will use the symbol S to represent an outcome set.

Definition 1. If $a_1, a_2, \ldots, a_n$ are outcomes of an experiment, then

$$S = \{a_1, a_2, \ldots, a_n\}$$

is called an outcome set of the experiment.

There may be more than one suitable outcome set for an experiment.

Following are some examples of experiments and suitable outcome sets that contain two outcomes. In Example 3 you must discover how many outcomes there are.

Example 1. Tossing a coin; S = {Heads, Tails}

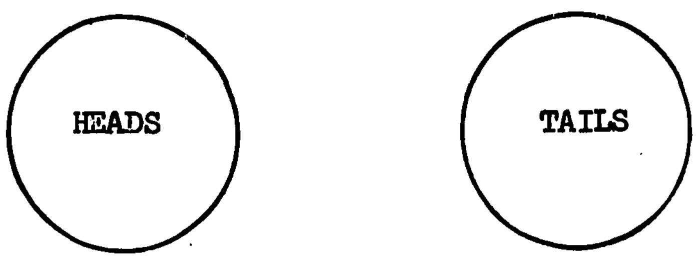


Figure 6.1

Example 2. Tossing a thumbtack; S = {Up, Down}.

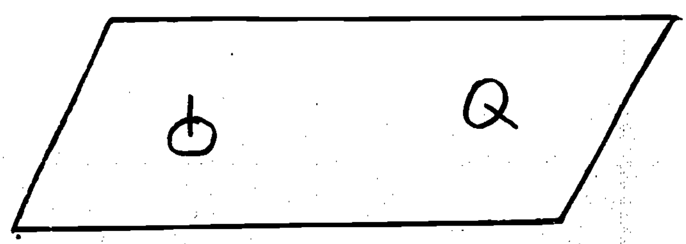
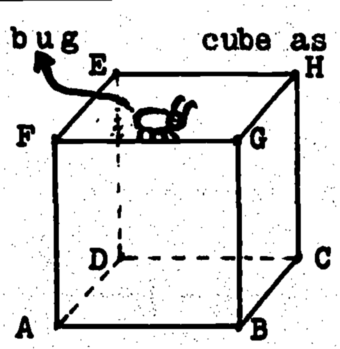


Figure 6.2

Example 3. A peripatetic bug takes walks on the edges of a cube as follows:



(a) He always starts at A and flies back to A after each walk.

(b) Each walk is exactly 3 edges long.

(c) He sometimes traverses the same edge 2 or 3 times in the same walk.

Figure 6.3

Questions. (1) List each possible trip by writing the vertices reached enroute; e.g. FAF, FEH, BGH, etc.

(2) Which vertices are possible destinations for a trip? (E. g. H is the destination of BCH.)

(3) Let each trip be a trial and each destination an outcome. At each vertex, except the last, there are three choices for continuing the trip, each of the 3 edges meeting there. Which outcomes do you think are most likely? Why?

Following are two examples of outcome sets with more than two outcomes and representations of these outcome sets as sets of points. In Example 6 you will perform an experiment.

Example 4. Tossing a die; $S = \{1,2,3,4,5,6\}$.

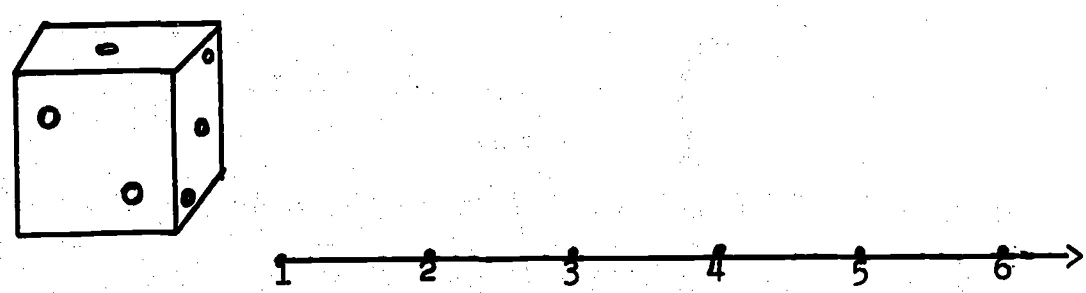


Figure 6.4

Example 5. Receiving a letter grade; $S = \{F, D, C, B, A\}$

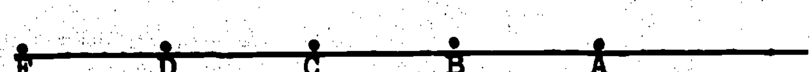


Figure 6.5

Example 6. Matching Cards.

(a) One or two players, two bridge decks, a pencil and paper are needed.

(b) Each player shuffles his deck, turns his deck back-up, turns the top card over and places it face-up on the table.

(c) If the cards match (see Figure 6.6) make a tally. Continue through the deck, card for card.

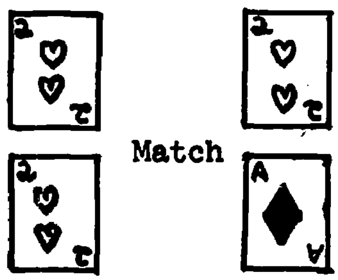


Figure 6.6

(d) After comparing the two decks, card for card, record the number of matches.

(e) Let each performance of steps b, c and d in sequence be a trial. Let the number of matches for each trial be an outcome. Repeat the trials until five have been performed.

Number of Matches	Tallies	Number of Tallies
0		
1		
2		
3		

Table 6.1

<u>Questions</u>. (1) Record your results for each experiment in a table like Table 6.1.

(2) Repeat the whole experiment using just the two sets of 13 hearts. Repeat with the first spades; also the first 3 clubs (i.e. A, 2, 3).

(3) Did the number of cards seem to influence the results?

The next two examples have outcome sets which are Cartesian products. The Cartesian product, A x B, of A and B is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$,

$$A \times B = \{(a, b): a \in A \text{ and } b \in B\}.$$

Example 7. Tossing a dime and a cent; $S = \{(H,H),(H,T),(T,H),(T,T)\}$, or equivalently $\{H,T\} \times \{H,T\}$

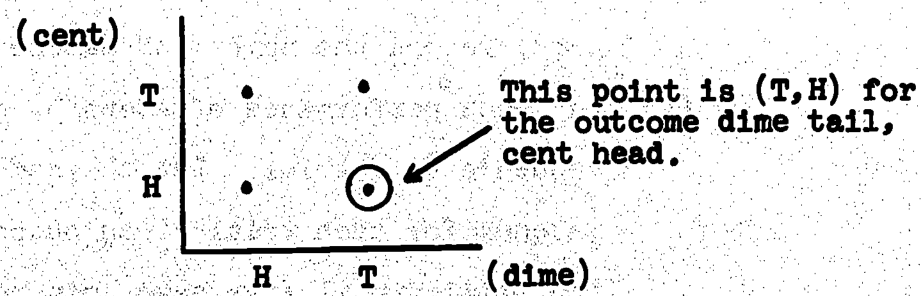


Figure 6.7

If you were interested in the number of heads, a suitable outcome set would be: {0 heads, 1 head, 2 heads}

Question. If each outcome in $\{(H,H), (H,T), (T,H), (T,T)\}$ were equally likely, which of the events 0 heads, 1 head, or 2 heads would be most likely?

Example 8. Tossing a pair of dice, one red and one green; $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$.

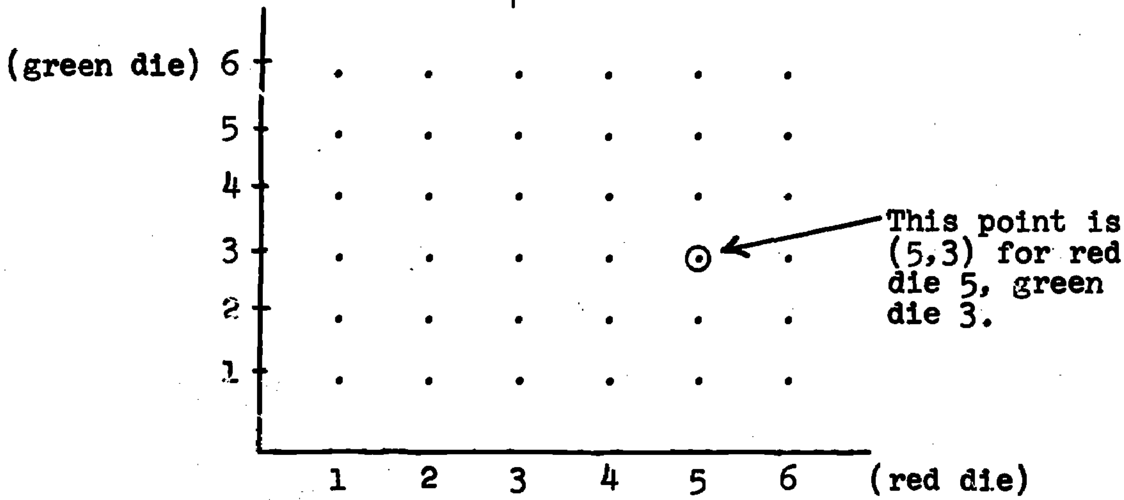
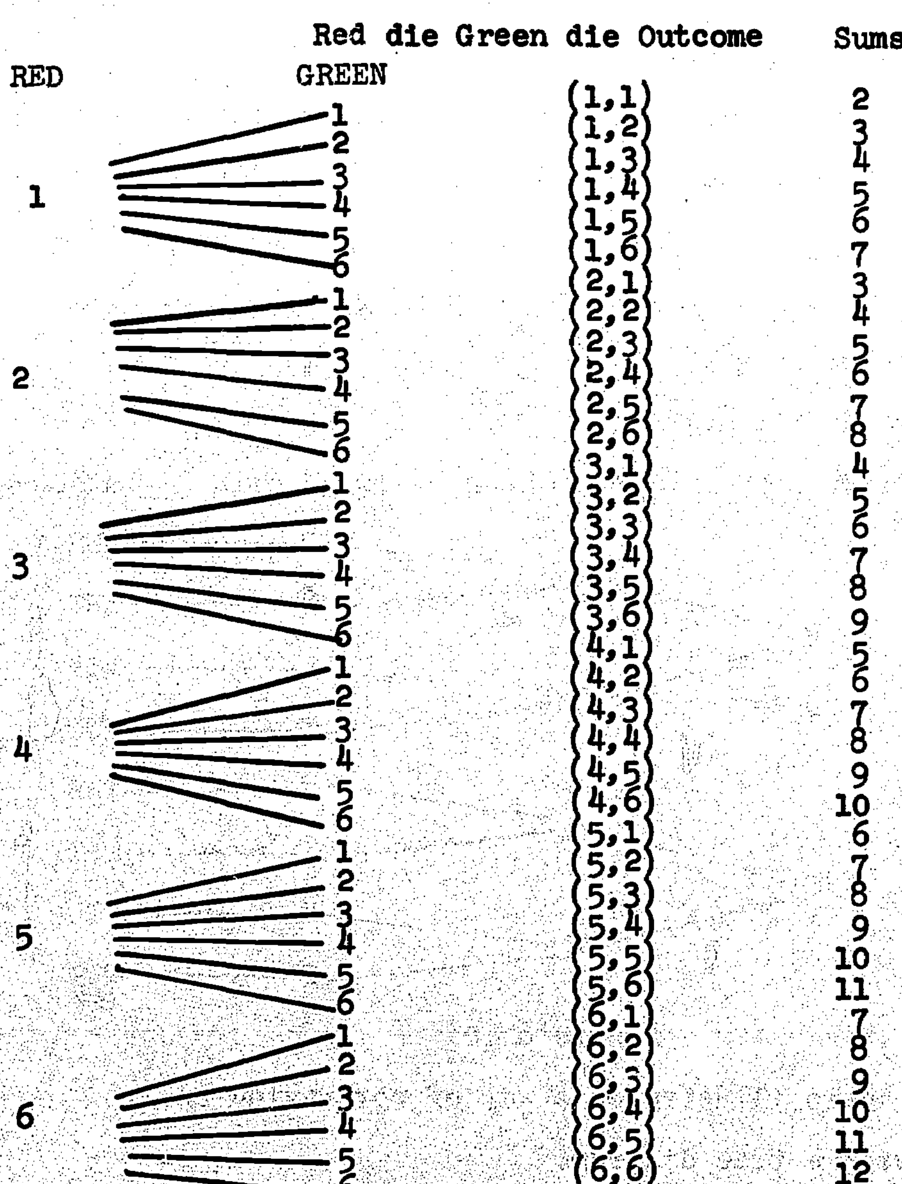


Figure 6.8



Tree diagram for die tossing experiment
Figure 6.9

Instead of the outcome set of Example 8, we could consider the outcome set to be the set of sums of the numbers of dots on the upper faces of the dice. E.G. (5,3) in Example 8 would correspond to the sum 8 in the new outcome set.

In this case a suitable outcome set would be:

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.\}$$

Notice, however, that the outcome set first used in Example 8 gives more detailed information.

<u>Question</u>. If each ordered pair in the first outcome set of Example 8 (i.e. $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$) is equally likely, which sum do you think will be most likely in the second outcome set (i.e. the set of sums)?

<u>Example</u> 9. Tossing 3 coins.

First coin   Second coin   Third coin   outcomes

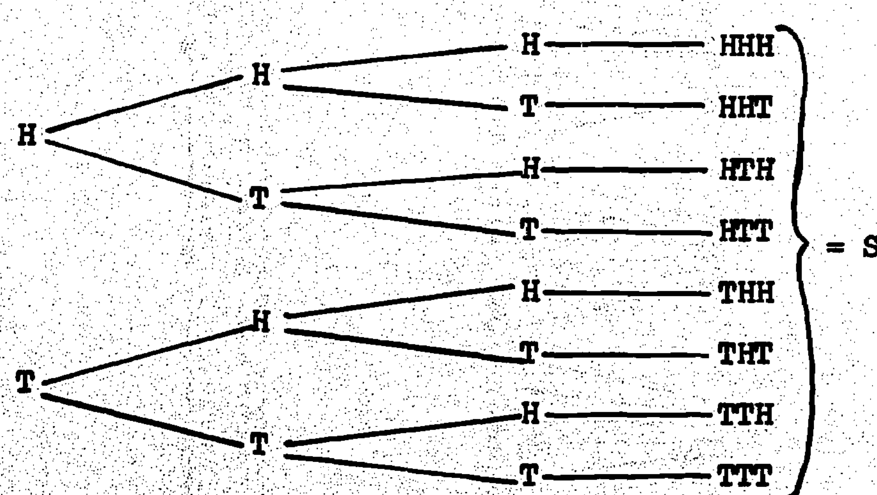


Figure 6.10

This example again illustrates a way of portraying an outcome set with a tree diagram.

It is also possible to graph the outcome set of Example 8 in 3 dimensions as shown in Figure 6.11.

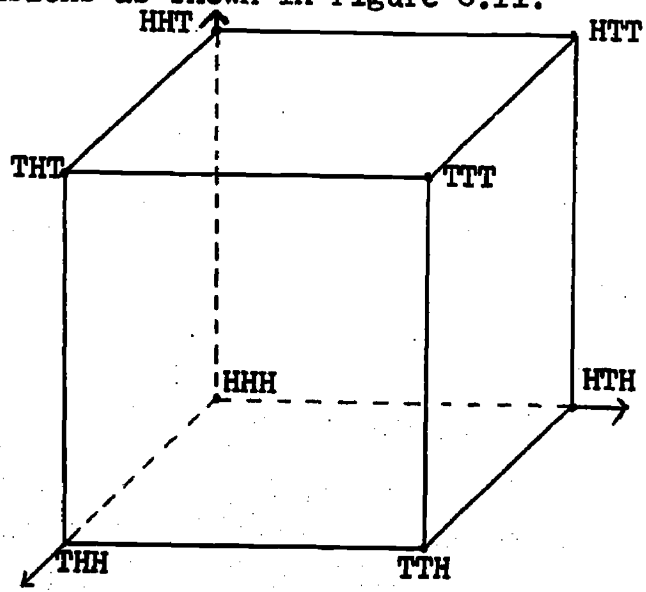


Figure 6.11

<u>Activity</u>. Join the points representing the outcomes in the event "exactly 2 heads" by line segements.

In Examples 1 to 9 you have seen descriptions of types of experiments and suitable outcome sets. Certain subsets of outcome sets are of interest in probability theory. In Example 8, one such subset is the set,

$$E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

<u>Question</u>. How could you describe the event E (above) with one simple sentence?

Each subset, A, of an outcome set, S, is called an event.

<u>Definition</u> 2. Let S be an outcome set. Then A is an <u>event</u> if and only if $A \subset S$, or equivalently, $A \in \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the power set (set of all subsets) of S.

<u>Question</u>. Is S a subset of S? Is ∅ a subset of S?
Are S and ∅ events?

The power set of S, where S = {H,T}, is:

$$\wp(S) = \{\emptyset, \{H\}, \{T\}, \{H,T\}\}$$

<u>Question</u>. List the events in the power set of S where
S = {0, 1, 2}.

(Hint: There should be 2^3 or 8 events in $\wp(S)$.)

<u>Definition</u> 3. A <u>singleton</u> is an event that contains exactly 1 outcome.

<u>Question</u>. How many singletons are there in $\wp(S)$ where
S = {H,T}?

The next two examples show how you can use the graph of an outcome set to graph an event. The graph of an event is a subset of the points in the graph of an outcome set that includes the event. The event can be shown by enclosing the subset of points as is done in Figure 6.12.

Example 10. Spinning a spinner twice; S = {1,2,3,4} x {1,2,3,4}

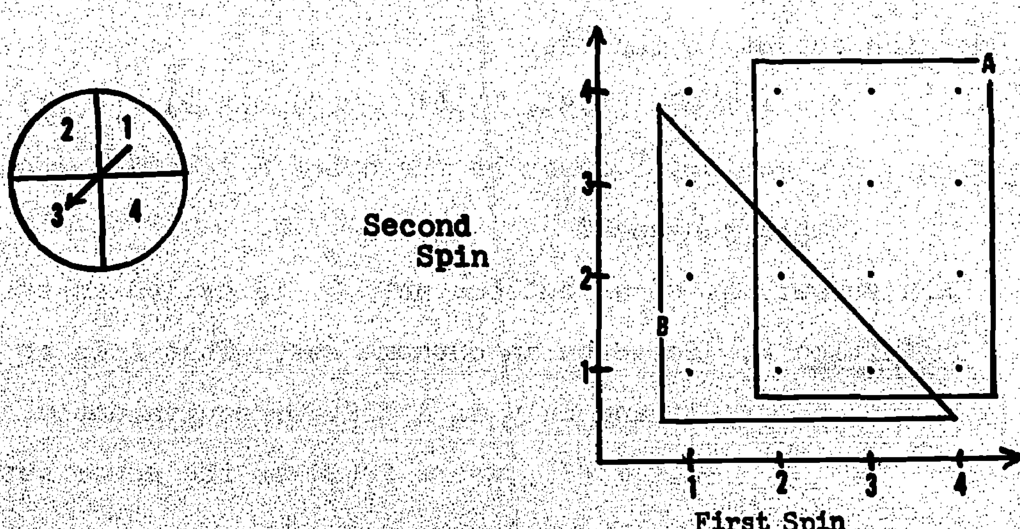


Figure 6.12

Subset A is the event that the outcome of the first spin is greater than 1. $A = \{(x,y): x > 1\} =$ $\{(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),$ $(4,3),(4,4)\}$ Subset B is the event that the sum of the outcomes on the first and second spin is less than 5.

$B = \{(x,y):x+y < 5\} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\}$

<u>Question</u>. Which points in Figure 6.12 are in both events A and B? (E.g. point (3,1) is in both A and B.)

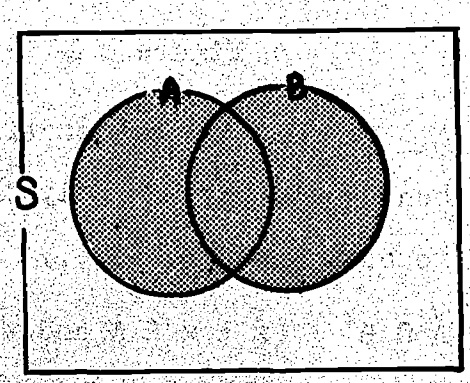
Which points are in either A or B (or both)?

You have previously encountered ideas about sets that are useful in probability. The remainder of this section will be devoted to relating these ideas to events.

<u>Definition</u> 4. <u>A ∪ B</u> (read "A union B") is the event that contains those and only those outcomes that belong to A or B (or both).

$$A \cup B = \{x: x \in A \text{ or } x \in B\}$$

A ∪ B is called the <u>union</u> <u>event</u> of A and B.



A ∪ B is shaded

Figure 6.13

In Example 10 the graph of A ∪ B is shown as a subset of the graph of the outcome set, S = {1,2,3,4} x {1,2,3,4}. The graph of A ∪ B includes 15 dots, 12 in event A and 6 in event B.
<u>Question</u>. What happened to the other 3 dots?

<u>Example</u> 11. Let S be an outcome set of the experiment of tossing two dice and observing the total number of dots obtained,

S = {2,3,4,5,6,7,8,9,10,11,12}

Let C and D be the events:

C = {2,3,4}; D = {4,5,6}.

Then C ∪ D = {2,3,4} ∪ {4,5,6} = {2,3,4,5,6}

which is the union event of C and D.

The graph of C ∪ D can be shown as in Figure 6.14.

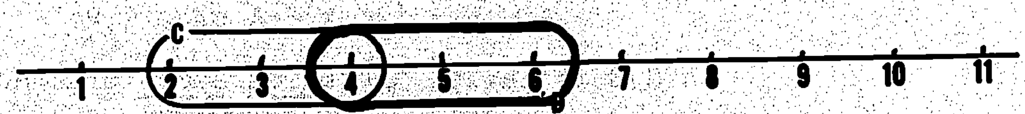


Figure 6.14

<u>Definition</u> 5. <u>A ∩ B</u> (read "A intersection B") is the event that contains those and only those outcomes that belong to both A and B; i.e.,

A ∩ B = {x : x ∈ A and x ∈ B}.

A ∩ B is called the <u>intersection</u> <u>event</u> of A and B.

A ∩ B occurs whenever A and B both occur.

Figure 6.15 illustrates the intersection event of A and B with Venn diagrams:

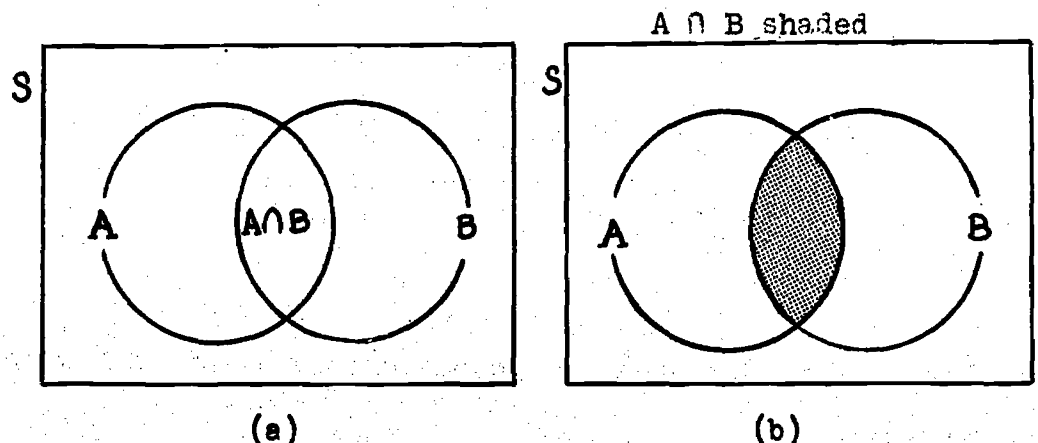


Figure 6.15

__Example__ 12. Toss three coins and observe the number of heads. $S = \{0, 1, 2, 3\}$. Let x be the number of heads. Let E and F be events defined by:

$$E = \{x: x \geq 2\},\ F = \{x: x \leq 2\},$$

$$\text{then } E \cap F = \{x: x = 2\}.$$

The graph in Figure 6.16 shows relations among events S, E, F and their intersections:

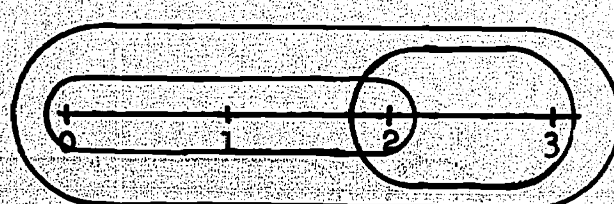


Figure 6.16

<u>Definition</u> 6. $\bar{A}$ (read "the complement of A") is the event that contains those and only those outcomes that are in S and not in A; i.e.,

$\bar{A} = \{x : x \in S \text{ and } x \notin A\}$.

$\bar{A}$ is called the <u>complementary</u> <u>event</u> of A.

$\bar{A}$ occurs whenever A does not occur.

Figure 6.17 illustrates the complementary event of A with a Venn diagram:

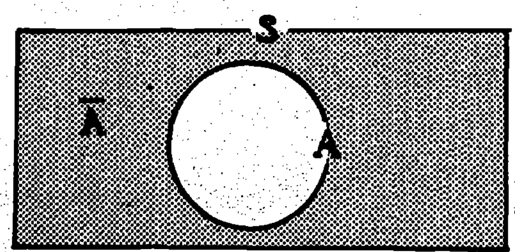


Figure 6.17

$\bar{A}$ is shaded.

In the example where the outcome set was a set of possible school grades, S = {F, D, C, B, A.} Let the event that you get a passing grade be G = {D, C, B, A}. Then the event that you fail is the complement of G. $\bar{G}$ = {F}. (See Figure 6.18.)

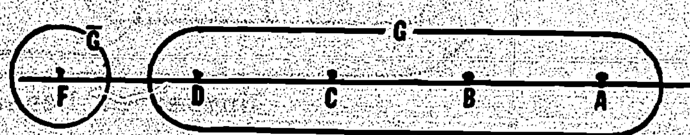


Figure 6.18

Definition 7. $A \setminus B$ (read "A minus B") is the set of all outcomes in S which are in A and not in B; i.e.,

$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

$A \setminus B$ is called the difference event of A and B.

$A \setminus B$ occurs whenever A occurs and B does not occur.

Figure 6.19 illustrates the difference event of A and B with a Venn diagram:

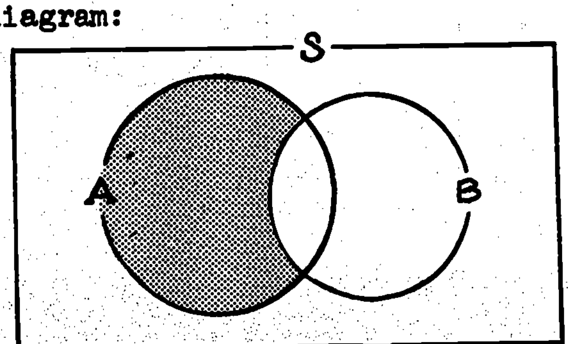


Figure 6.19

$A \setminus B$ is shaded.

In the outcome set of Example 11 which included the sums of the numbers of dots on the upper faces of the dice, let H and J be the following events:

$$H = \{2, 4, 6, 8\}; \quad J = \{6, 8, 10, 12\}.$$

The difference event is $H \setminus J = \{2, 4\}$.

Two events in the same outcome set may have no outcomes in common. In the outcome set of Example 11, the set of sums, let $C = \{2,4,6\}$ and let $D = \{3,5,7\}$. C and D have no members in common. In other words the intersection set of C and D is the empty set.

<u>Definition</u> 8. Two events C and D, for which $C \cap D = \emptyset$, are called <u>disjoint</u> <u>events</u>.

The diagrams in Figure 6.20 show two ways of portraying two disjoint events.

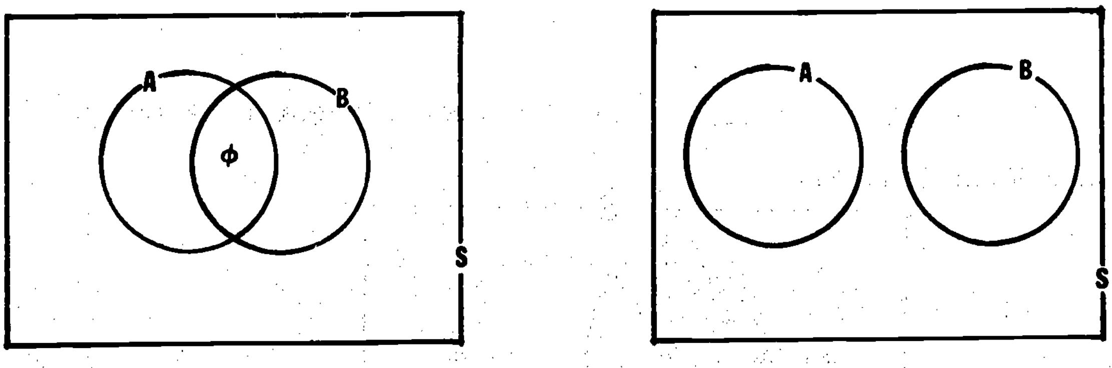


Figure 6.20

The notion of disjoint events can be extended to three or more events.

<u>Definition</u> 8a. Three events A, B and C are disjoint, if and only if $A \cap B = B \cap C = A \cap C = \emptyset$.

Figure 6.21 illustrates three events that are disjoint.

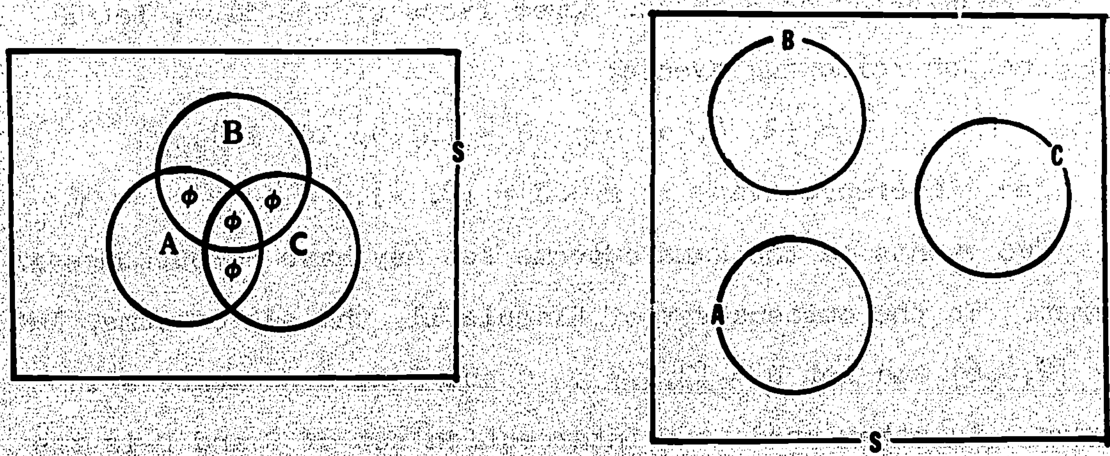


Figure 6.21

<u>Definition</u> 8b. In general, n events $A_1, A_2, \ldots, A_n$ are disjoint if and only if they are pairwise disjoint.

Figure 6.22 illustrates the event E, that exactly one of the events A, B and C occurs:

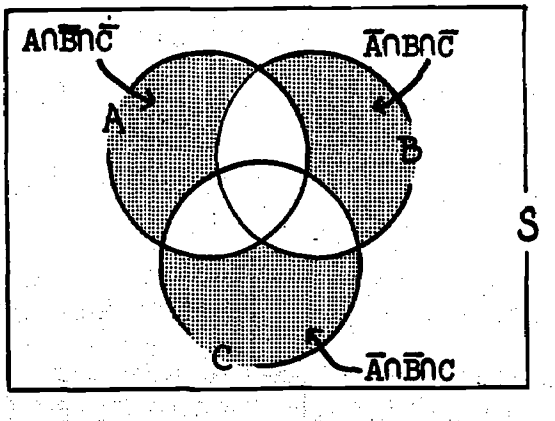


Figure 6.22

$E = (A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$

Another event, F, is illustrated and described in Figure 6.23.

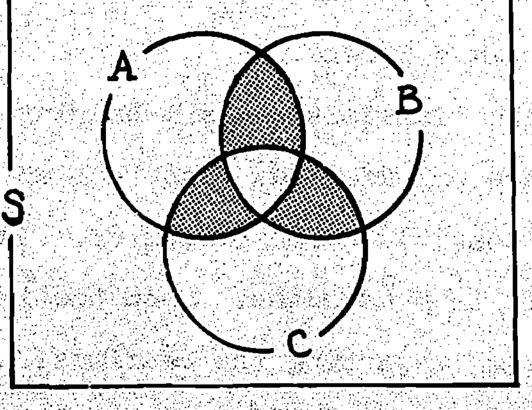


Figure 6.23

$F = (A \cap B \cap \bar{C}) \cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C)$

F is the event that exactly two of the events A, B and C occur.

<u>Questions</u>. (a) Event E in Figure 6.22 is expressed as the union of 3 events. Are these 3 events disjoint?

(b) Change event F in Figure 6.23 by interchanging the role of intersection and union. Draw a Venn diagram to illustrate this new event.

6.3 Exercises

1. Give roster names and a graph of suitable outcome sets for each of the following experiments and draw tree diagrams for each: (A roster name is a listing of outcomes enclosed by braces.)

   (a) Tossing a coin and a thumbtack.

   (b) Choosing a snack, where there is a choice of rye or whole wheat bread and a choice of honey, marmalade or caviar as a spread.

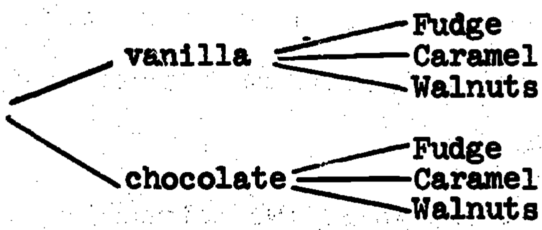
   (c) Testing the diameters of ball-bearings on an assembly line, where the diameter must be 1 cm. with a greatest possible error of .01 cm. Use a number line calibrated in hundredths. Show the intervals for accepting or rejecting the bearings.

   (d) Selecting a girl friend on the basis of hair color (red, brown or blonde), eye color (blue or brown) and height h in feet with 3 possibilities, $h < 5$, $5 \leq h \leq 6$, and $h > 6$.

2. Describe a practical experiment for which the following are suitable outcome sets.

   (a) S = {(urn I, red bead), (urn I, blue bead), (urn I, white bead), (urn II, black bead), (urn II, yellow bead)}.

   (b)



   (c) $S = \{w < 100, 100 \leq w \leq 120, w > 120\}$

   (d) S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

   (e) S = {ae, ai, ao, au, ea, ei, eo, eu, ia, ie, io, iu, oa, oe, oi, ou, ua, ue, ui, uo}

3. From a bridge deck select the cards of each suit showing numbers 2, 3 or 4. Draw one card from the 12 selected cards. A suitable outcome set is:

   S = {H2, H3, H4, D2, D3, D4, S2, S3, S4, C2, C3, C4}

   H, D, S, and C are obvious abbreviations for the 4 suits. Hearts and diamonds are the red suits. Spades and clubs are the black suits.

   Event A is described by, "The card is from a red suit."

   Event B is described by, "The number on the card is 3."

   Give the roster names of events in parts (a) to (f) and follow the instructions in (g) and (h):

   (a) Event A (b) Event B (c) The union event of A and B.

   (d) The intersection event of A and B.

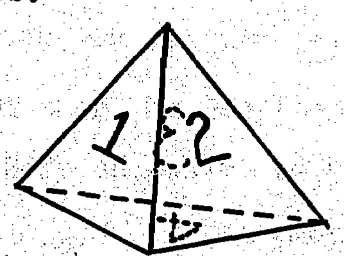
(e) The complementary event of A.

(f) The difference event of A and B.

(g) Which three of the above 6 events A, B, $A \cup B$, $A \cap B$, $\bar{A}$ and $A \setminus B$ are disjoint?

(h) Represent the outcome set and events A and B as sets of points in a plane with the suits assigned points on a horizontal axis and the numbers assigned points on a vertical axis.

4. Consider the experiment where two tetrahedra, one blue and one yellow, are tossed. Each tetrahedron (a three dimensional figure consisting of four triangular faces) has the numerals 1, 2, 3, 4 on its faces.



The outcome of each toss is an ordered pair of numbers, the first of which is the number on the down-face of the blue tetrahedron and the second by that on the down face of the yellow tetrahedron.

(a) Give a roster name of the outcome set, i.e. list the outcomes within braces.

(b) Give a roster name of the event C: "The sum of the numbers on the down-faces is 6."

(c) Give a roster name of the event D: "The number on the down-face on the blue tetrahedron is 4."

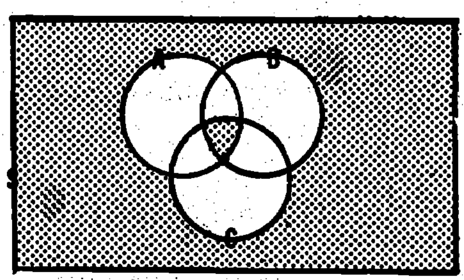
(d) Draw a graph of the outcome set.

(e) On the graph in part (d), encircle event C of part (b).

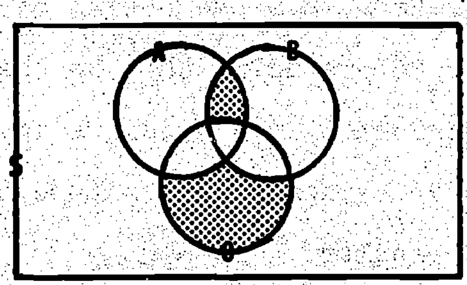
(f) On the same graph encircle event D of part (c).

5. Use set notation involving union, intersection and complement to describe the events shaded in the following Venn diagrams.

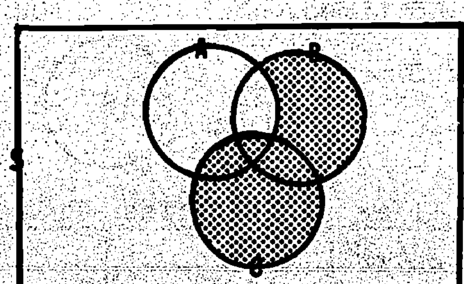
(a)



(b)



(c)



6. Draw Venn diagrams to illustrate the events:

(a) $(A \cup B) \cap C$ (b) $\bar{A} \cap \bar{B} \cap \bar{C}$

(c) $\overline{A \cap B}$ (d) $\bar{A} \cup \bar{B}$

(e) $\overline{A \cup B}$ (f) $\bar{A} \cap \bar{B}$

(g) $(A \cap C) \cup (B \cap C)$

7. Compare the Venn diagrams for parts (c) and (d) of Exercise 6. What do you notice about the two diagrams? Can you formulate an explanation regarding the relationship between these two events?
8. Repeat Exercise 7 for parts (e) and (f) of Exercise 6. Repeat Exercise 7 for parts (a) and (g) of Exercise 6.
9. Let the outcome set S be given by S = {a, b, c}. Suppose that a trial results in the outcome a. Under these conditions, which events of $\mathcal{P}(S)$ have occurred?

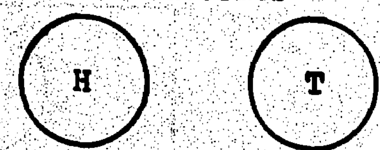
6.4 Probability Measure

In this section we shall formulate the notion of probability more precisely. Before doing this we should recall the experiments we performed in a previous course. The following example gives some idea of the nature of these experiments and the related results.

Example 1. Toss a coin 50 times.

Outcome set: S = {H, T}

Outcomes



Outcomes	Tallies	Frequencies	Relative frequencies
H	卌 卌 卌 卌 卌 卌 ||	32	32/50
T	卌 卌 卌 |||	18	18/50
		50	50/50 = 1

Figure 6.24

All of the experiments we performed seem to have certain things in common.

Use the table in Figure 6.24 to verify the following statements:

1. Each of the relative frequencies is a real number between 0 and 1 inclusive.
2. The sum of the relative frequencies of the outcomes in S is 1.
3. The relative frequency of the union of 2 disjoint events is the sum of the relative frequencies of the two events. In Example 1 you can use $\{H\} \cup \{T\} = \{H, T\}$.

In these experiments, there is nothing hypothetical about the relative frequencies, one only has to count and compose a fraction. However the word probability is used as a prediction of relative frequency. The experiments sometimes influence the prediction and sometimes the symmetry of the experimental objects influence the prediction.

Question: Why do relative frequencies have the 3 properties mentioned on the previous page? I.e., give an argument based on arithmetic.

With the properties of relative frequencies in mind the following definition should not seem quite as abstract:

<u>Definition</u> 9. Let S be a finite outcome set and $\mathcal{P}(S)$ the power set of S. A <u>probability</u> <u>measure</u> P

is a function with domain $\mathcal{P}(S)$, and codomain R, with the following properties:

(1) $0 \leq P(A) \leq 1$ for every $A \in \mathcal{P}(S)$.

(2) $P(S) = 1$.

(3) If A and B are disjoint (i.e., $A \cap B = \emptyset$), then:

$P(A \cup B) = P(A) + P(B)$.

We call P(A) the probability of A, and the ordered pair (S,P), consisting of the outcome set S and the probability measure P, a probability space.

Thus a probability measure P is a function that assigns a real number P(A) to every event A in such a way that properties (1), (2), and (3) are satisfied. Before we develop some logical consequences of the definition, let us consider some examples.

The first example uses a spinner experiment to motivate a function t which assigns real numbers to the singleton events in $\mathcal{P}(S)$. The strategy in designing the function t is to make its assignments to singleton events in such a way that t will play a role in determining suitable assignments to the events in $\mathcal{P}(S)$ for a probability measure P. There are two simple requirements for t:

1. $t(\{a_i\}) \geq 0$ for $i = 1, 2, 3$.

2. $\sum_{i=1}^{3} [t(\{a_i\})] = 1$.

Example 2. Spinner experiment. $S = \{a_1, a_2, a_3\}$

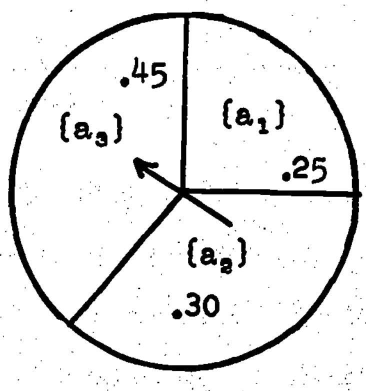


Figure 6.25

{a_1}, {a_2}, and {a_3} are called <u>singleton events</u>, or sometimes just <u>singletons</u>, since they each contain just one outcome. {a_1} is the event that the arrow stops on sector a_1 of the spinner, and similarly for {a_2} and {a_3}. (See Figure 6.25.)

The power set of S or $\mathcal{P}(S)$ is:

$$\mathcal{P}(S) = \{\emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1,a_2\}, \{a_1,a_3\}, \{a_2,a_3\}, \{a_1,a_2,a_3\}\}.$$

Let t be the real valued function defined on the singleton events of $\mathcal{P}(S)$ by Table 6.2.

{a_i}	{a_1}	{a_2}	{a_3}
t({a_i})	.25	.30	.45

Table 6.2

Question: Use the table for function t to see if t satisfies requirements 1 and 2 listed above the example.

The next example develops a specific case of a probability measure in connection with the spinner experiment.

<u>Example</u> 3. Let S be the outcome set of Example 2, $S = \{a_1, a_2, a_3\}$. Let P be a real valued function defined on the events A of $\mathcal{P}(S)$ by Table 6.3.

A	$\emptyset$	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$	$\{a_1, a_2\}$	$\{a_1, a_3\}$	$\{a_2, a_3\}$	$\{a_1, a_2, a_3\}$
P(A)	0	.25	.30	.45	.55	.70	.75	1.00

Question: Check to see that the assignments made by P in Example 3 satisfy the three properties of a probability measure as defined in Definition 9. Just select a few case of property 3 to verify. Do you see how the function P is built up from the function t?

We make some informal summarizing statements about the assignments made by the function P in Example 3:

1. $P(\emptyset) = 0$. This will be proved later as a consequence of the definiton of a probability measure.
2. The assignments made to the singletons by P were the same as those made by t.
3. All of the events in $\mathcal{P}(S)$ containing more than 1 outcome can be formed by the union of 2 or more singletons. Every pair of distinct singletons are disjoint. Therefore, property 3 and the assignments to singletons can be used to find an appropriate assignment for the other events in $\mathcal{P}(S)$ except $\emptyset$.

Example 4. This example shows the derivations of specific probabilities for certain events in $\mathcal{P}(S)$ of

Example 3 by using unions of singletons in $\mathcal{P}(S)$, property 3 and an extension of property 3 to more than 2 events. Notice that we omit the braces in the probability statements to simplify the notation:

1. $\{a_1, a_2\} = \{a_1\} \cup \{a_2\}$

   $P(a_1, a_2) = P(a_1) + P(a_2) = .25 + .30 = .55$

2. $\{a_1, a_3\} = \{a_1\} \cup \{a_3\}$

   $P(a_1, a_3\} = P(a_1) + P(a_3) = .25 + .45 = .70$

3. $\{a_2, a_3\} = \{a_2\} \cup \{a_3\}$

   $P(a_2, a_3) = P(a_2) + P(a_3) = .30 + .45 = .75$

4. $\{a_1, a_2, a_3\} = \{a_1\} \cup \{a_2\} \cup \{a_3\}$

   $P(a_1, a_2, a_3) = P(a_1) + P(a_2) + P(a_3)$

   $= .25 + .30 + .45 = 1.00$

From the 4 parts of example 4 one might suspect that the procedure displayed can be generalized to include events which contain any number of outcomes. This, in fact, is the case:

<u>Theorem</u> 1. Let (S, P) be a probability space, with a finite outcome set S. Then for every $A \in \mathcal{P}(S)$ such that A is the union of 2 or more singleton events, $P(A)$ is the sum of the probabilities of those singletons.

More briefly: Let (S, P) be a probability space and S a finite outcome set. For every $A \in \mathcal{P}(S)$ such that $n(A) \geq 2$, $P(A) = \sum_{a_i \in A} P(a_i)$

(n(A) denotes the number of elements in set A.) The proof of Theorem 1 requires mathematical induction.

We shall call the probabilities $P(a_i)$ for $a_i \in S$, <u>elementary probabilities</u>.

Theorem 1 tells us that the probability measure P is determined by the elementary probabilities. In fact, this is a common way to give a probability measure for finite outcome sets. In Examples 2, 3, and 4 we saw how a probability measure is given in this way.

<u>Example</u> 5. The experiment of counting the number of customers entering a certain post office during one minute has the outcomes and elementary probabilities shown in Table 6.4. (Let 1c represent one customer, 2c two customers, etc.)

Outcomes	0c	1c	2c	3c	more than 3c
Singletons	{0c}	{1c}	{2c}	{3c}	{more than 3c}
Elementary Probabilities	.05	.15	.22	.22	.36

Table 6.4

Let A be the event that at most 2 customers arrive during one minute. Then:

$$A = \{0c, 1c, 2c\}$$

From Theorem 1 we obtain:

$$P(A) = P(0c) + P(1c) + P(2c)$$
$$= .05 + .15 + .22$$
$$= .42$$

In a similar way we can get the probability that at least 3 customers arrive during one minute:

$.22 + .36 = .58$

Example 6. The experiment of tossing a certain die has the elementary probabilities displayed in Table 6.5.

Outcomes	1	2	3	4	5	6
Singletons	{1}	{2}	{3}	{4}	{5}	{6}
Elementary Probabilities	.14	.17	.16	.18	.19	.16

Table 6.5

The probability for more than 2 but less than 6 is given by:

$.16 + .18 + .19 = .53$

Example 7. Suppose that all singletons in the die tossing experiment have the same elementary probability (See Table 6.6):

Outcomes	1	2	3	4	5	6
Singletons	{1}	{2}	{3}	{4}	{5}	{6}
Elementary Probabilities	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Table 6.6

In this problem, the probability that the number of dots is more than 2 but less than 6 is:

$$P(3) + P(4) + P(5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

In cases where all the elementary probabilities are equal, we say that the probability measure is uniform. Section 6.6 will be devoted to uniform probability measures.

In the following theorem we have collected some additional consequences of our definition of a probability measure.

Theorem 2. Let (S, P) be a probability space, and let $A, B \in \mathcal{P}(S)$. Then:

(a) $P(A) + P(\overline{A}) = 1$

(b) $P(\emptyset) = 0$

(c) $P(A \setminus B) = P(A) - P(A \cap B)$

(d) If $B \subset A$,
$P(B) \leq P(A)$.

Proof. (a) The events A and $\overline{A}$ are disjoint and $A \cup \overline{A} = S$.
Then using the properties of P:
$1 = P(S) = P(A \cup \overline{A}) = P(A) + P(\overline{A})$.

(b) Since $\overline{S} = \emptyset$, it follows from (a) by replacing A by S that $P(S) + P(\emptyset) = 1$. But from $P(S) = 1$, it now follows that $P(\emptyset) = 0$.

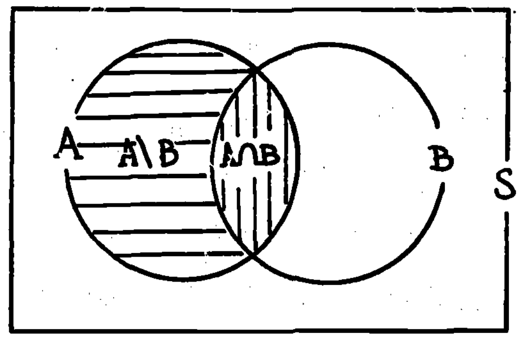


Figure 6.25

(c) From Figure 6.25, we can see that

$A = (A \setminus B) \cup (A \cap B)$,

$(A \setminus B) \cap (A \cap B) = \emptyset$.

Then from property 3 of a probability measure,

$$P(A) = P(A \setminus B) + P(A \cap B), \text{ or}$$

$$P(A \setminus B) = P(A) - P(A \cap B).$$

(d) If $B \subset A$, then $A \cap B = B$. Thus

$P(A) - P(A \cap B) = P(A) - P(B) = P(A \setminus B)$,

by part (c) of this theorem. But

$P(A \setminus B) \geq 0$. Thus $P(A) - P(B) \geq 0$ or

$P(B) \leq P(A)$.

Some parts of Theorem 2 will be used more often than others. It is sometimes easier to compute $P(\bar{A})$ than $P(A)$. But since $P(A) = 1 - P(\bar{A})$, from part (a) of Theorem 2, we can obtain $P(A)$ easily once we know $P(\bar{A})$.

Example 8. In a certain game Allan tosses darts at a dart-board. The dart-board contains rectangular regions as shown in Figure 6.27. The probability that he will hit region A is .4, and the probability that he will hit region B is .3. The probability that he will hit both regions is .1. Figure 6.27 illustrates the various regions of the dartboard as indicated by $A\setminus B$, $B\setminus A$, $A \cap B$ and $\overline{A \cup B}$.

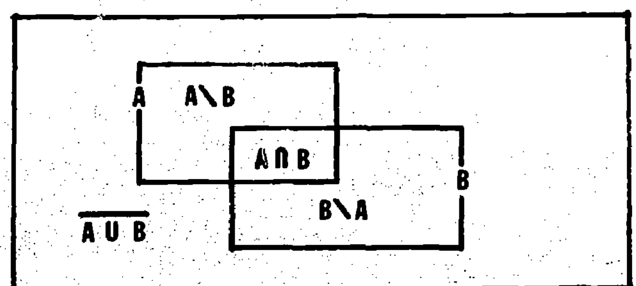


Figure 6.27

We can obtain many probabilities related to the dart game by using Figure 6.27 and our formulas. Thus:

$$P(A \setminus B) = P(A) - P(A \cap B) = .4 - .1 = .3$$

Therefore .3 is the probability that Allan hits A but not B.

The probability that he hits B but not A is obtained in a similar manner:

$$P(B \setminus A) = P(B) - P(A \cap B) = .3 - .1 = .2$$

The probability that he does not hit A is:

$$P(\overline{A}) = 1 - P(A) = 1 - .4 = .6$$

Figure 6.28 summarizes, with a Venn diagram, some probabilities derived by the formulas of Theorem 2.

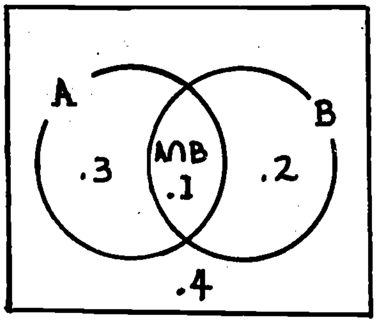


Figure 6.28

Check to see that the results obtained from Theorem 2 agree completely with our Venn diagram.

Property 3 of a probability measure P has to do with the probability $P(A \cup B)$ where A and B are disjoint. In the following theorem we consider the probability of $A \cup B$ where A and B need not be disjoint.

<u>Theorem</u> 3. Let (S, P) be a probability space. For all events $A, B \in \mathcal{P}(S)$ we have:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(See Figure 6.29.)

<u>Proof</u>.

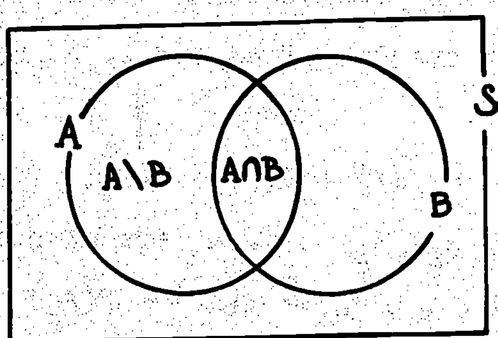


Figure 6.29

We observe that $A \cup B = (A\setminus B) \cup B$, where $(A\setminus B)$ and B are disjoint. Hence, from property 3 of a probability measure we have:

$$(1) \quad P(A \cup B) = P(A\setminus B) + P(B)$$

From Theorem 2(c) we have:

$$(2) \quad P(A\setminus B) = P(A) - P(A \cap B)$$

Substituting the right side of equation (2) for $P(A\setminus B)$ in equation (1):

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

<u>Question</u>: How can one prove Theorem 3 with the aid of Theorem 1?

<u>Example</u> 9. Using the dart board example again, let us find the probability that Allan hits region A or region B (that is, at least one of A and B). From Theorem 3 we get:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= .4 + .3 - .1 = .6$$

This agrees with our previous result.

Property 3 of a probability measure can be extended to more than two disjoint events. Thus, if A, B and C are disjoint, then:

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$. Even more generally, if the events $A_1, A_2, A_3, \ldots, A_n$ are disjoint, then:

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

If we abbreviate $A_1 \cup A_2 \cup \ldots \cup A_n$ by writing $\bigcup_{i=1}^{n} A_i$ and use the summation symbol Σ, we can write:

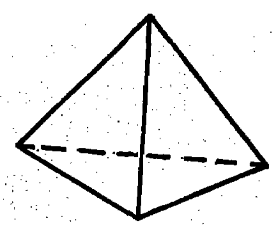
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

In Section 6.5 you will derive several other interesting properties of a probability measure.

6.5 Exercises

1. The faces of a tetrahedron are painted red, yellow, green, and blue. For the experiment of tossing this tetrahedron and observing the color of the bottom face, we can use the following outcomes and elementary probabilities:

Outcomes	Red	Yellow	Green	Blue
Singletons	{Red}	{Yellow}	{Green}	{Blue}
Elementary probabilities	.2	.3	.2	.3



What is the probability for each of the following events:

(a) {blue}

(b) {yellow or green}

(c) {not red}

(d) {The color is red, green, or blue}

(e) {pink}

(f) {not pink}

2. In the experiment of counting the number of yellow cars that pass a certain street corner during a two minute interval, the following outcomes and probabilities are given:

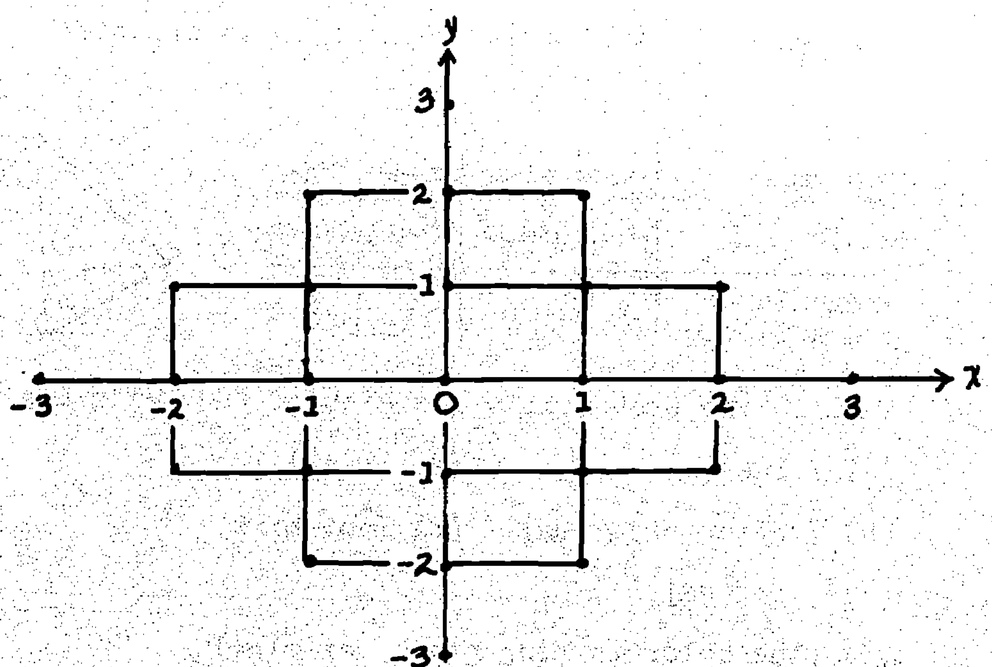
Outcomes	0 cars	1 car	2 cars	3 cars	4 cars	more than 4 cars
Elementary probabilities	.13	.27	.27	.18	.09	.06

Find the probability that the number of yellow cars passing during a two minute interval for this street corner is:

(a) at most 1. (b) more than 2. (c) between 1 and 4, exclusive.

3. For the experiment of tossing three symmetric coins and counting the number of heads, we can use the elementary probabilities 1/8, 3/8, 3/8, and 1/8 for the events 0, 1, 2, and 3 heads respectively. Find the probability that there is at least one head.

4.



The figure above shows a maze for experimenting with rats.

We assign coordinates to the intersections. The rat starts at the origin and may jump in any direction where there is a path shown in the diagram. Each jump is 1 unit in length. Starting at the origin the rat takes exactly 3 jumps reaching 1 of the other 24 points.

(a) Use the counting principle and the fact that the rat chooses directions so that each of the four directions possible are equally likely to find the total number of 3-jump trips.

(b) Find the 9 trips that terminate at (1,0), e.g., $(0,0) \longrightarrow (0,1) \longrightarrow (1,1) \longrightarrow (1,0)$.

(c) What is the probability of the rat terminating a trip at (1,0).

(d) Find the probability of the rat terminating a trip at each of the points where this is possible, e.g., $P(0,3) = \frac{1}{64}$.

(Hint for (d): use symmetry, e.g., the trips to (1,0) are symmetric to trips to (-1,0).

5. What is the probability that the rat of Exercise 4 stops on the line with the equations:

(a) $x + y = 1$ (c) $y - x = 1$

(b) $x + y = 3$ (d) $x - y = 3$

6. What is the probability that the rat of Exercise 4 stops at a point whose coordinates belong to the set:

(a) $\{(x,y) : |x| + |y| = 1\}$

(b) $\{(x,y) : |x| + |y| = 3\}$

7. Let $S = \{a_1, a_2, a_3, a_4\}$ be an outcome set and $P(a_1)$, $P(a_2)$, $P(a_3)$, $P(a_4)$ the elementary probabilities. Show that from the definition of a probability measure P, it follows that the elementary probabilities must fulfill the two conditions:

   (a) $P(a_i) > 0$ for $i \in \{1,2,3,4\}$ and

   (b) $\sum_{i=1}^{4} P(a_i) = 1$

   Try to generalize the above statements to a general probability space, (S,P).

8. Let p be a number between 0 and 1, exclusive:

   (a) Show that if $S = \{0, 1, 2\}$, the numbers,
   $P(x) = \binom{2}{x} p^x(1 - p)^{2 - x}$, $x \in S$,
   fulfill the two conditions in Exercise 7 above.

   (b) Show that if $S = \{0, 1, 2, \ldots, n\}$, the numbers,
   $P(x) = \binom{n}{x} p^x(1 - p)^{n - x}$, $x \in S$,
   satisfy the two conditions in Exercise 7 above.

9. In one of the classical experiments performed by Mendel, the founder of genetics, the probability of getting a yellow pea was $\frac{3}{4}$, a wrinkled pea $\frac{1}{4}$, and the probability of getting both yellow and wrinkled was $\frac{3}{16}$. Find the probability that a pea is

   (a) not yellow.
   (b) not wrinkled.
   (c) yellow but not wrinkled.
   (d) wrinkled but not yellow.

10. An engineer in a transistor factory finds that the probability that a transistor has defect A is .1; the probability that a transistor has defect B is .05; and the probability that a transistor has both defect A and defect B is .03. Find the probability that a transistor has:

(a) at least one of defect A or defect B;

(b) neither defect A nor defect B.

11. Let (S,P) be a probability space. Prove that for every A, B $\in \mathcal{P}(S)$:

(a) $P(A \cap B) \leq P(A)$

(b) $P(A) \leq P(A \cup B)$

(c) $P(A \cup B) \leq P(A) + P(B)$

(Hint: use Theorem 2(d) for parts (a) and (b) and Theorem 3 for part (c).)

12. When going home from work, Marshall can take either one of two busses, A or B. He finds that when he gets to the bus stop, the probability that bus A will be there is .2. The probability that bus B will be there is .3, and the probability that both bus A and bus B will be there is .1. Find the probability that one of the busses but not the other will ue there.

13. Show that if A and B are events, then the probability that exactly one of A and B will occur is:

$$P(A) + P(B) - 2P(A \cap B).$$

14. The odds favoring event A, written O(A), is defined $O(A) , \frac{P(A)}{P(\overline{A})}$; $P(\overline{A}) \neq 0$. show that $P(A) > .5$ if and only if $O(A) > 1$.

*15. Let A, B and C be events. Show that:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

*16. Prove that $P(A \cup B) = 1 - P(\overline{A} \cap \overline{B})$.

6.6 Uniform Probability Measure

We saw an instance of a uniform probability measure in Example 7 of Section 6.4. We now state:

Definition 10. Let (S,P) be a finite probability space. The probability measure P is called a uniform probability measure, if and only if all of the elementary probabilities are the same. In symbols: Let $S = \{a_1, \ldots, a_n\}$. Then P is a uniform probability measure if and only if $P(a_i) = P(a_j)$ for every pair of subscripts (i, j) for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

We now derive a formula that is used to calculate probabilities, when the probability measure is uniform. As a preparation for the proof of the theorem, suppose that the outcome set is $S = \{a_1, a_2, a_3, a_4\}$. If the probability measure is uniform, then the four elementary probabilities are equal:

$$P(a_1) = P(a_2) = P(a_3) = P(a_4).$$

Since the sum of the elementary probabilities must be 1, each must equal $\frac{1}{4}$. The same kind of reasoning can be used to show that, if the outcome set, $S = \{a_1, a_2, \ldots, a_n\}$ has n outcomes and the probability measure is uniform, each of the elementary probabilities is $\frac{1}{n}$.

<u>Theorem</u> 4. Let (S, P) be a probability space with a uniform probability measure.
Let N(S) and N(A) be the number of elements in S and A. Then the probability P(A) of the event A is then given by:

$$P(A) = \frac{N(A)}{N(S)}.$$

<u>Proof.</u> We know from Theorem 1 that $P(A) = \sum_{a_i \in A} P(a_i)$.

But in this case all of the elementary probabilities are $\frac{1}{n}$, and $n = N(S)$. Thus

$$P(A) = \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{N(A) \text{ terms}} = \frac{N(A)}{N(S)}.$$

From this we see that if the probability measure is uniform, P(A) is the quotient of the number of outcomes in A and the total number of outcomes in S. The formula $P(A) = \frac{N(A)}{N(S)}$ was for a long time the only definition of probability. It was, for instance, the definition that Pascal and Fermat developed in their correspondence. This method of calculating probabilities is therefore often referred to as the "classical" method.

We now show by examples how probability is calculated using $P(A) = \frac{N(A)}{N(S)}$. We shall make use of what you learned in Chapter 5, Combinatorics. You will find that the counting principle is especially useful. To refresh your memory, we restate this principle:

CP If a first activity can be completed in r_1 ways, and then a second activity can be completed in r_2 ways, and so on until a kth activity can be completed in r_k ways, then the sequence of k activities can be completed, one after the other, in $r_1 \cdot r_2 \cdot \ldots \cdot r_k$ ways. Figure 6.30 illustrates the CP by a tree diagram in the case where $r_1 = 3$, $r_2 = 2$ and $r_3 = 2$:

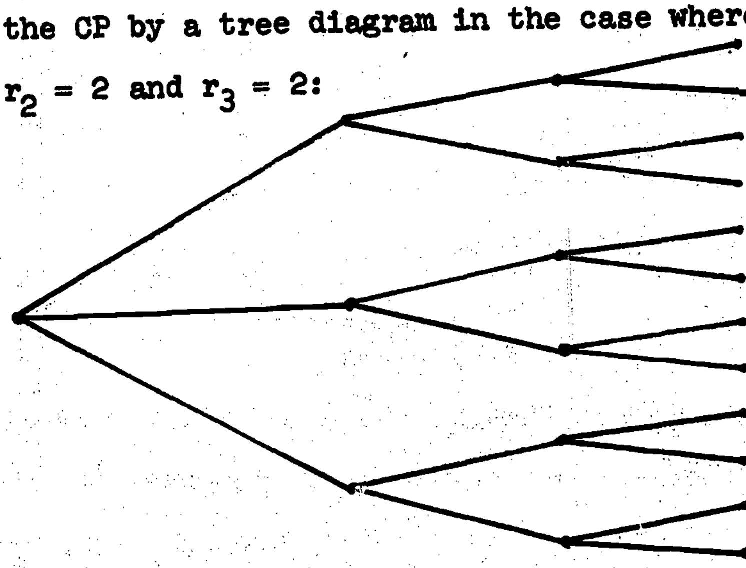


Figure 6.30

We shall also have occasion to use the following formulas that you learned in Combinatorics:

(a) The number of permutations of a set of n elements is given by:

$(n)_n = n! = n(n - 1)(n - 2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$

(b) The number of subsets with r elements of a set with n elements is given by:

$$\binom{n}{r} = \frac{(n)_r}{r!} = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}$$

When a selection is said to be made <u>at random</u>, this means that each possible selection has the same probability. In this case a uniform probability is being used.

<u>Example</u> 1.

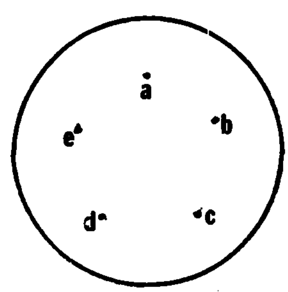


Figure 6.31

There are five chairs at a round table. (See Figure 6.31.) Two of these are selected at random and in such a way that the same chair cannot be chosen twice. What is the probability that these two chairs are next to each other?

Before reading an explanation of a solution to this problem, try to decide for yourself what a suitable outcome set might be, and keeping in mind that a uniform probability distribution is implied, decide what the probability of the event that the selected two chairs are next to each other. There are two ways to make the decision, one by combinatorics, the other by guessing (and then using combinatorics.)

First by using the brute force method one could also:

(a) List all of the two-member subsets of the set of chairs, {a, b, c, d, e};

(b) List all of the two-membered subsets of the event "two chairs are next to each other."

(c) Then use the formula developed in Theorem 4.

Using combinatorics is quicker:

(a) The number of two-member subsets of a set with five members is $\binom{5}{2} = 10$;

(b) Then you can look at Figure 6.31 to see that pairing each chair with the one on its right will give the two-member subsets that are next to each other, i.e., 5 of the subsets.

(c) Using Theorem 4 then gives $\frac{5}{10} = \frac{1}{2}$ as the probability of selecting 2 chairs next to each other at random from 5 chairs at a round table.

Question: Generalize this example to the case where there are 6, 7, 8 and in general n chairs around the table.

Suppose that the problem in Example 1 had been worded:

"There are 5 chairs at a round table. One of the 5 is selected at random. Then from the remaining 4 a second chair is selected at random. What is the probability that these two chairs are next to each other?"

Now it seems appropriate to use a set of ordered pairs as an outcome set. You can use the idea of permutations to find N(S). $N(S) = (5)_2 = 5 \times 4 = 20$. You can use the Counting Principle to find the number of ordered pairs of chairs that are next to each other. You have 5 choices for the first chair and, since there are 2 chairs next to any given chair around the table, you have 2 choices for the second. This gives 5 x 2 or 10 ordered pairs of chairs next to each other. By now you must know that the probability is $\frac{5 \cdot 2}{5 \cdot 4} = \frac{10}{20} = \frac{1}{2}$.

<u>Example</u> 2. There are nine marbles numbered 1, 2, ... 9 in a bag. Marbles 1, 2, 3, 4 are blue and marbles 5, 6, 7, 8, 9 are red. One of the marbles is selected at random from the 9. Then a second marble is selected from the 8 remaining. Find the probability that both are blue.

In Example 2 the wording clearly suggests that a suitable outcome set would be a set of ordered pairs of marbles and that since the selection is <u>without replacement,</u> ordered pairs with equal components would be ruled out.

Question: For the sake of variety (as well as your edification) use Figure 6.32 to find the answer to Example 2. Also answer the following:

(a) Which set of dots represents the outcome set?

(b) Which set of dots represents the outcomes in the event "both marbles are blue."

(c) What is the probability of the event described in (b).

(d) Why were the dots on the main diagonal deleted?

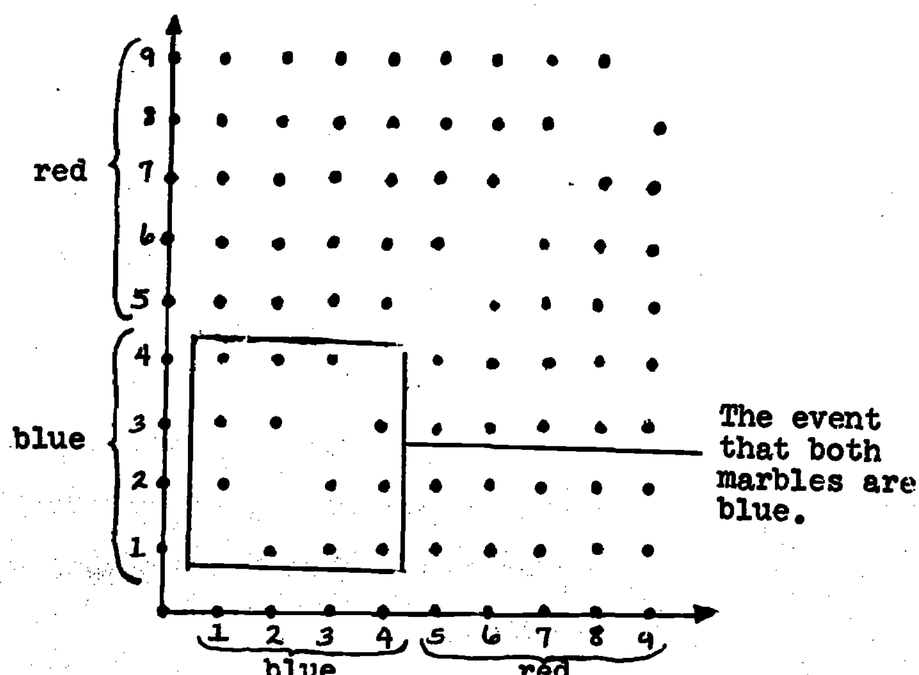


Figure 6.32

Example 2 can be reworded to suggest that a suitable outcomes set would be all two-member subsets of the set of marbles. Then the problem is easy to solve using what you learned in combinatorics about the number of r-member subsets of a set with n members.

Question: Solve Example 2 after rewording to the effect ... "Two marbles are selected at random from the 9." Select as an outcome set the two-member subsets of the set of 9 marbles.

Example 3. There are 10 boys and 12 girls in a class. Eight students are selected at random to constitute a committee. What is the probability that there will be 4 boys and 4 girls on the committee?

In problems like this we use subsets as outcomes. The total number of outcomes is $\binom{22}{8}$. By using the counting principle, it is easy to see that the number of subsets with 4 boys and 4 girls is $\binom{10}{4} \cdot \binom{12}{4}$. Thus our probability is:

$$\frac{\binom{10}{4} \cdot \binom{12}{4}}{\binom{22}{8}} = \frac{105}{323}$$

Check the computation of $\frac{105}{323}$.

Table 6.7 is a table of random numbers. The table consists of digits arranged in rows and columns. The digits have been selected by some random process. This means, among other things, that they have been selected in such a way that in choosing single digits from the table, each of the ten digits is equally likely to be selected. Each of the digits has the probability of 0.1 for selection.

If you select a single digit from a table of random numbers, the probability that it is 7 is 0.1. The probability that it is even is 0.5. (Can you explain why?)

A TABLE OF RANDOM NUMBERS

23018	70826	40641	52659	27607	43739	88519	58374
76576	38158	55842	70050	49196	29696	19015	39833
61272	23923	30483	02163	43236	05158	81197	13871
04659	04404	04615	14601	73036	20220	49825	73845
53947	02640	97591	66940	95692	15892	34629	14693
76609	15914	29821	04270	20023	23018	12681	89036
10984	40554	53947	88347	14830	05692	54180	91296
23239	17914	38787	64198	45800	21676	53625	99121
61816	50784	26248	69071	06675	15816	69541	55431
50444	04703	57949	09636	43844	46311	84748	05681
27393	33875	03451	09213	78952	14240	65983	92745
64701	78416	71670	19167	70741	69647	61359	94968
54254	78919	83984	95656	08613	99006	27117	90957
30390	92457	75943	57616	60085	84104	84104	38543
30160	35707	31830	97344	58501	19138	08198	15358
01623	37762	68295	28263	06426	97642	84557	82071
00663	70293	80716	51129	63563	27720	60938	99814
39637	11770	35081	15498	86244	88062	11338	74725
18346	01929	65329	91095	75364	58584	16248	68897
36015	72437	84726	53714	98790	69562	81759	29184
57055	61920	08487	33754	95846	06857	74128	33143
12733	19076	26961	69330	50226	35582	04502	30822
40373	36704	98704	68382	14892	71309	35475	05946
65166	26844	57745	18627	48103	79279	43066	90043
47672	97430	18926	79538	09071	46137	65030	33138
64339	78870	25752	82806	38829	55168	74956	52220
38755	08583	14761	17431	48456	79060	32894	86947
55758	10329	94997	26071	74940	24416	61540	75853
51298	27096	54768	30607	68410	99269	70619	03272
73346	82246	90929	79535	90512	26472	16414	99265
34585	74481	55659	57038	84156	44410	25201	18125
53552	08747	56609	43607	19132	60515	79963	13922
82328	58653	22192	95497	31143	19645	78500	17521
97266	75002	73747	36318	40114	45888	12867	32185
91129	75202	58706	31831	80194	19698	23459	04664
78980	84195	33147	36963	54818	57770	32179	43227
55475	02174	37177	58609	15883	18556	49509	41139
66562	92193	05353	00615	10525	42541	87590	01688
16801	02719	00230	11109	84054	58919	96896	90261
21476	66002	37371	03472	76458	74387	25362	58586

Table 6.7

<u>Example</u> 4. There are many ways to use a table of random numbers. This example will illustrate one particular way of using such a table. Out of a group of 100 grade 9 students, 40 are to be selected at random for a special course in game theory. Each of the students are initially assigned a two digit number from sequence 00, 01, 02,..., 99.

Question: If the probability of selecting any one of the 10 digits is .1, what is the probability of selecting one of the above two-digit numbers in a table of random numbers? If you were to use Table 6.7 to select the group of 40 you might start by using just the first 2 columns of digits as shown below:

Student chosen
1st → $\boxed{23}$018
2nd → $\boxed{76}$676
$\boxed{61}$272
etc. $\boxed{04}$659
$\boxed{53}$947

The first student chosen for the group of 40 would be the one to whom 23 was initially assigned. Then 76, 61, 04, 53, etc. would follow. This procedure would continue until 40 students were selected. Notice that although the original assignment of numbers to students was not a random one, the use of the random number table made the selection of the

40 for the experimental group a random one.

Some important considerations are:

(a) If a pair of digits occurs a second time you simply skip it and go on.

(b) If you reach the bottom of the two columns before 40 are selected, you start at the top of columns 6 and 7 and continue.

(c) Rows could be used instead of columns by selecting consecutive mutually exclusive pairs.

(d) You should make a note of the page, row and column when you finish using the table for the next problem. When you start using the table for the next problem, you should start where you left off. If you keep repeating the use of the same set of digits you could be biasing your selection.

Questions: (a) How would you choose 25 persons at random among 67 with the aid of random numbers?

(b) How would you use random numbers to simulate:

(1) tossing a symmetric die.

(2) spinning a spinner  .

(3) tossing a fair coin.

If your experiment is to select two adjacent digits from the table, the sample space is $\{00, 01, 02, \ldots, 98, 99\}$. The outcomes then represent two-digit random numbers.

In this case also you can assign a uniform probability measure to the set of 100 outcomes. For instance, the probability that a chosen two-digit number is less than 36 is $\frac{36}{100} = 0.36$. In some of the following exercises, and in Section 6.8 you will work with random numbers.

6.7 Exercises

1. Five chairs are in a row. Two chairs are selected at random. What is the probability that they are next to each other?

2. A two-digit random number is selected from a table of random two-digit numbers. What is the probability that the number (a) is even? (b) is greater than 25? (c) has 9 as its last digit?

3. A three digit random number is selected from a table of random three digit numbers: (a) What is the outcome set? (b) What is the probability that the chosen number is odd? (c) ... is less than 100? (d) ... starts with a 9?

4. Two symmetric dice are tossed. Construct a graph of the outcome set such as the one in Example 8, Section 6.2 and illustrate the following events by circling the appropriate sets of dots. Use the graphs to find the probabilities of

the events:

(a) the total number of dots (on the upper faces) is 7.

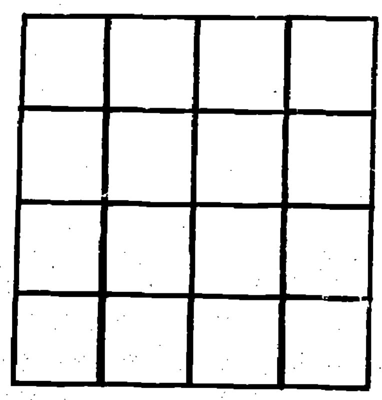
(b) the total number of dots is less than 6.

(c) at least one die shows 2 dots.

(d) both dice show more than 3 dots.

(e) at least one die shows at least 3 dots.

5. Four of the smallest squares in the lattice are selected at random. Find the probability that the four squares



(a) are in the same row. (b) are in the same diagonal.

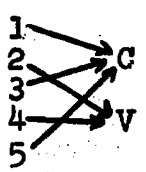
(c) are distributed so that there is exactly one in each row and exactly one in each column.

6. Derive the elementary probabilities in Exercise 3 in Section 6.5.

7. Derive the elementary probabilities in Exercise 8(b) in Section 6.5 for $n = 5$ and $p = 1/3$.

8. If you draw 5 cards from a deck of playing cards, what is the probability of getting 3 aces and 2 kings? (The deck consists of 52 cards, 4 of which are aces, and 4 kings.)

*9. We will temporarily define a five-letter word as any ordered quintuple of letter with 2 vowels and 3 consonants. We will define a vowel as any member of {a, e, i, o, u}, and a consonant as any of the other 21 letters in the alphabet. We will use a symbol like CVCVC to represent the form of words with an alternating consonant-vowel arrangement. You may repreat letters in a word in parts (a) to (d) below. The form of a word, as used here, is a mapping. It shows which positions get consonants (C), and which get vowels (V), e.g.,



(a) Use the counting principle to find the number of words with form CVCVC. (Express answer as factors of product.)

(b) Do the same for CCCVV.

(c) How many different forms are there for a five-letter word?

(d) What is the total number of five-letter words?

(e) If you use a uniform probability measure, what is the probability of selecting the word "teded?"

(f) Repeat (a) and (b) with the restriction that no letter may be repeated within the same word.

10. Consider the experiment of tossing three symmetric dice. Use the set of ordered triples of numbers from 1 to 6 as

an outcome set. Also use a uniform probability measure.

(a) What is the probability of getting 3 sixes on a toss?

(b) In how many ways can an outcome have 2 fives and 1 six?

(c) What is the probability of tossing 2 fives and a six?

(d) What is the probability of getting 0 sixes on a toss?

11. Ten cards were numbered from 1 to 10 and placed in a hat. A set of 3 cards was drawn at random from the hat, i.e., a three-member subset of a ten-membered set. What is the probability that one of the cards drawn was the 5 card? State a theorem about probability that you used to solve this problem.

12. Toss 3 symmetric dice. Let the outcome set be the set of 216 ordered triples of the cartesian product, $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$.

(a) Give the roster name for the set of sums of the triples of numbers.

(b) How many of the triples have a sum of 17?

(c) What is the probability of getting a sum of 17 on a toss of the three dice?

(d) Find the probability for each of the possible sums starting with $P(3) = 1/216$.

6.8 Looking Back

When you studied probability in Course I Chapter 5, you performed trials of experiments and you studied the behavior of relative frequencies. For instance, in connection with a tack-tossing experiment you obtained Figure 6.33.

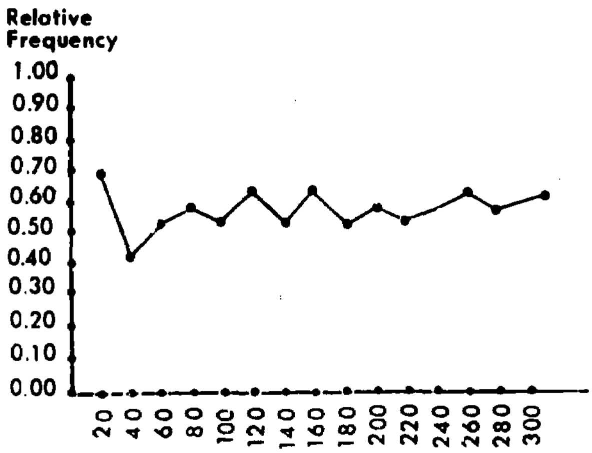


Figure 6.33

You saw in connection with this and similar experiments that the relative frequency stabilized. This means that as the number of trials continue to increase the relative frequency tends to stay close to some number between 0 and 1.

In this chapter you have learned, among other things, to calculate probabilities. Since the properties of probabilities are similar to those of relative frequencies, it seems reasonable to ask if there is any relation between probability and relative frequency. We consider this question in connection with a concrete experiment.

Example. Let us return to random numbers. The experiment is to select a two-digit number from Table 6.7. A is the event that the number selected is less than 40. By the method of Section 6.6 you can easily find:

$$P(A) = 0.40$$

Since this probability was obtained theoretically, it is interesting to see if our experiment yields a relative frequency close to 0.40. See Section 6.6 Example 4 for some suggestions on the use of a table of random numbers.

Using the format of Table 6.8 and the suggestions of Example 4, Section 6.6, record the results of 50 trials of the experiment described in Example 1 above. Also record the frequency and relative frequency for the event A.

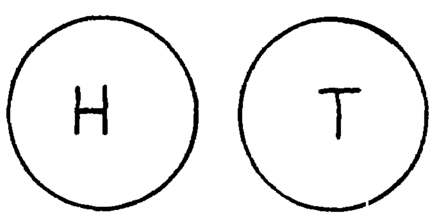
Question. Did the relative frequency of A tend to 0.40 as the number of trials approached 50? If not, what would you guess the reason might be?

TRIAL NUMBER	OUTCOME	FREQUENCY OF A	RELATIVE FREQUENCY OF A
1	52	0	0.00
2	99	0	0.00
3	46	0	0.00
4	14	1	0.25
5	42	1	0.20
6	22	2	0.33
7	87	2	0.29
8	20	3	0.38
9	39	4	0.44
10	51	4	0.40
11	49	4	0.36
12	22	5	0.42
13	56	5	0.38
14	83	5	0.36
15	24	6	0.40
16	02	7	0.44
17	27	8	0.47
18	73	8	0.44
19	00	9	0.47
20	86	9	0.45

Table 6.8

Questions. Make a graph like the one in Figure 6.33 showing your results. What conclusions follow from your experiment? What relationship exists between the probability 0.40 and the relative frequencies that you have observed? Do you agree that the probability 0.40 was a good prediction of the relative frequency for the event A in 50 trials?

For some situations the symmetry of the experimental objects or the results of previous experiments might convince us that a uniform probability measure is the best model. But in certain other situations the lack of symmetry or experimental evidence may lead us to believe that a uniform probability measure is not appropriate. Consider, for example, the two experiments in Figure 6.34:



Tossing a Coin

Tossing a Thumbtack

Outcome	H	T
Elementary Probability	0.5	0.5

(a)

Outcome	Pin Up	Pin Down
Elementary Probability	?	?

(b)

Figure 6.34

Do you agree that it seems more appropriate to use a uniform probability measure in the coin tossing experiment than in the tack tossing experiment? If you think about the two experiments, you will see that there is some kind of symmetry in the coin tossing experiment that is not found in the tack tossing experiment. You might ask how we assign the elementary probabilities in the tack tossing experiment. One way is to perform trials and use the observed relative frequencies as elementary probabilities. This is a method that is studied extensively within the field of statistics. You will learn more about it in later courses.

6.9 Exercises

1. Suppose that you selected 50 family names at random from a telephone book. The outcome set would the set of possible lengths of the names, i.e., the number of letters in the name. Do you think that with a large sample you would use a uniform probability measure to predict the length of names? Design an experiment and perform about 30 trials where you would use a table of random numbers to select a page of the telephone book, then use the table to select a column, and then use the table to select a name within the column. Make a table with the headings:

   Name length   Frequency   Relative frequency

Examine the relative frequency distribution and make a decision about what kind of a probability measure you would use for the lengths of names in the telephone book.

2. Consider the experiment of selecting a card at random from a deck of playing cards and recording the suit. Let the outcome set be:

   S = {heart, diamond, club, spade}

   Shuffle the deck well each time before making a selection and perform 64 trials.

   (a) Record the frequency of each outcome.

   (b) Record the relative frequency of each outcome.

   (c) What is the sum of the relative frequencies?

   (d) Would you use a uniform probability measure for prediction in this experiment?

   (e) What was the relative frequency for each of the following events: "not a spade;" "a red suit;" "red suit or black suit " ?

   (f) What probability would you assign the events in part (e), using a uniform probability measure?

3. Classify each of the following experiments with related outcome sets according to whether or not you would choose a uniform probability measure:

   (a) Tossing two coins; outcome set is the cartesian product, {H, T} X {H, T}.

   (b) Tossing two coins; outcome set based on number of heads: S = {no heads, one head, two heads}

(c) Selecting an item from a production line:

$S = \{\text{defective, non-defective}\}$.

(d) Selecting a card from a deck of playing cards:

$S = \{\text{face card, not face card}\}$.

(e) Selecting a card from a deck of playing cards:

S is the set of 52 different outcomes.

(f) Selecting a marble from a box which contains 3 red and 5 white marbles:

$S = \{\text{red, white}\}$.

(g) Selecting a marble from a box which contains 4 yellow and 4 blue marbles:

$S = \{\text{yellow, blue}\}$.

4. Suppose for a certain coin we have evidence that probability 1/3 is a reasonable assignment for heads in the experiment of tossing the coin. What would you predict as the frequency for tails in 51 tosses? What would you predict as the relative frequency of tails?

6.10 Looking Ahead

In this chapter you have been introduced to these basic concepts of probability theory: outcome set, events, and probability measure. This is not your last contact with this theory. In this section we shall give you a preview of things to come.

One of the most important conepts in probability theory is that of independence. Briefly and somewhat loosely, two events A and B are independent if the fact that one of them has occurred does not affect the probability of the other occurring. This doesn't quite tell the whole story, but it will do for now. You will meet this concept next year and also the related concept of conditional probability.

We next illustrate in an example an important concept that you will study at length in future courses, the concept of a random variable which you have encountered before without this name. This concept will bring together many ideas about experiments, mappings and probability.

<u>Example</u>. Consider the experiment of tossing a symmetric coin twice. Let (H, T) represent the outcome that the first toss turns up heads and the second tails. The outcome set is then:

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

$$\text{or} \quad S = \{H, T\} \times \{H, T\}.$$

We illustrate this outcome set in Figure 6.35.

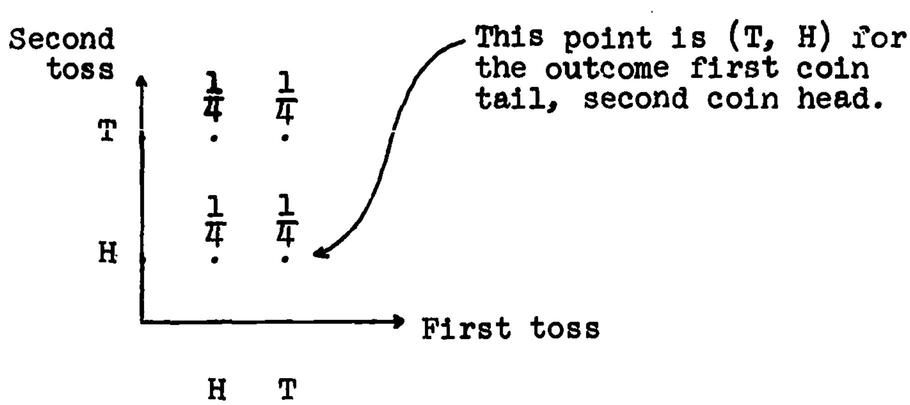


Figure 6.35

It is reasonable to use a uniform probability measure for this experiment. In other words we assign the elementary probability 1/4 to each outcome.

Suppose we are interested in the number of tails obtained. We would then define a mapping X with domain S and codomain R, which assigns values as in Figure 6.36. This mapping X is called a random variable. Any function from an outcome set to the real numbers is called a <u>random variable</u>.

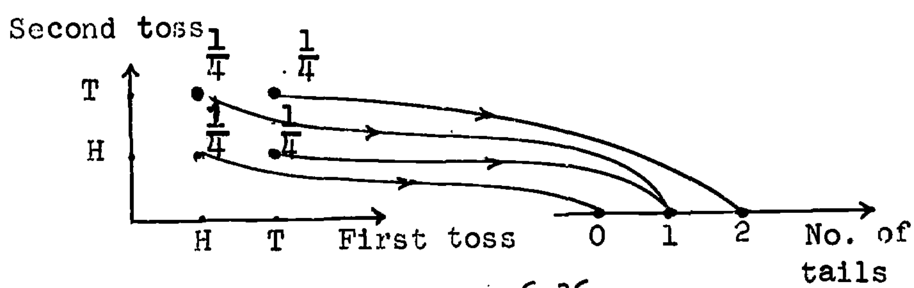


Figure 6.36

What is the probability of exactly 1 tail? There are two outcomes, (H, T) and (T, H), that result in exactly one tail and each has probability 1/4. Therefore the probability of exactly one tail is $1/4 + 1/4 = 1/2$. One easy way to find this and similar probabilities is to let the elementary probabilities "go along" with the mapping. In other words, the probability of a certain number of tails is the probability of the event in the original outcome set that maps onto this certain number of tails as pictured in Figure 6.36.

It is now easy to find the following probabilities:

The probability of exactly two tails.

The probability of at most two tails.

The probability of at least two tails.

We will have more to say about random variables in the future.

6.11 Exercises

1. Consider the experiment of tossing three coins and the outcome set

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

Let the random variable X be the mapping that maps each outcome on the number of heads in the outcome. For example, $X(THH) = 2$.

(a) Find the image of each outcome in S.

(b) Tabulate the range of X.

(c) Draw a mapping diagram to illustrate the random variable. (See Figure 6.36.)

(d) Assign to each image a probability which is the sum of the probabilities of the outcomes in S which map onto that image.

(e) Make another mapping diagram showing the assignment of probabilities to the images under the random variable X.

2. Let an experiment be to select a card at random from a deck of playing cards, and the outcome simply the selected card.

Let Y be a random variable which maps an outcome onto 1 if is a red card, and maps the outcome onto 0 otherwise.

(a) Using the same procedure as in Exercise 1, find the probability of 1; find the probability of 0.

(b) Does the assignment of probabilities to {1, 0} in (a) satisfy the 3 requirements of a probability measure?

3. Five cards are placed in a hat. The cards are numbered from 1 to 5. A card is drawn at random and not replaced. A second card is drawn at random from those remaining.

(a) Make up an outcome set for the experiment based on ordered pairs.

(b) Assign probabilities to the outcomes.

(c) Find the images of the outcomes for the random variable which assigns the absolute difference of each pair to the pair. (Define the absolute difference of (a, b) to be $|a - b|$.)

(d) Assign probabilities to the images in (c) based on the assigned probabilities in (b).

6.12 Summary

Let S be an outcome set. A is an event iff $A \subset S$; or equivalently, iff $A \in \mathcal{P}(S)$, where $\mathcal{P}(S)$, the power set of S, is the set of all subsets of S. If A and B are events, then:

(a) $A \cup B$ is the union event of A and B.

(b) $A \cap B$ is the intersection event of A and B.

(c) $\overline{A}$ is the complementary event of A.

(d) $A \setminus B$ is the difference event of A and B.

(e) A and B are disjoint if and only if $A \cap B = \emptyset$.

(f) A, B and C are disjoint if and only if $A \cap B = \emptyset$, $B \cap C = \emptyset$ and $A \cap C = \emptyset$.

A probability measure P is a real valued function with $\mathcal{P}(S)$ as its domain, and it has the properties:

(1) $0 \leq P(A) \leq 1$ for every $A \in \mathcal{P}(S)$.

(2) $P(S) = 1$.

(3) If A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$.

We call P(A) the probability of A, and the ordered pair (S,P), consisting of an outcome set S and a probability measure P, a probability space.

Events that contain exactly one outcome of outcome set S are called singleton events. Probabilities of singleton events are called elementary probabilities.

Let (S, P) be a finite probability space. The probability measure P is called a uniform probability measure if and only if all the elementary probabilities are the same.

Theorem 1. Let (S, P) be a probability space, with S a finite outcome set. Then for every event $A \in \mathcal{P}(S)$ we have:

$$P(A) = \sum_{a_i \in A} P(a_i).$$

Theorem 2. Let (S, P) be a probability space, and let $A, B \in \mathcal{P}(S)$. Then:

(a) $P(\emptyset) = 0$

(b) $P(A) + P(\bar{A}) = 1$

(c) $P(A \setminus B) = P(A) - P(A \cap B)$

(d) If $B \subset A$, $P(B) \leq P(A)$

Theorem 3. For all events $A, B \in \mathcal{P}(S)$ we have:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Theorem 4. Let (S, P) be a probability space with a uniform probability measure. Let S have n outcomes and let the number of outcomes in the event A be (A). The probability P(A) of the event A is then given by:

$$P(A) = \frac{N(A)}{N(S)} .$$

A table of random numbers was included and some problems were done to illustrate the use of such tables.

A review of the stability of relative frequencies and the relation between probability and relative frequency was included in a "looking back" section.

A preview of ideas to presented from probability theory in later course included independence and random variables.

A random variable is a mapping from an outcome set to the real numbers. The probabilities for the images under the mapping are determined by the probabilities of the singletons from the original outcome set.

6.13 Review Exercises

1. A coaxial cable from a radio station to its trasmitting antenna is 5,000 feet in length, and has a break which must be located. Assuming that each foot of cable is equally likely to have the break, what is the probability that the break is within 2,000 feet of the station? What is the probability that the break is not within 2000 feet of the station? What is the probability that it is within 2000 feet of the station or within 2000 feet of the antenna? What is the probability that it is within 4000 feet of the antenna and within 4000 feet of the station?

2. In studying the three possible outcomes of an experiment, it was found that the second outcome was twice as likely as the first and the third was 3 times as likely as the second. What is the probability of the singleton event that contains the first outcome?

3. If you select a four-digit random number from a table of random numbers, what is the probability that:
   (a) all four digits will be the same?
   (b) no digit appears twice in the number?

4. If you have six purple socks and four yellow socks mixed and dress in such a hurry that you do not observe the colors, what is the probability that the socks you select will be of the same color?

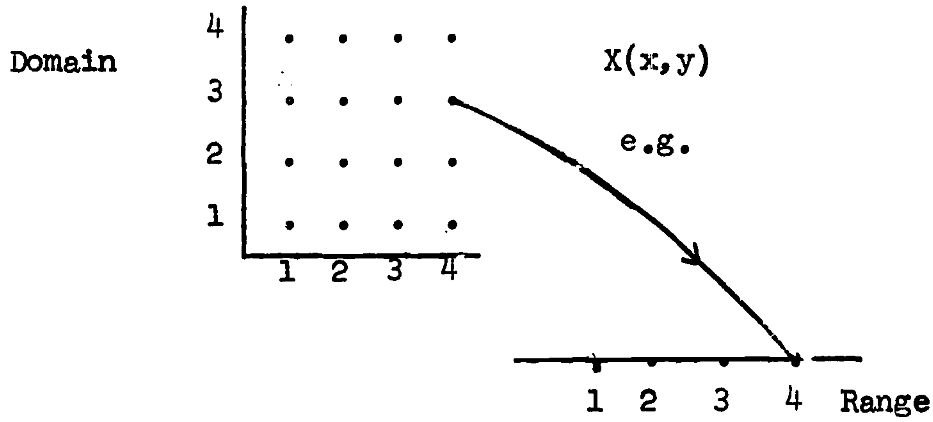
5. Seven men checked their hats when they entered a restaurant. A friend of the hat check girl removed the tags as a prank. When three of the men returned, the hat check girl chose three of the hats at random. What is the probability that they belonged to those three men?

6. If you toss 5 symmetric coins, what is the probability that:
   (a) exactly three will be heads?
   (b) at least three will be heads?
   (c) at most three will be heads?

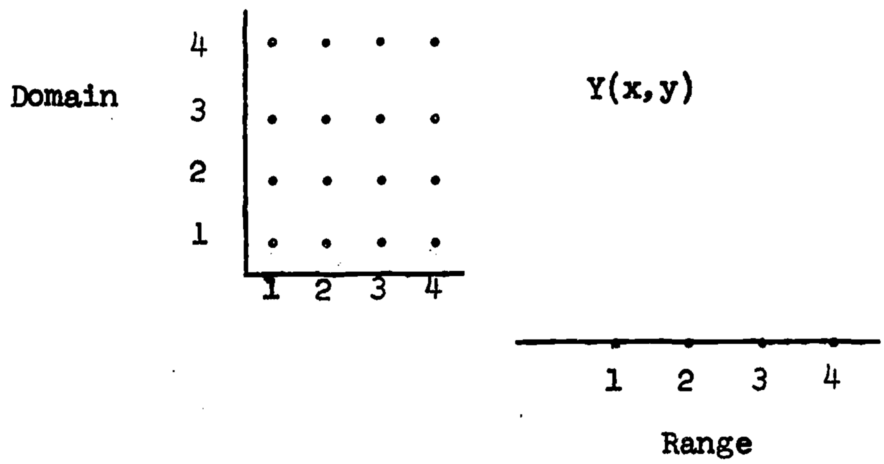
7. An experiment consists of selecting 2 cards, one after the other, from a set of 7 cards numbered from 1 to 7 (replacing the first card before selecting the second.)
   (a) How many members are in an outcome set (which contains every member of $\{1,2, \ldots, 7\} \times \{1,2 \ldots, 7\}$)?
   (b) Draw a graph of the outcome set letting the first card selected be represented by x and the second by y.

8. Define $\max(x,y) = x$ if $x \geq y$ and y if $y \geq x$.
   Define $\min(x,y) = x$ if $x \leq y$ and y if $y \leq x$.
   Let the random variable X map (x,y) onto $\max(x,y)$.
   Let the random variable Y map (x,y) onto $\min(x,y)$.

For example $X(5,2) = 5$; $Y(5,2) = 2$; $X(4,4) = Y(4,4) = 4$.
Make graphs like those below and use them to show a mapping diagram of X:



Make graphs like those below and use them to show a mapping diagram of Y:



Chapter 7

POLYNOMIAL AND RATIONAL FUNCTIONS

7.1 Polynomial Functions

If a spherical object is thrown upward at a velocity of 32 feet per second, its height at a later time can be found by the formula

$$h = 32t - 16t^2,$$

where $\underline{h}$ represents the height and $\underline{t}$ the number of seconds after the object was thrown.

Questions. (1) How many seconds will it take for the object to return to the ground?
(2) How high will the object go?

The above problem, while oversimplified, indicates the importance of mathematics in studying physical phenomena such as motion. Actually the formula "$h = 32t - 16t^2$" describes a function

$$f: \ t \longrightarrow 32t - 16t^2$$

for $t \geq 0$. Of course the variable used is of no importance; so we could just as easily describe the function in the following way:

$$f: \ x \longrightarrow 32x - 16x^2$$

with domain $\{x : x \in R_0^+\}$. (R_0^+ is the set of non-negative real numbers.) This particular kind of function is very important not only in applications of mathematics but in mathematics itself, and in this chapter we shall work with many such functions.

Consider for instance the function f defined by

$$f: \ x \longrightarrow x^2 + 3x$$

where the domain and codomain are both the set R of real numbers. (Unless otherwise specified all functions have domain and codomain R.) Some of the assignments made by this function are as follows:

$$3 \longrightarrow 3^2 + 3(3) = 18$$

$$-2 \longrightarrow (-2)^2 + 3(-2) = -2$$

$$100 \longrightarrow (100)^2 + 3(100) = 10300$$

This function f can be "built" or generated from some real functions already familiar to you:

$$c_3: \quad x \longrightarrow 3$$

$$j_R: \quad x \longrightarrow x.$$

The steps in this generation may be described in the following way:

1) What is $[c_3 \cdot j_R]$ (x)?

$$\begin{aligned} [c_3 \cdot j_R]\,(x) &= c_3(x) \cdot j_R(x) \\ &= 3 \cdot x \\ &= 3x \end{aligned}$$

Now we have already generated a new function; call it g. Thus,

$$g: \quad x \longrightarrow 3x.$$

2) What is $[j_R \cdot j_R]$ (x) ?

$$\begin{aligned} [j_R \cdot j_R]\,(x) &= j_R(x) \cdot j_R(x) \\ &= x \cdot x \\ &= x^2 \end{aligned}$$

We have still another function; call it h. That is,

$$h: \quad x \longrightarrow x^2.$$

3) Having generated the g and h functions, we can now ask:

What is $[h + g]\,(x)$?

$$[h + g]\,(x) = h(x) + g(x)$$
$$= x^2 + 3x$$

It is important to notice that the function f was generated by only addition and multiplication of functions, one the identity function and the other a constant function. It is possible of course to use more than two functions at the outset, as in Example 1.

<u>Example</u> 1. Using addition and multiplication only, generate a new function from the following:

c_5: $x \longrightarrow 5$

$c_{\frac{1}{2}}$: $x \longrightarrow \frac{1}{2}$

j_R: $x \longrightarrow x$

Here are some of the functions that can be generated:

(a) $[c_5 + c_{\frac{1}{2}}](x) = c_5(x) + c_{\frac{1}{2}}(x)$
$= 5 + \frac{1}{2}$
$= 5$

Hence: $[c_5 + c_{\frac{1}{2}}]$: $x \longrightarrow 5\frac{1}{2}$

(b) $[c_5 \cdot j_R \cdot j_R](x) = c_5(x) \cdot j_R(x) \cdot j_R(x)$
$= 5 \cdot x \cdot x$
$= 5x^2$

Thus: $[c_5 \cdot j_R \cdot j_R]$: $x \longrightarrow 5x^2$

(c) From the two previous functions, we see that:

$[[c_5 \cdot j_R \cdot j_R] + [c_5 + c_{\frac{1}{2}}]](x) =$
$5x^2 + 5\frac{1}{2}$

Let us call this new function p; that is,

$$p: \quad x \longrightarrow 5x^2 + 5\tfrac{1}{2}.$$

An important feature of the function p in the example is that it was generated by using only addition and multiplication of constant functions and the identity function. Such a function is called a polynomial function.

Definition 1. j_R is a real polynomial function.
c_a, $a \in R$, are real polynomial functions.
Any function generated from a finite number of the above functions, using no operation other than addition and multiplication of functions, is a real polynomial function.

Example 2. Is g: $x \longrightarrow -4x^3 + 5x$ a polynomial function?
Since $-4x^3 + 5x = (-4)(x)(x)(x) + (5)(x)$, the function $g = [[c_{-4} \cdot j_R \cdot j_R \cdot j_R] + [c_5 \cdot j_R]]$,

$$\text{where } c_{-4}: \quad x \longrightarrow -4$$
$$j_R: \quad x \longrightarrow x$$
$$c_5: \quad x \longrightarrow 5.$$

Since g may be generated from the identity function and constant functions, using only addition and multiplication, it is a polynomial function.

From Example 2, we know that the function g: $x \longrightarrow -4x^3 + 5x$ is a polynomial function. The expression "$-4x^3 + 5x$" is referred to simply as a polynomial. Every polynomial function has a polynomial associated with it.

The real polynomial function g may also be written $g: y \longrightarrow -4y^3 + 5y$, or $g: t \longrightarrow -4t^3 + 5t$. We refer to the corresponding polynomial $-4y^3 + 5y$ as a <u>polynomial</u> <u>in</u> <u>y</u>, and to $-4t^3 + 5t$ as a <u>polynomial</u> <u>in</u> <u>t</u>, etc.

<u>Example</u> 3. The function h: $x \longrightarrow \frac{1}{3}x^4 - 7x^3 + 14x - 9$ is a polynomial function.

The polynomial associated with this function is "$\frac{1}{3}x^4 - 7x^3 + 14x - 9$."

<u>Example</u> 4. Is $x \longrightarrow \frac{3}{x}$ a polynomial function?

The constant function c_3 and the identity function j_R seem to be involved here. However, in order to generate the given function, it is necessary to divide c_3 by j_R. Since this is something other than addition or multiplication, the given function is not a polynomial function.

<u>Example</u> 5. What is the polynomial associated with function

$$p = [[c_3 \cdot j_R] + [c_{-5} \cdot j_R \cdot j_R \cdot j_R] + c_6 + [c_{-8} \cdot j_R \cdot j_R]] \ ?$$

$$[c_3 \cdot j_R](x) = c_3(x) \cdot j_R(x) = 3 \cdot x$$

$$[c_{-5} \cdot j_R \cdot j_R \cdot j_R](x) = c_{-5}(x) \cdot j_R(x) \cdot j_R(x) \cdot j_R(x) = -5 \cdot x \cdot x \cdot x$$

$$c_6(x) = 6$$

$$[c_{-8} \cdot j_R \cdot j_R](x) = c_{-8}(x) \cdot j_R(x) \cdot j_R(x) = -8 \cdot x \cdot x$$

Therefore, the associated polynomial is

$$3x - 5x^3 + 6 - 8x^2.$$

Incidentally, since addition of functions is associative and commutative, it is permissible to write the polynomial in Example 5 as "$-5x^3 - 8x^2 + 3x + 6$." It is in fact quite common to write polynomials in this manner, so that the exponents involved appear in descending order from left to right, going from the greatest exponent to the least.

7 .2 Exercises

1. Write the polynomial associated with each of the following polynomial functions.

(a) $[c_5 \cdot c_2]$

(b) $[c_5 + c_2]$

(c) $[c_1 \cdot j_R]$

(d) $[c_1 + j_R]$

(e) $[[c_{-4} \cdot j_R \cdot j_R] + [c_{\sqrt{2}} \cdot j_R] + c_{-10}]$

(f) $[[j_R \cdot j_R \cdot j_R \cdot j_R \cdot j_R] + j_R]$

(g) $[[j_R + j_R + j_R + j_R + j_R] \cdot j_R]$

(h) $[[c_7 \cdot j_R \cdot j_R] + [c_{-1} \cdot j_R] + c_0]$

(i) $[[c_{-1} \cdot j_R \cdot j_R \cdot j_R] + [c_{12} \cdot j_R \cdot j_R] + [c_4 \cdot j_R] + c_9]$

(j) $[c_0 \cdot j_R]$

(k) $[[j_R \cdot j_R] + c_1]$

(l) $[[c_8 \cdot j_R \cdot j_R \cdot j_R \cdot j_R] + [c_{-3} \cdot j_R \cdot j_R] + c_7]$

2. Write the following polynomials in descending order of exponents.

(a) $3x + x^5 + 4$

(b) $-x^2 - 7x^6 + 4x$

(c) $x^9 + x^6 + x^3$

(d) $-3x^3 + 7 - 2x$

(e) $4 + 3x - 7x^2 + 8x^3$

(f) $\frac{2}{3}x + 7x^2 - \frac{5}{4}$

(g) $\sqrt{3}x^2 - x + 17$

3. Show how a polynomial function can be generated by constant functions and the identity function by addition and multiplication. Which of the following functions are not polynomial functions? Explain why not.

(a) $f: x \longrightarrow 2x^7 + 3x^5 - 8$
(b) $g: x \longrightarrow \frac{x+3}{x}$
(c) $h: x \longrightarrow \frac{1}{3}x + 1$
(d) $x \xrightarrow{r} \frac{2}{x}$
(e) $x \xrightarrow{s} \frac{x}{2}$
(f) $x \xrightarrow{t} -x^2 - 5x - 10$

4. Which of the following are polynomial functions? For each polynomial function, write the associated polynomial.

(a) $[j_R + c_2]$
(b) $[c_2 - j_R]$
(c) $[c_2 \cdot j_R]$
(d) $[c_2 \div j_R]$
(e) $[c_{\frac{1}{2}} \cdot j_R]$

5. $p: x \longrightarrow a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

where all of the a's are real numbers, and $n \in W$, is a polynomial function. Explain how it is generated from the identity function and constant functions.

The expression "$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$" is often used when discussing polynomials in general.

6. Using the following functions (identity and constant functions) generate five different polynomial functions.

$$j_R, \quad c_{-7}, \quad c_{\frac{1}{4}}, \quad c_{\sqrt{2}}$$

7. $p: x \longrightarrow \frac{2}{3}x^2 + 7x - 8$ is a polynomial function.

Find the following:

(a) $p(-1)$ (b) $p(0)$ (c) $p(3)$ (d) $p(-4)$ (e) $p(10)$

8. (a) Explain how the definition of polynomial function includes c_0 as a polynomial function. What is the range of c_0?

(b) Explain how the definition of polynomial function includes c_1 as a polynomial function. What is the range of c_1?

(c) $[c_0 + j_R] = ?$ (d) $[c_0 \cdot j_R] = ?$ (e) $[c_1 \cdot j_R] = ?$

9. (a) Are $[c_2 \cdot j_R]$ and $[j_R + j_R]$ polynomial functions? Does $[c_2 \cdot j_R](x) = [j_R + j_R](x)$ for all $x \in R$?

(b) True or false: $[c_3 \cdot j_R] = [j_R + j_R + j_R]$

(c) Describe another way to generate the polynomial function

$$[j_R + j_R + \dots + j_R]$$

m addends, where $m \in W$.

10. (a) $[c_2 + c_{-2}] = ?$ (d) $[[c_{-2} \cdot j_R \cdot j_R] + [c_2 \cdot j_R \cdot j_R]] = ?$

(b) $[c_0 + c_8] = ?$ (e) $[c_1 \cdot [c_2 \cdot j_R \cdot j_R]] = ?$

(c) $[c_0 + [c_2 \cdot j_R \cdot j_R]] = ?$

7.3 Degree of a Polynomial

We have already noted that a polynomial is associated with every polynomial function, and we shall study the polynomial functions largely by means of their associated polynomials. First, we need to define some words commonly used in discussing polynomials.

As in Exercise 5 of the preceding section, we shall use

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0,$$

where the a's are real numbers and $n \in W$, to represent a real polynomial. a_n is called the coefficient of x^n, a_{n-1} is called the coefficient of x^{n-1}, etc. More generally, if "$a_i x^i$" appears in a polynomial, "a_i" is called the coefficient of "x^i". The following examples should help to make this clear.

Example 1. In the polynomial "$5x^3 + 7x^2 - 3x + 2$,"

$n = 3$, $n-1 = 2$, etc.,

$a_n = a_3 = 5$, so that the coefficient of x^3 is 5;

$a_{n-1} = a_2 = 7$, so that the coefficient of x^2 is 7;

$a_{n-2} = a_1 = -3$, so that coefficient of x is -3;

$a_o = 2$.

"a_o" is called the constant term, since it comes from the constant function c_{a_o}, without multiplication by j_R. Thus, in the polynomial "$5x^3 + 7x^2 - 3x + 2$," of Example 1, the constant term is "2." "a_n" ($a_n \neq 0$) is called the leading coefficient. The leading coefficient of "$5x^3 + 7x^2 - 3x + 2$" is "5."

Example 2. In the polynomial "$4x^2 + 7$,"

$a_n = a_2 = 4$, so that the coefficient of x^2 is 4;

$a_{n-1} = a_1 = 0$, so that the coefficient of x is 0;

$a_o = 7$, so that the constant term is "7."

Definition 2. The degree of the polynomial function $p: x \longrightarrow a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o$ ($a_n \neq 0, n \in W$) is n. This is abbreviated as $\deg(p) = n$. The degree of the associated polynomial is also n. The constant polynomial function c_o and its associated polynomial "0" have no degree.

Note: When dealing with exponents, we define $x^0 = 1$. Thus in Definition 2, $a_o = a_o x^o$. Therefore, when $n = 0$, p: $x \longrightarrow a_o$ so that $\deg(p) = 0$, and $\deg(a_o) = 0$.

Example 3. The degree of "$-2x^3 + 5x - 10$" is 3.

Example 4. $\mathrm{Deg}(c_5) = 0$, since c_5: $x \longrightarrow 5 = 5x^0$.
Also of course the degree of the polynomial "5" is zero.

Notice that in the definition of degree of a polynomial function (and polynomial) it was stated that $a_n \neq 0$.

Example 5. Find the degree of the polynomial function

$$f:\ x \longrightarrow 0x^2 + 5x - 2 .$$

At first glance, it might seem as though we have a polynomial function of degree 2. However, the first non-zero coefficient is 5, so that "$5x - 2$" is of degree one. Therefore, $\deg(f) = 1$.

7.4 Exercises

1. Find the degree of each of the following polynomial functions.

   (a) p: $x \longrightarrow 7x - 3 + 5x^3$

   (b) q: $x \longrightarrow x^5 - 82x^2 + 14$

   (c) r: $x \longrightarrow 0x^4 + 7x^3 + 0x^2 - 10x - 13$

   (d) c_8: $x \longrightarrow 8$

   (e) c_0: $x \longrightarrow 0$

2. Find the degree of each of the following polynomials.

   (a) $-x^2 + 2x + 3$

   (b) $2x + 3$

   (f) $0x^4 + 0x^3 + x^2 - 30$

   (g) $x^4 + 0x^3 + 0x^2 + 8x + 7$

(c) 3

(d) 0

(e) $15x^7 + 2x^5 - \frac{3}{4}$

(h) $\sqrt{7}x^{10}$

(i) $2\pi r$

(j) πr^2

3. In the polynomial "$-5x^3 + \frac{1}{2}x^2 + \sqrt{7}x - \frac{2}{3}$":

(a) What is the coefficient of x?

(b) What is the coefficient of x^3?

(c) What is the constant term?

(d) What is the greatest power of x appearing in the polynomial?

(e) What is the degree of the polynomial?

(f) What is the leading coefficient?

4. In the polynomial "$0x^3 - 8x^2 + 6x + 15$":

(a) What is the coefficient of x?

(b) What is the coefficient of x^2?

(c) What is the greatest power of x appearing in the polynomial?

(d) What is the greatest power of x having a non-zero coefficient?

(e) What is the degree of the polynomial?

(f) What is the leading coefficient?

5. Remembering that $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is used to represent a real polynomial, answer the following questions:

(a) In the polynomial "$5x^3 - 7x^2 + 4x - 8$," what is a_2?

(b) In the polynomial "$x^5 - 10$," what is a_0?

(c) In "$x^2 - 2$," what is a_1?

(d) In "7," what is a_1?

(e) In "$-4x^7 + 5x^6 + 7x^4 - 8x^3 + 18x - 5$," what is a_7?

6. Consider the polynomial "$5x^2 - 3x - 3$."

(a) What is a_0?

(b) What is a_1?

(c) What is a_2?

(d) What is a_3?

(e) What is a_i, $i \in W$ and $i > 2$?

7. Find the degree of each of the following polynomial functions.

(a) c_5

(b) j_R

(c) $[c_5 + j_R]$

(d) c_2

(e) c_{-2}

(f) $[c_2 + c_{-2}]$

(g) $[j_R + j_R]$

8. If the functions used to generate polynomial functions are restricted to the identity function and constant functions of type c_a, $a \in Q$, then a subset of the real polynomials, called <u>polynomials over the rational numbers</u>, is obtained. For instance,

$$\tfrac{1}{3}x + 5$$

belongs to the set of polynomials over the rational numbers, since it is generated by j_R, $c_{\frac{1}{3}}$, and c_5. However,

$$\sqrt{2}x + 5$$

does not belong to the set of polynomials over the rationals, since $c_{\sqrt{2}}$ was used in its generation.

Similarly, the set of <u>polynomials over the integers</u> is that subset of the real polynomials whose elements are generated by j_R and constant functions c_a, $a \in Z$. For instance,

$$2x + 5$$

is a polynomial over the integers, while

$$\frac{1}{2}x + 5$$

is not.

Classify each of the following by making checks in all appropriate columns.

	Polynomial over Integers	Polynomial over Rationals	Real Polynomial
$-2x^5 + 7x^2 - 8$			
$\frac{3}{4}x^2 - \sqrt{2}x + 5$			
$-x - 1$			
$x^2 + 4x + 7$			
7			
$5x + 5$			
$5x + \frac{1}{3}$			
$5x + \sqrt{3}$			
$\sqrt{4}x^2 + \sqrt{9}x + \sqrt{16}$			

7.5 **Addition of Polynomials: (P,+)**

If

$$f:\ x \longrightarrow 9x^2 + 3x - 2 \text{ and}$$
$$g:\ x \longrightarrow -5x^2 - 6x + 8$$

are real polynomial functions, what is $[f + g]$?

$$[f + g](x) = f(x) + g(x)$$
$$= (9x^2 + 3x - 2) + (-5x^2 - 6x + 8)$$
$$= 9x^2 - 5x^2 + 3x - 6x - 2 + 8$$

$= (9 + -5)x^2 + (3 + -6)x + (-2 + 8)$

$= 4x^2 - 3x + 6.$

Therefore: $[f + g]: x \longrightarrow 4x^2 - 3x + 6.$

Example 1. Using the functions above, find f(3), g(3), [f + g](3).

$$f(3) = 9(3)^2 + 3(3) - 2 = 81 + 9 - 2 = 88$$

$$g(3) = -5(3)^2 - 6(3) + 8 = -45 - 18 + 8 = -55$$

[f + g](3) then must be 88 + (-55), or 33.

Using the polynomial found above for [f + g]:

$$[f + g](3) = 4(3)^2 - 3(3) + 6 = 36 - 9 + 6 = 33$$

Since every polynomial function has a unique polynomial associated with it, addition of polynomial functions may be expressed simply by addition of polynomials; for instance:

$$(2x^2 + 3x + 7) + (-x^2 + 5) = x^2 + 3x + 12.$$

Example 2 is another illustration of addition of polynomials. You should be able to interpret it as addition of polynomial functions.

Example 2. $(5x^3 + 7x^2 + 8) + (-2x^3 - 5x^2 + 7x + 8) =$
$3x^3 + 2x^2 + 7x + 16.$

If we let P denote the set of all real polynomial functions, is (P,+) an operational system? That is, is it true that the sum of two polynomial functions is a polynomial function? Let f: $x \longrightarrow a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ and

g: $x \longrightarrow b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0$

be two polynomial functions. Then we can make the following observations about $[f + g]$:

(1) First, do you see that "$a_0 + b_0$" will appear in the associated polynomial?

(2) Also, "$a_1 x + b_1 x$," or "$(a_1 + b_1)x$" will appear in the associated polynomial. In other words, the coefficient of x in the polynomial of $[f + g]$ is simply the sum of the coefficient of x in the polynomial of f and the coefficient of x in the polynomial of g.

(3) In fact, it is not difficult to see that every power of x in the polynomial of $[f + g]$ will have a coefficient determined by adding the coefficients of that power in the polynomials of f and g. For instance, the coefficient of x^2 will be $(a_2 + b_2)$.

This sort of informal argument should make it reasonable to conclude that the sum of two polynomial functions is a polynomial function. Therefore, $(P,+)$ is an operational system.

Operational systems are best described by their properties. One property of $(P,+)$ is suggested by Examples 3 and 4.

<u>Example</u> 3. If f: $x \longrightarrow 2x + 5$ is a real function,

$$\begin{aligned} \text{then } [f + c_0](x) &= f(x) + c_0(x) \\ &= (2x + 5) + 0 \\ &= 2x + 5 = f(x). \end{aligned}$$

therefore, $[f + c_0] = f$.

Example 4. If $g: x \longrightarrow -x^7 + \frac{1}{3}x^5 + \sqrt{3}$ is a real function,

then $[c_0 + g](x) = c_0(x) + g(x)$
$= 0 + (-x^7 + \frac{1}{3}x^5 + \sqrt{3})$
$= -x^7 + \frac{1}{3}x^5 + \sqrt{3} = g(x)$

Therefore, $[c_0 + g] = g$.

The property which the examples illustrate is the identity property of $(P,+)$, the polynomial function c_0 being the identity element of the system. Also, the polynomial "0" is an identity element for addition of polynomials. (While there is a distinction between the polynomial "0" and the real number 0, we do not use different symbols for them; it should always be clear which one is intended.)

Example 5. $(7x^4 - \frac{2}{3}x + 6) + 0 = 7x^4 - \frac{2}{3}x + 6$.

Other properties of $(P,+)$ are investigated in the exercises; notice especially exercises 20, 21, 22, 23, 29.

We make one more observation about addition of polynomial functions. If $\deg(p) = n$ and $\deg(q) = m$, what is $\deg([p + q])$? The question is quite easily answered, and a specific example should make it clear. Let p and q be as follows:

$p: \quad x \longrightarrow x^5 + 3x^4 + 7x^3 + 2x^2 + 8x + 3$

$q: \quad x \longrightarrow 5x^2 + 9x + 2$

Thus, $\deg(p) = 5$, and $\deg(q) = 2$. Since the polynomial for q may also be written as "$0x^5 + 0x^4 + 0x^3 + 5x^2 + 9x + 2$," the polynomial for $[p + q]$ is

$$(1 + 0)x^5 + (3 + 0)x^4 + (7 + 0)x^3 + (2 + 5)x^2 + (8 + 9)x$$
$$+(3 + 2), \text{ or } x^5 + 3x^4 + 7x^3 + 7x^2 + 17x + 5.$$

Therefore, deg([p + q]) = 5. It could not be greater than 5, since all coefficients of both polynomials are zero for powers of x greater than 5. Thus, it might seem reasonable to assert that deg([p + q]) = max (deg(p), deg(q)). However, this is not always the case. Suppose p and q are as follows:

$$p:\quad x \longrightarrow 2x^2 + 3x + 7$$

$$q:\quad x \longrightarrow -2x^2 + 8x - 3$$

Here, max (deg(p), deg(q)) = max (2,2) = 2. However, deg ([p + q]) = 1. Sometimes, therefore, the degree of the sum of two polynomials is less than the maximum degree of the polynomials being added. We may however make the following statement:

$$\deg([p + q]) \leq \max(\deg(p), \deg(q)),$$

provided that neither p nor q nor [p + q] is the function c_o.

7.6 Exercises

1. Let f: $x \longrightarrow x^3 - 7x^2 + 3x + 4$ and

   g: $x \longrightarrow 2x^3 + 3x^2 - 7x - 4$

   be two real polynomial functions.

   (a) Find the associated polynomial for [f + g].

   (b) What is [f + g](2) ?

   Check by finding f(2) + g(2). (See Example 1 in Section 7.5.)

   (c) What is [f + g](0) ?

   Check by finding f(0) + g(0).

   (d) What is [f + g](-5) ?

   Check by finding f(-5) + g(-5).

2. If $f(x) = 17x^2 - 13x + 22$ and $g(x) = -13x^2 - 11x - 39$, then $[f + g](x) =$

3. $(-7x^3 - 12x^2 + 6x + 8) + (-6x^3 - 11x + 9) =$

4. $(-14x + 6) + (-16x - 6) =$

5. $(\frac{1}{2}x^3 - \frac{2}{3}x^2 + 7) + (\frac{3}{4}x^3 + \frac{5}{3}x - 7) =$

6. $(4x^2 - 6x + 3) + (-4x^2 + 6x - 3) =$

7. $(\sqrt{2}x^2 + \frac{3}{5}x - \sqrt{7}) + (\sqrt{2}x^2 - \sqrt{5}x + \frac{1}{2}) =$

8. $(x^{10} + 1) + (x^{10} - 1) =$

9. $(.2x^4 + .7x^3 - .4x) + (.8x^4 + .5x^3 + x^2 + .7) =$

10. $(5x^2 + \frac{4}{5}x + \frac{3}{5}) + (\frac{1}{5}x^2 + \frac{1}{5}x + \frac{1}{5}) =$

11. $(10x^4 + \frac{1}{2}x^3 + 6) + (-17x^4 + \frac{5}{6}x^2 - 7x) =$

12. $(a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) =$

13. Add the polynomials:
$$\begin{array}{r} 7x^2 - 3x + 5 \\ \underline{-2x^2 + 4x + 6} \end{array}$$

14. Add:
$$\begin{array}{rrrr} -13x^3 & - 7x^2 & & + 5 \\ \hline 2x^3 & + 3x^2 & + 4x & + 9 \end{array}$$

15. Add:
$$\begin{array}{rrrrr} 17x^4 & + 3x^3 & - 6x^2 & + 4x & - 10 \\ \hline 3x^4 & & + 5x^2 & & - 7 \end{array}$$

16. $(2x^2 + 1) + (-2x^2 - 1) =$

17. $(29x^3 - 16x^2 + 42) + (13x^3 + 6x - 17) =$

18. $(-\frac{3}{4}x^4 + \frac{1}{2}x^3 - \frac{2}{3}x^2 + \frac{1}{4}x + \frac{5}{3}) + (\frac{5}{4}x^4 + \frac{1}{2}x^3 - \frac{1}{3}x^2 + \frac{1}{2}x + \frac{5}{6}) =$

19. (a) $(-3x^2 + 5x - 9) + (-2x^2 - 18x + 2) + (-15x^2 + 22x - 8) =$

(b) $(14x^{10} + 2) + (-7x^8 + \sqrt{6}) + (6x^6 + 14) =$

(c) $(9x^2 - 14x + 3) + (-2x^2 + 11x + 5) =$

(d) $(-2x^2 + 11x + 5) + (9x^2 - 14x + 3) =$

(e) Add:

$$\begin{array}{rrrr} 3x^3 & +\ 14x^2 & -\ 8x & +\ 6 \\ & -2x^2 & +\ 15x & -\ 13 \\ 7x^3 & -\ 12x^2 & & -\ 10 \\ \hline & 8x^2 & -\ 14x & -\ 2 \end{array}$$

(f) Add the following:

$-7x^3 + x + 4$; $14x^2 - 8$; $9x^3 - 17$; $24x^3 - x^2 - x - 1$; $15x^3 + 6x^2$

20. (a) $(-3x^2 + 5x - 7) + (3x^2 - 5x + 7) =$

(b) $(-2x - 7) + (2x + 7) =$

(c) $(\frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{2}{3}) + (-\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{2}{3}) =$

(d) $(x^2 + 1) + (-x^2 - 1) =$

21. (a) $(3x^2 - 4x + 6) + (-3x^2 + 4x - 6) =$

(b) If $f: x \longrightarrow 3x^2 - 4x + 6$ is a real polynomial function, find a function g such that $[f + g] = c_0$. (Remember that c_0 is the identity element for $(P,+)$. Therefore, we may say $g = [-f]$, the inverse of f.)

22. Let $f: x \longrightarrow \frac{1}{2}x^3 - 3x + 7$ be a real function.

(a) Find $[-f]$.

(b) What is $[f + [-f]]$?

23. Let $g: x \longrightarrow 3x^8 + 14x^3 - 35x - 19$ be a real function.

(a) Find $[-g]$.

(b) What is $[g + [-g]]$?

24. $-(3x^8 + 14x^3 - 35x - 19) =$ (See Exercise 23)

25. $-(-x^2 - 7x + 5) =$

26. $-(7x^4 - 5x^3 + 8x^2 - 14x - 8) =$

27. $-(17x^3 - 8x + 9) =$

28. $-(-(-4x^2 + 9x - 10))$

29. (a) What are the properties of a commutative group (S,+)?

(b) Is addition of real polynomial functions associative?

(c) Is (P,+), where P is the set of real polynomial functions and "+" is addition of functions, a commutative group?

30. (a) Is $(P_Z,+)$, where P_Z is the set of polynomial functions over the integers, a commutative group?

(b) Is $(P_Q,+)$, where P_Q is the set of polynomial functions over the rational numbers, a commutative group?

(c) Is $(P_N,+)$, where P_N is the set of polynomial functions over the natural numbers, a commutative group?

31. If f: $x \longrightarrow 3x^2 - 7x + 14$ and g: $x \longrightarrow -2x^2 + 5x - 25$ are real polynomial functions, find the polynomial associated with [f - g].

(Hint: Since (P,+) is a group, f - g = f + (-g).)

32. $(2x^3 - 7x^2 + 15x + 3) - (-4x^3 + 7x^2 + 2x - 8) =$

33. $(1.2x^4 - 3.6x^2 - 5.4) - (3.7x^4 + 1.8x^2 - 2.6) =$

34. $(\frac{3}{4}x^2 + \frac{1}{2}x - \frac{2}{3}) - (\frac{1}{4}x^2 - \frac{5}{4}x - \frac{1}{2}) =$

35. $(\sqrt{2}x + 3) - (-\sqrt{2}x - 6) =$

36. Subtract:

$$\begin{array}{r} -2x^4 + 5x^3 - 12x^2 - 7x + 2 \\ \underline{15x^4 - 3x^3 + 7x^2 + 5x - 8} \end{array}$$

37.

$$\begin{array}{r} 5x^3 \quad\quad + 14x - 18 \\ \underline{3x^2 - 6x \quad\quad} \end{array}$$

38. Consider the following real polynomial functions:

f: $x \longrightarrow 5x^3 - 7x + 5$

$g: \; x \longrightarrow -3x^2 + 4x - 7$

$h: \; x \longrightarrow x^3 - x^2 + x - 1$

Find the polynomial associated with each of the following functions:

(a) $[[f + g] - h]$ (d) $[[f - g] - h]$

(b) $[f + [g - h]]$ (e) $[[g - h] + f]$

(c) $[f - [g - h]]$ (f) $[[h - g] + f]$

39. For each of the following, give the degree of $[f + g]$. If, in any case, it is not possible to determine the degree with the information given, explain why.

(a) $\text{def}(f) = 5$, $\deg(g) = 2$

(b) $\deg(f) = 0$, $\deg(g) = 3$

(c) $\deg(f) = 6$, $\deg(g) = 6$

40. In the inequality $\deg([p + q]) \leq \max(\deg(p), \deg(q))$, why is it necessary to require that neither p nor q nor $[p + q]$ be the function c_o?

41. What is the subgroup relationship among the groups $(P,+)$, $(P_Z,0)$, and $(P_Q,+)$?

7.7 Multiplication of Polynomial Functions: $(P,+,\cdot)$

Suppose that

$f: \; x \longrightarrow x + 2$

and $g: \; x \longrightarrow x + 3$

are two real polynomial functions. Then

$$\begin{aligned}[f \cdot g](x) &= f(x) \cdot g(x) \\ &= (x + 2) \cdot (x + 3) \\ &= (x + 2)x + (x + 2)3 \\ &= x^2 + 2x + 3x + 6\end{aligned}$$

$$= x^2 + 5x + 6$$

Thus, the product function $[f \cdot g]$ may be characterized as follows:

$$[f \cdot g]: \quad x \longrightarrow x^2 + 5x + 6.$$

In the problem considered above, we may say that the product of the two given polynomial functions is a polynomial function; or the product of the polynomials "$x + 2$" and "$x + 3$" is the polynomial "$x^2 + 5x + 6$." Is the product of every two polynomial functions a polynomial function? Two special cases are discussed below in order to suggest an answer to this question.

1) Let

$$r: \quad x \longrightarrow a_1 x + a_0 \text{ and}$$

$$s: \quad x \longrightarrow b_1 x + b_0$$

be two real polynomial functions of the first degree. That is, a_1, a_0, b_1, and b_0 are real numbers, with a_1 and b_1 not zero. Then,

$[r \cdot s](x) = r(x) \cdot s(x) =$

$$(a_1 x + a_0)(b_1 x + b_0) = (a_1 x + a_0)(b_1 x) + (a_1 x + a_0)(b_0)$$
$$= (a_1 b_1)x^2 + (a_0 b_1)x + (a_1 b_0)x + (a_0 b_0)$$
$$= (a_1 b_1)x^2 + (a_0 b_1 + a_1 b_0)x + (a_0 b_0).$$

Since all of the coefficients in the result are real numbers and $a_1 b_1 \neq 0$ (why?) we can say that the product of two real polynomials of first degree is a real polynomial of second degree. Also, of course, the product of two real polynomial functions of first degree is a real polynomial function of second degree.

2) $g: \quad x \longrightarrow a_3 x^3 + a_2 x^2 + a_1 x + a_0$ and

$h: \quad x \longrightarrow b_2 x^2 + b_1 x + b_0$

are two real polynomial functions with a_3 and b_2 not zero. Then

$[g \cdot h](x) = g(x) \cdot h(x) =$

$(a_3x^3 + a_2x^2 + a_1x + a_0)(b_2x^2 + b_1x + b_0) =$

$(a_3x^3 + a_2x^2 + a_1x + a_0)(b_2x^2) + (a_3x^3 + a_2x^2 + a_1x + a_0)(b_1x) +$

$(a_3x^3 + a_2x^2 + a_1x + a_0)(b_0) =$

$a_3b_2x^5 + a_2b_2x^4 + a_1b_2x^3 + a_0b_2x^2 + a_3b_1x^4 + a_2b_1x^3 + a_1b_1x^2 +$

$a_0b_1x + a_3b_0x^3 + a_2b_0x^2 + a_1b_0x + a_0b_0 =$

$(a_3b_2)x^5 + (a_3b_1 + a_2b_2)x^4 + (a_3b_0 + a_2b_1 + a_1b_2)x^3 +$

$(a_2b_0 + a_1b_1 + a_0b_2)x^2 + (a_1b_0 + a_0b_1)x + (a_0b_0).$

Since all of the coefficients in the result are real numbers, with $a_3b_2 \neq 0$, we see that the product of a third degree real polynomial function and a second degree real polynomial function is a real polynomial function of degree five.

Although the above two particular proofs do not constitute a general proof, perhaps they do make reasonable the conclusion that the product of two real polynomials is a real polynomial, and therefore:

If f and g are real polynomial functions,

then $[f \cdot g]$ is a real polynomial function.

Furthermore, $\deg([f \cdot g]) = \deg(f) + \deg(g)$, provided that neither f nor g is the function c_0.

Example 1. "$\frac{1}{2}x^4 + 3x^2 - 5$" is a polynomial of degree 4.

"$x^2 + \frac{1}{4}$" is a polynomial of degree 2.

$(\frac{1}{2}x^4 + 3x^2 - 5)(x^2 + \frac{1}{4}) =$

$(\frac{1}{2}x^4 + 3x^2 - 5)(x^2) + (\frac{1}{2}x^4 + 3x^2 - 5)(\frac{1}{4}) =$

$\frac{1}{2}x^6 + 3x^4 + (-5)x^2 + \frac{1}{8}x^4 + \frac{3}{4}x^2 + (-5)(\frac{1}{4}) =$

$\frac{1}{2}x^6 + 3x^4 - 5x^2 + \frac{1}{8}x^4 + \frac{3}{4}x^2 - \frac{5}{4} =$

$\frac{1}{2}x^6 + \frac{25}{8}x^4 - \frac{17}{4}x^2 - \frac{5}{4}$.

Note that the degree of the result is $4 + 2 = 6$.

<u>Example</u> 2. $(x + 3)(x + 2) = x^2 + 5x + 6$, as discussed earlier. Sometimes the following "vertical" arrangement is useful.

$$\begin{array}{l}
x + 3 \\
\underline{x + 2} \\
x^2 + 3x \\
\underline{\quad\;\; + 2x + 6} \\
x^2 + 5x + 6
\end{array}$$

<u>Example</u> 3. Find the product $(-5x^3 - x + 6)(x^2 - 2)$.

We must remember that we are actually finding the product of two functions:

$$f: \quad x \longrightarrow -5x^3 - x + 6 \text{ and}$$

$$g: \quad x \longrightarrow x^2 - 2.$$

For every x, $[f \cdot g](x) = (-5x^3 - x + 6)(x^2 - 2)$.

$$\begin{array}{l}
-5x^3 - x + 6 \\
\underline{\qquad\quad x^2 - 2} \\
-5x^5 - x^3 + 6x^2 \\
\underline{\qquad\;\; +10x^3 \qquad\qquad + 2x - 12} \\
-5x^5 + 9x^3 + 6x^2 + 2x - 12
\end{array}$$

With our acceptance of the fact that the product of every two real polynomials is a real polynomial, $(P, \cdot)$, where P is the set of real polynomial functions, is an operational system. In Course II the associative property for multiplication of functions was indicated. The commutative property can be similarly displayed. Thus for arbitrary functions f, g, h:

$$[[f \cdot g] \cdot h] = [f \cdot [g \cdot h]]$$

$$[f \cdot g] = [g \cdot f]$$

That is, multiplication of functions is commutative and associative; and since polynomial functions are simply a subset of all real functions, $(P, \cdot)$ certainly possesses these two properties. Another property is suggested by Examples 4 and 5 below.

Example 4. Let $f: \quad x \longrightarrow x + 5$ be a real function.

$$\begin{aligned} \text{Then } [f \cdot c_1](x) &= f(x) \cdot c_1(x) \\ &= (x + 5) \cdot 1 \\ &= x + 5 = f(x) \end{aligned}$$

Therefore, $[f \cdot c_1] = f$.

Example 5. Let $g: \quad x \longrightarrow x^2 - 3$ be a real function.

$$\begin{aligned} \text{Then } [c_1 \cdot g](x) &= c_1(x) \cdot g(x) \\ &= 1 \cdot (x^2 - 3) \\ &= x^2 - 3 = g(x). \end{aligned}$$

Therefore, $[c_1 \cdot g] = g$.

Do you see that the constant function c_1 is an identity element for $(P, \cdot)$? That is,

For every $f \in P$, $[c_1 \cdot f] = [f \cdot c_1] = f$.

Is $(P, \cdot)$ a commutative group? We have already established associativity, commutativity, and existence of an identity element. In order to have a group structure, we must show the existence of an inverse for each element in P. Let us take for instance the polynomial function

$$p: \quad x \longrightarrow x^2.$$

Does this function have an inverse in $(P, \cdot)$? If there is an inverse polynomial function -- call it q -- then we must have:

$$[p \cdot q] = c_1.$$

However, this is impossible. p is of degree 2, and c_1 is of degree 0. Therefore, the degree of q -- call it n -- would have to be such that $2 + n = 0$. However, there is no polynomial of degree -2, and therefore we conclude that the polynomial function p has no inverse in $(P,\cdot)$. Therefore, $(P,\cdot)$ is not a group.

<u>Question.</u> Can you identify some polynomial functions that do have inverses in $(P,\cdot)$?

We have now (in this section and in Sections 7.5 and 7.6) discussed two operations on the set P of real polynomial functions. We have therefore a two-fold operational system $(P,+,\cdot)$. Let us summarize all of the properties we have discussed together with the following property discussed in Course II:

For every $f, g, h \in P$, $[f\cdot[g + h]] = [[f\cdot g] + [f\cdot h]]$

That is, multiplication distributes over addition.

Let f, g, and h be elements of the set P of real polynomial functions. Then:

(1) $[[f + g] + h]=[f + [g + h]]$

(2) $[c_0 + f]=[f + c_0]= f$

(3) $[f + [-f]] =[[-f]+ f]= c_0$

(4) $[f + g]=[g + f]$

(5) $[[f \cdot g] \cdot h]=[f \cdot [g \cdot h]]$

(6) $[c_1 \cdot f]=[f \cdot c_1]= f$

(7) $[f \cdot g]=[g \cdot f]$

(8) $[f \cdot [g + h]]=[[f \cdot g] + [f \cdot h]]$

$(P,+,\cdot)$ is not a field. (Which property is missing?)

Properties (1), (2), (3), (4) yield that $(P,+)$ is a commutative group. Properties (7) and (8) imply

(9) $[[g + h]\cdot f] = [[g\cdot f] + [h\cdot f]]$

We can now see that $(P,+,\cdot)$ is a ring with unity, as defined in Chapter 3. A ring in which multiplication is commutative (property (7) above) is called a <u>commutative ring</u>. Thus $(P,+,\cdot)$ is a commutative ring with unity.

7.8 Exercises

1. Let

$f: \; x \longrightarrow x - 2$ and

$g: \; x \longrightarrow x^2 + 3x + 4$

be two real functions

(a) Find the polynomial associated with $[f \cdot g]$.

(b) Find $[f \cdot g](2)$. Check by finding $f(2) \cdot g(2)$.

(c) Find $[f \cdot g](-3)$. Check by finding $f(-3) \cdot g(-3)$.

(d) Find $[f \cdot g](\frac{1}{2})$. Check by finding $f(\frac{1}{2}) \cdot g(\frac{1}{2})$.

2. $(x + 2)(x + 5) =$

3. $(x + 2)(x - 5) =$

4. $(x - 2)(x + 5) =$

5. $(x - 2)(x - 5) =$

6. $(2x + 3)(x + 7) =$

8. $(2x^2 + x + 1)(x - 8)$

9. $(x^2 + 7x + 8)(x^2 - 3x + 5) =$

10. $(4x^3 - 7)(3x^2 + 7x + 8)$

11. $(\frac{1}{2}x + \frac{1}{4})(\frac{2}{3}x - \frac{1}{5}) =$

12. $(\frac{7}{8}x + \frac{1}{2})(\frac{3}{8}x - \frac{1}{4}) =$

13. $(.2x + .5)(.3x - .7) =$

14. $(2x^6 + 5)(x^4 - 3x^2 + 5)$

15. $(x^7 - 2)(x^8 + 8) =$

16. $(x + 7)^2 =$

17. $(x - 8)^2 =$

18. $(3x - 10)^2 =$

19. $(2x + 5)^2 =$

20. $(y + 4)^2 =$

21. $(a - 9)^2 =$

22. $(t + \frac{1}{2})^2 =$

23. $(x + \sqrt{2})^2 =$

24. $(t + 16)^2 =$

25. $(x + b)^2 =$

26. $(ax + b)^2 =$

27. $(x^2 + 2x + 1)^2 =$

28. $(ax^2 + bx + c)^2 =$

29. $(y - 4)(y + 4) =$

30. $(x + 6)(x - 6)$

31. $(t + \frac{1}{3})(t - \frac{1}{3}) =$

32. $(a + .6)(a - .6) =$

33. $(2x + 7)(2x - 7) =$

34. $(3x + 4)(3x - 4) =$

35. $(6a + 7)(6a - 7) =$

36. $(\frac{1}{2}x + \frac{2}{5})(\frac{1}{2}x - \frac{2}{5}) =$

37. $(x + \sqrt{5})(x - \sqrt{5}) =$

38. $(3t + \sqrt{6})(3t - \sqrt{6}) =$

39. $(x + b)(x - b) =$

40. $(ax + b)(ax - b) =$

41. $(x + 2)(x^2 - 2x + 4) =$

42. $(a - 5)(a^2 + 5a + 25) =$

43. $(z + 2)(z - 3)(z^2 - 4z + 16) =$

44. $(n + 7)(2n + 1)(4n^2 - 2n + 1) =$

45. $(a + b)(a^2 + ab + b^2) =$

46. $(a - b)(a^2 + ab + b^2) =$

47. **Multiply:**

$$\begin{array}{r} 3x^6 - 2x^5 + 5x^4 + 3x^3 - 7x^2 + 7x + 8 \\ \underline{10x^3 + 6x^2 - 3x - 5} \end{array}$$

48. **Multiply:**

$$\begin{array}{r} -5x^{10} + 3x^7 - 2x^4 - 12 \\ \underline{x^5 - x^3 - 9} \end{array}$$

49. $-(3x^5 - 2x + 7) =$

50. $(-1)(3x^5 - 2x - 7) =$

51. $(-1)(-x^2 + \sqrt{5}) =$

52. $-(-x^2 + \sqrt{5}) =$

53. $(-1)(2x - 7)(3x + 8) =$

54. Let

$f: x \longrightarrow x - 4$

$g: x \longrightarrow x^2 + 5x + 6$

$h: x \longrightarrow -x^3 + 3$

be real functions. Find the polynomial associated with each of the following functions.

(a) $[[g \cdot h] + f]$

(b) $[g \cdot [h + f]]$

(c) $[c_0 \cdot g]$

(d) $[c_1 \cdot g]$

(e) $[[f - g] \cdot h]$

(f) $[c_{-1} \cdot g]$

(g) $[-g]$

(h) $[g + c_1]$

(i) $[g + c_0]$

(j) $[[f \cdot g] \cdot h]$

(k) $[h \cdot h]$

(l) $[[h \cdot h] - [g \cdot g]]$

(m) $[-h]$

(n) $[c_{-1} \cdot h]$

55. Let f, g, and h be the following real polynomial functions:

$$f: \quad x \longrightarrow x^3$$

$$g: \quad x \longrightarrow x^2 + 1$$

$$h: \quad y \longrightarrow 2x^5 + x$$

(a) Find the polynomial associated with $f \circ g$. ("o" means composition.)

(b) Find the polynomial associated with $j_R \circ g$.

(c) Find the polynomial associated with $h \circ j_R$.

(d) Find the polynomial associated with $g \circ h$.

(e) Find the polynomial associated with $h \circ g$.

(f) Is (P, o) an operational system?

(g) What is the identity element for (P, o)?

(h) If deg(p) = m and deg(q) = n, what is deg(poq)? deg(qop)?

56. Look in the text at the properties of $(P,+,\cdot)$, which are the defining properties of a commutative ring with unity. Then decide which of the following two-fold operational systems are commutative rings with unity.

(a) $(W,+,\cdot)$

(b) $(Z,+,\cdot)$

(c) (Even integers,$+,\cdot$)

(d) $(Q,+,\cdot)$

(e) $(Z_4,+,\cdot)$

(f) $(Z_8,+,\cdot)$

(g) (2 x 2 matrices,$+,\cdot$)

(h) any field

(i) $(N,+,\cdot)$

*57. For the systems in Exercise 58 which are not commutative rings with unity, state which of the eight defining properties hold and which do not.

7.9 Division of Polynomial Functions

Addition, subtraction, and multiplication are operations on the set P of real polynomial functions. We study these operations in terms of polynomials. What about division? How, for instance can we interpret

$$(x^2 + 3x) \div x \text{ ?}$$

In a multiplicative group $(G,\cdot)$, any division $a \div b$ can be interpreted as $a \cdot b^{-1}$, where b^{-1} is the multiplicative inverse of b. (For example, in $(Q,\cdot)$, $\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \cdot \frac{5}{4}$.) However, $(P,\cdot)$ is not a group (see Section 7.7), and so such an interpretation here is without meaning.

We also have interpreted

$$a \div b$$

in other multiplicative systems in the following way:

$a \div b = c$ if and only if $c \cdot b = a$.

This interpretation is a sensible one in the case of the polynomial division problem above. We may reason as follows:

$$(x^2 + 3x) \div x = q(x)$$

if and only if

$$q(x) \cdot x = x^2 + 3x.$$

The distributive property makes it easy to see that $q(x)$ must be $x + 3$. That is,

$$(x^2 + 3x) \div x = x + 3$$

since

$$(x + 3) \cdot x = x^2 + 3x$$

The same division is shown below in a form that will be useful in some later examples.

$$\begin{array}{r|l} & x + 3 \\ \hline x & x^2 + 3x \\ & \underline{x^2 + 3x} \end{array}$$

Suppose we alter the above problem slightly, as follows:

$$(x^2 + 3x + 2) \div x.$$

There is in fact no polynomial function q such that $q(x) \cdot x = x^2 + 3x + 2$ for all $x \in R$.

This situation is something like that in $(W, \cdot)$, where not every whole number divides every other. For example, given $14 \div 3$, we can say that there is no whole number a such that $a \cdot 3 = 14$; thus, 3 does not divide 14, and 3 is not a factor of 14. We do however make use of the following division algorithm:

$$\begin{array}{rl} & \;\;4 \quad \text{Quotient} \\ \text{Divisor} \;\; \underline{3} & \overline{\big)\;14} \\ & \;\;\underline{12} \\ & \;\;\;2 \quad \text{Remainder} \end{array}$$

Thus, $14 = (4 \cdot 3) + 2$. Recall that in this algorithm, we demand that the remainder be less than the divisor; in this case, $2 < 3$.

A similar algorithm exists for polynomial division, as illustrated below for $(x^2 + 3x + 2) \div x$.

$$\begin{array}{rl} & \;\;x + 3 \qquad \text{Quotient} \\ \text{Divisor} \;\; \underline{x} & \overline{\big)\;x^2 + 3x + 2} \\ & \;\;\underline{x^2 + 3x \qquad} \\ & \qquad\qquad 2 \quad \text{Remainder} \end{array}$$

Notice that, for all $x \in R$, $x^2 + 3x + 2 = ((x + 3) \cdot x) + 2$. We cannot say that the remainder, which is the polynomial "2," is less than the divisor, which is the polynomial "x," since we have not ordered the polynomial functions. We can however say that the degree of the polynomial "2" is less than the degree of the polynomial "x."

Example 1. Let $f: x \longrightarrow x^2 + 8x + 5$ and

$p: x \longrightarrow x + 2$

be two real polynomial functions.

Find two polynomial functions q and r such that

$$f = [[q \cdot p] + r]$$

and

$$\deg(r) < \deg(p).$$

$$\begin{array}{r|l} & \quad x \\ \hline x + 2 & x^2 + 8x + 5 \\ & \underline{x^2 + 2x} \\ & \qquad 6x + 5 \end{array}$$

$x^2 + 8x + 5 = x(x + 2) + (6x + 5)$. However, $\deg(6x + 5) \not< \deg(x + 2)$. So the process is continued.

$$\begin{array}{r|l} & \quad x + 6 \\ \hline x + 2 & x^2 + 8x + 5 \\ & \underline{x^2 + 2x} \\ & \qquad 6x + 5 \\ & \qquad \underline{6x + 12} \\ & \qquad\quad -7 \end{array}$$

$x^2 + 8x + 5 = (x + 6)(x + 2) + (-7)$. And $\deg(-7) < \deg(x + 2)$.

Thus, the two desired functions are

$q: x \longrightarrow x + 6$ and

$r: x \longrightarrow -7$.

The preceding example and discussion suggest the following theorem whose proof is omitted.

Theorem 1. Given two real polynomial functions f and p, $p \neq c_0$, there exist unique real polynomial functions q and r, with $r = c_0$ or $\deg(r) < \deg(p)$, such that $f = [(p \cdot q) + r]$.
Thus, for all $x \in R$, $f(x) = (p(x) \cdot q(x)) + r(x)$.

Notice the word "unique" in the statement of the theorem. This is a word we have used several times before; it means that it is not possible to find more than one pair of functions, q and r, meeting the required conditions.

Example 2. p: $x \longrightarrow 2x^2 - 5$

f: $x \longrightarrow 3x^3 - 4x^2 + 7x + 10.$

Find two real polynomial functions, q and r, with $r = c_0$ or $\deg(r) < \deg(p)$, such that $f = [(q \cdot p) + r]$.

$$\begin{array}{r|l}
 & \frac{3}{2}x - 2 \\ \hline
2x^2 - 5 & 3x^3 - 4x^2 + 7x + 10 \\
 & 3x^3 \qquad\quad - \frac{15}{2}x \\ \hline
 & \qquad -4x^2 + \frac{29}{2}x + 10 \\
 & \qquad -4x^2 \qquad\quad + 10 \\ \hline
 & \qquad\qquad \frac{29}{2}x
\end{array}$$

For all $x \in R$,

$3x^3 - 4x^2 + 7x + 10 = (2x^2 - 5)(\frac{3}{2}x - 2) + (\frac{29}{2}x).$

$\deg(\frac{29}{2}x) = 1$; $\deg(2x^2 - 5) = 2$; thus $\deg(r) < \deg(p)$.

Example 3.
$$\begin{array}{r|l}
 & 0 \\ \hline
x^2 - 7x + 8 & x - 5 \\
 & 0 \\ \hline
 & x - 5
\end{array}$$

Notice that the quotient here is the zero polynomial, and r = f. Clearly $\deg(x - 5) < \deg(x^2 - 7x + 8)$.

Example 4.
$$\begin{array}{r|l}
 & x + 5 \\ \hline
x - 3 & x^2 + 2x - 15 \\
 & x^2 - 3x \\ \hline
 & \qquad 5x - 15 \\
 & \qquad 5x - 15 \\ \hline
 & \qquad\qquad 0
\end{array}$$

Notice here the remainder is the zero polynomial. And we have, for all $x \in R$,

$$x^2 + 2x - 15 = (x + 5)(x - 3) + 0.$$

Since c_0 is the identity in $(P,+)$, we may write simply:

$$x^2 + 2x - 15 = (x + 5)(x - 3).$$

Looking at Example 4, we may say that "x - 3" <u>is a factor of</u> "$x^2 + 2x - 15$." We may also say that "x - 3" <u>divides</u> "$x^2 + 2x - 15$."

<u>Definition</u> 3. A polynomial function p <u>divides</u> a polynomial function f if and only if there exists a polynomial function q such that $f = [q \cdot p]$.

<u>Example</u> 5. Does $(x - r)$ divide $x^4 - r^4$, where $r \in R$?

$$
\begin{array}{r|lllll}
\multicolumn{1}{r}{} & x^3 & + x^2r & + xr^2 & + r^3 & \\ \cline{2-6}
x - r & x^4 & & & & - r^4 \\
 & \underline{x^4} & \underline{- x^3r} & & & \\
 & & x^3r & & & - r^4 \\
 & & \underline{x^3r} & \underline{- x^2r^2} & & \\
 & & & x^2r^2 & & - r^4 \\
 & & & \underline{x^2r^2} & \underline{- xr^3} & \\
 & & & & xr^3 & - r^4 \\
 & & & & \underline{xr^3} & \underline{- r^4} \\
 & & & & & 0
\end{array}
$$

Therefore, $x^4 - r^4 = (x^3 + x^2r + xr^2 + r^3)(x - r)$.

$x - r$ divides $x^4 - r^4$.

Example 5 in fact suggests a general theorem which may be stated in the following way:

<u>Theorem</u> 2. $(x - r)$ divides $(x^n - r^n)$, where "$x - r$" and "$x^n - r^n$" are real polynomials, $n \in N$.

While we shall not give a formal proof of this theorem, it is easy to give an informal argument based on multiplication. Consider the product $(x - r)(x^{n-1} + x^{n-2}r + x^{n-3}r^2 + \dots + xr^{n-2} + r^{n-1})$.

$$\begin{array}{lllllll}
x^{n-1} & + x^{n-2}r & + x^{n-3}r^2 & + \dots & + xr^{n-2} & + r^{n-1} & \\
 & & & & & \underline{x - r} & \\
\hline
x^n & + x^{n-1}r & + x^{n-2}r^2 & + \dots & + x^2r^{n-2} & + xr^{n-1} & \\
 & - x^{n-1}r & - x^{n-2}r^2 & - \dots & - x^2r^{n-2} & - xr^{n-1} & - r^n \\
\hline
x^n & & & & & & - r^n
\end{array}$$

<u>Example</u> 6. $(x - r)$ divides $(x^5 - r^5)$.

$x^5 - r^5 = (x^4 + x^3r + x^2r^2 + xr^3 + r^4)(x - r)$.

7.10 <u>Exercises</u>

1. Let $f\colon\ x \longrightarrow x^2 + 7x + 5$ and

   $p\colon\ x \longrightarrow x - 3$

   be real polynomial functions. Find two real polynomial functions q and r such that $f = [[q \cdot p] + r]$, and $r = c_0$ or $\deg(r) < \deg(p)$.

   What is $\deg(p)$? What is $\deg(r)$?

2. Let $f\colon\ x \longrightarrow 3x^4 - x^3 + 5x^2 - 7$ and

   $p\colon\ x \longrightarrow x^2 + 2x + 5$

   be real polynomial functions. Find two real polynomial functions q and r such that $f = [[q \cdot p] + r]$, and $r = c_0$ or $\deg(r) < \deg(p)$.

   What is $\deg(p)$? What is $\deg(r)$?

3. Let $f: x \longrightarrow x^5 + 12$ and

$p: x \longrightarrow x^2$

be real polynomial functions. Find two real polynomial functions q and r such that $f = [[q \cdot p] + r]$, and $r = c_0$ or $\deg(r) < \deg(p)$.

In Exercises 4 - 20, find $q(x)$ and $r(x)$ so that for all $x \in R$, $f(x) = (q(x)p(x)) + r(x)$, and $r = c_0$ or $\deg(r) < \deg(p)$.

4. $f(x) = x^3$ $\quad p(x) = x$
5. $f(x) = x$ $\quad p(x) = x^3$
6. $f(x) = x^2 - 5$ $\quad p(x) = x - 2$
7. $f(x) = x^3 - 8$ $\quad p(x) = x - 2$
8. $f(x) = x - 2$ $\quad p(x) = x^3 - 8$
9. $f(x) = x^6 - 7x^5 + 14x^4 - 5x^3 + 8x^2 - 3x + 5$ $\quad p(x) = 2$
10. $f(x) = 3x^2 + 7x - 2$ $\quad p(x) = 3x + 1$
11. $f(x) = x^2 - 6x + 9$ $\quad p(x) = x - 3$
12. $f(x) = x^5 - 3x^4 + 8$ $\quad p(x) = x^5 - 3x^4 + 8$
13. $f(x) = 4x^2 + 7x - 3$ $\quad p(x) = 2x + 3$
14. $f(x) = 6x^3 + 5x^2 - 8x + 4$ $\quad p(x) = 3x - 2$
15. $f(x) = 5x^6 - 2x^5 + 5x^4 - 17x^3 + 41x^2 - 19x - 2$

    $p(x) = x^5 + x^3 - 3x^2 + 7x + 1$
16. $f(x) = 5x^6 - 2x^5 + 5x^4 - 17x^3 + 41x^2 - 19x - 2$ $\quad p(x) = 5x - 2$
17. $f(x) = 4x^2 + 12x + 9$ $\quad p(x) = 2x + 3$
18. $f(x) = 4x^2 + 10x + 9$ $\quad p(x) = 2x + 3$
19. $f(x) = 2x^4 - \frac{23}{2}x^3 + 14x^2 + 4x - 16$ $\quad p(x) = 4x^3 - 7x^2 + 8$
20. $f(x) = x^3 + 27$ $\quad p(x) = x + 3$
21. Let $f: x \longrightarrow x^2 - 7x + 3$ and

    $p: x \longrightarrow x - 2$

    be two real polynomial functions.

(a) Find polynomial functions q and r, with $r = c_0$ or deg(r) < deg(p), such that $f = [[q \cdot p] + r]$

(b) Show that $f(x) = (q(x)p(x)) + r(x)$ when $x = 5$

(c) Show that $f(x) = (q(x)p(x)) + r(x)$ when $x = -2$

(d) Show that $f(x) = (q(x)p(x)) + r(x)$ when $x = 2$

22. Let f: $x \longrightarrow 2x^3 - 5x^2 - 8x + 14$ and

p: $x \longrightarrow x + 5$

be two real polynomial functions.

(a) Find polynomial functions q and r, with $r = c_0$ or deg(r) < deg(p), such that $f = [[q \cdot p] + r]$.

(b) Show that $f(x) = (q(x)p(x)) + r(x)$ when $x = 1$.

(c) Show that $f(x) = (q(x)p(x)) + r(x)$ when $x = 0$.

(d) Show that $f(x) = (q(x)p(x)) + r(x)$ when $x = -5$.

23. f: $x \longrightarrow x^3 - 12x^2 + 38x + 8$ and

p: $x \longrightarrow x - 5$

are two real polynomial functions.

If q: $x \longrightarrow x^2 - 7x$ and r: $x \longrightarrow 3x + 8$, then $f = [[q \cdot p] + r]$.

also,

if q: $x \longrightarrow x^2 - 7x + 3$ and r: $x \longrightarrow 23$, then $f = [[q \cdot p] + r]$.

Explain why this does not contradict the word "unique" in Theorem 1.

24. Recall what is meant by saying that a polynomial p divides a polynomial f. (See the definition in Section 7.9). Then answer "true" or "false" to each of the following statements concerning real polynomials.

(a) x^2-3 divides x^2-3

(b) $x+6$ divides $x^2+12x+36$

(c) $x+6$ divides $x^2+12x+30$

(d) $x-2$ divides x^2-4

(e) $x+2$ divides x^2+4

(f) 7 divides x^2+4

(g) 0 divides x^2+4

(h) $x-5$ divides $x^2-10x+25$

(i) $x^2-10x+25$ divides x^3-10x^2+25x

(j) $x-5$ divides x^3-10x^2+25x

(k) $x-a$ divides x^3-a^3

(l) $x+a$ divides x^3+a^3

(m) $x-\frac{1}{2}$ divides $x^2-\frac{1}{4}$

(n) $x+\frac{1}{2}$ divides $x^2+\frac{1}{4}$

(o) If $\deg(p) > \deg(f)$, then p does not divide f.

(p) If $\deg(p) = \deg(f)$, then p divides f.

25. Answer "true" or "false" to the following statements about the relation "divides" in the set W of whole numbers.

(a) The whole number 1 divides every whole number.

(b) Every whole number except 0 divides itself.

(c) If a divides b, then b divides a.

(d) The whole number 5 divides every whole number.

(e) If a divides b, and b divides c, then a divides c.

26. Answer "true" or "false" to the following statements about the relation "divides" in the set P of real polynomial functions.

(a) The function c_1 divides every real polynomial function.

(b) Every real polynomial function except c_0 divides itself.

(c) If f divides g, then g divides f.

(d) The function c_5 divides every real polynomial function.

(e) If f divides g, and g divides h, then f divides h.

27. Show that "$x - r$," $r \in R$, divides "$x^6 - r^6$" by finding $q(x)$ such that $q(x)\cdot(x - r) = x^6 - r^6$.

28. Show that "$x - r$," $r \in R$, divides "$x^7 - r^7$" by finding $q(x)$ such that $q(x)\cdot(x - r) = x^7 - r^7$.

29. From Theorem 2 we know that $x - r$ divides $x^3 - r^3$, where $r \in R$.

    (a) If $r = 2$, we have: $x - 2$ divides $x^3 - 8$

    Find $q(x)$ such that $q(x)\cdot(x - 2) = x^3 - 8$.

    (b) If $r = -2$, we have: $x + 2$ divides $x^3 + 8$.

    Find $q(x)$ such that $q(x)\cdot(x + 2) = x^3 + 8$.

    (c) If $r = 0$, we have: x divides x^3.

    Find $q(x)$ such that $q(x)\cdot(x) = x^3$.

30. Find, either by the division algorithm or by using Theorem 2, $q(x)$ such that $q(x)\cdot(x - r) = x^{10} - r^{10}$.

7.11 Polynomial Factors and The Factor Theorem

In Chapter 4 of Course II, entitled "Fields," certain expressions -- which we may now call real polynomials of degree 2 -- were factored. For instance, the polynomial

$$x^2 + 3x - 10$$

may be expressed as the product

$$(x + 5)(x - 2).$$

This means that for every $x \in R$:

$$x^2 + 3x - 10 = (x + 5)(x - 2).$$

It also means of course that the real polynomial function

$$f: \quad x \longrightarrow x^2 + 3x - 10$$

is the product of the following two real polynomial functions:

$$g: \quad x \longrightarrow x + 5$$

$$h: \quad x \longrightarrow x - 2$$

Factoring can also be used in solving certain equations, a procedure studied in Course II and reviewed in Example 1 below.

Example 1. Solve "$x^2 + 3x - 10 = 0$."

(The domain is to be taken as the set R of real numbers.)

For every $x \in R$, $x^2 + 3x - 10 = (x + 5)(x - 2)$.

Therefore,

$$x^2 + 3x - 10 = 0$$
if and only if
$$(x + 5)(x - 2) = 0$$
if and only if
$$x + 5 = 0 \text{ or } x - 2 = 0$$
if and only if
$$x = -5 \text{ or } x = 2.$$

The solution set is $\{-5, 2\}$.

Question. Is it possible to solve the equation

$$3x^2 + 14x + 8 = 0$$

by making use of factoring?

The answer to the question above depends upon whether or not we can find useful factors of "$3x^2 + 14x + 8$." We could, for instance, express "$3x^2 + 14x + 8$" as "$\frac{1}{2}(6x^2 + 28x + 16)$." However this is not useful since solving "$\frac{1}{2}(6x^2 + 28x + 16) = 0$" is no easier than the original problem. What kinds of factors then are useful? The original polynomial is a polynomial over the integers (that is, the coefficients are integers). In such a case we most frequently want the factors to also be polynomials with integral coefficients. In particular, in factoring a second degree polynomial over the integers, we look for factors of form $(ax + b)(cx + d)$, where a, b, c, and d are integers.

Since factoring is multiplication "in reverse," let us begin by multiplying

$$(ax + b)(cx + d),$$

where "$ax + b$" and "$cx + d$" are first degree polynomials over the integers.

$$\begin{aligned}(ax + b)(cx + d) &= (ax + b)(cx) + (ax + b)(d)\\ &= (ac)x^2 + (ad + bc)x + (bd).\end{aligned}$$

We may think of the coefficients of this product in terms of two integers R and S such that:

$$R = ad$$
$$S = bc.$$

Then we have:

$$\begin{aligned}R + S &= ad + bc \quad (= \text{coefficient of "}x\text{"})\\ R \cdot S &= (ad)(bc)\\ &= (ac)(bd) \quad (= \text{product of coefficient of "}x^2\text{" and the constant term})\end{aligned}$$

<u>Example</u> 2. Find factors of the form

$$(ax + b)(cx + d) \qquad (a, b, c, d \in Z)$$

for the polynomial $3x^2 + 14x + 8$.

If there are such factors then there must be two integers R and S such that

$$\begin{aligned}R + S &= 14 && (\text{coefficient of "}x\text{"})\\ R \cdot S &= 3 \cdot 8 && (\text{product of coefficient of "}x^2\text{" and the constant term.})\\ &= 24\end{aligned}$$

Since $12 + 2 = 14$, and $12 \cdot 2 = 24$, there are two such integers. Let $R = 12$, $S = 2$. Then we have

$$\begin{aligned} 3x^2 + 14x + 8 &= 3x^2 + (R + S)x + 8 \\ &= 3x^2 + (12 + 2)x + 8 \\ &= 3x^2 + 12x + 2x + 8 \\ &= 3x(x + 4) + 2(x + 4) \\ &= (3x + 2)(x + 4) \end{aligned}$$

We have therefore "factored" the polynomial "$3x^2 + 14x + 8$."

Question. In Example 2, would it matter if you let $R = 2$ and $S = 12$? Try it and see!

Example 2 allows us to answer a question asked earlier. We may now solve the equation "$3x^2 + 14x + 8 = 0$" by considering the equivalent equation "$(3x + 2)(x + 4) = 0$." The solution set is easily seen to be $\{-\frac{2}{3}, -4\}$.

Example 3. Factor "$6x^2 + x - 35$."

If "$6x^2 + x - 35$" is the product of factors of the form $(ax + b)(cx + d)$, $a, b, c, d \in Z$, then there must be integers R and S such that

$$R + S = 1; \quad R \cdot S = -210.$$

15 and -14 are two such numbers. Let $R = 15$ and $S = -14$. Then

$$\begin{aligned} 6x^2 + x - 35 &= 6x^2 + (R + S)x - 35 \\ &= 6x^2 + (15 + (-14))x - 35 \\ &= 6x^2 + 15x - 14x - 35 \\ &= 3x(2x + 5) - 7(2x + 5) \\ &= (3x - 7)(2x + 5). \end{aligned}$$

Example 4. Factor "$x^2 + 2x + 3$."

Are there two integers R and S such that:

$$R + S = 2 \text{ and } R \cdot S = 3\ ?$$

Since no such integers exist, "$x^2 + 2x + 3$" has no factors of form $(ax + b)(cx + d)$, with $a, b, c, d \in Z$. Therefore, "$x^2 + 2x + 3$" is said to be prime over the integers.

It is important to understand in Example 3 that the polynomial "$x^2 + 2x + 3$" is called prime because it does not have factors of a certain type. A similar situation exists in the set N of natural numbers. We say for instance that the number 5 is prime because there are no natural numbers a and b (other than 1 and 5) such that $ab = 5$. If we use rational numbers, however, 5 does have factors -- for instance, $5 = \frac{1}{2} \cdot 10$.

Does "$x^3 - 3x^2 - 4x + 12$" have a factor of form "$ax + b$," $a, b \in Z$? All of the preceding examples have dealt with polynomials of second degree; the present question is about a polynomial of degree three. There is a fairly easy way to answer certain questions of this kind. See if you can follow the steps:

Let p be the real function such that, for all $x \in R$,

$$p(x) = x^3 - 3x^2 - 4x + 12.$$

$$\begin{aligned} \text{Then } p(2) &= (2)^3 - 3(2)^2 - 4(2) + 12 \\ &= 8 - 12 - 8 + 12 \\ &= 0. \end{aligned}$$

Since $p(2) = 0$, $x - 2$ is a factor of $x^3 - 3x^2 - 4x + 12$.

The steps above indicate that $x - 2$ is a factor of $p(x) = x^3 - 3x^2 - 4x + 12$ because $p(2) = 0$. This is a specific application of the following theorem:

<u>Theorem</u> 3. (Factor Theorem)

Let $p: \ x \longrightarrow a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a real polynomial function. Then $x - r$, $r \in R$, is a factor of $p(x)$ if and only if $p(r) = 0$.

<u>Proof</u>. Suppose $x - r$ is a factor of $p(x)$; that is, $x - r$ divides $p(x)$. Then there is a polynomial function q such that

$$p(x) = (x - r) \cdot q(x) \text{ for all } x \in R.$$

Then
$$\begin{aligned} p(r) &= (r - r) \cdot q(r) \\ &= 0 \cdot q(r) \\ &= 0 \end{aligned}$$

On the other hand, suppose $p(r) = 0$. Then

$$\begin{aligned} p(x) &= p(x) - 0 \\ &= p(x) - p(r) \text{ (SPE)} \\ &= (a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0) \\ &\quad - (a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0) \\ &= a_n(x^n - r^n) + a_{n-1}(x^{n-1} - r^{n-1}) \\ &\quad + \ldots + a_1(x - r). \end{aligned}$$

Since $x - r$ divides $x^n - r^n$, $x^{n-1} - r^{n-1}$, etc., we see that $x - r$ divides $p(x)$. (Theorem 2)

<u>Example</u> 5. Is $x - 3$ a factor of $p(x) = x^3 + 2x^2 - 9x - 18$? Since $p(3) = (3)^3 + 2(3)^2 - 9(3) - 18 = 27 + 18 - 27 - 18 = 0$, the answer is "yes." In fact, division shows that

$$x^3 + 2x^2 - 9x - 18 = (x - 3)(x^2 + 5x + 6).$$

And since $x^2 + 5x + 6 = (x + 2)(x + 3)$, we can write finally

$$x^3 + 2x^2 - 9x - 18 = (x - 3)(x + 2)(x + 3).$$

7.12 Exercises

1. $p: \quad x \longrightarrow x^2 + 3x - 28$

   is a real polynomial function. Find two first degree polynomial functions f and g such that $p = [f \cdot g]$.

2. $q: \quad x \longrightarrow 3x^2 + 7x - 20$

   is a real polynomial function. Find two first degree polynomial functions f and g such that $q = [f \cdot g]$.

In Exercises 3 -- 15, express each second degree polynomial as the product of factors $(ax + b)(cx + d)$, $a, b, c, d \in Z$, if possible.

3. $x^2 - 11x + 24$
4. $x^2 + 14x + 33$
5. $x^2 - 7x - 8$
6. $x^2 + 2x - 35$
7. $2x^2 - 11x - 21$
8. $4x^2 + 17x - 15$
9. $5x^2 + 12x + 4$
10. $14x^2 + 17x - 6$
11. $15x^2 - 7x - 2$
12. $4x^2 + 3x + 2$
13. $6x^2 - 55x - 50$
14. $6x^2 - 7x - 24$
15. $9x^2 + 25x - 6$

16. Factor each of the following polynomials.

    (a) $x^2 - 4$ (Hint: $x^2 - 4 = x^2 + 0x - 4$. Here $R + S = 0$.)

    (b) $x^2 - 16$

    (c) $n^2 - 100$

    (d) $4x^2 - 9$

    (e) $25y^2 - 49$

    (f) $x^2 - b^2$

    (g) $a^2x^2 - b^2$

17. Factor each of the following polynomials.

(a) $x^2 + 6x + 9$ (d) $x^2 - 24x + 144$

(b) $a^2 + 10a + 25$ (e) $x^2 - 2ax + a^2$

(c) $x^2 - 8x + 16$ (f) $x^2 + 2ax + a^2$

18. Each of the polynomials in Exercise 17 is called a perfect square polynomial since it may be factored in the form

$$(x + a)^2$$

Tell what must be added to each of the following polynomials so that the result is a perfect square polynomial.

(a) $x^2 + 14x$ (d) $x^3 + x$

(b) $x^2 + 18x$ (e) $x^2 + bx$

(c) $x^2 + 5x$ (f) $x^2 + \frac{b}{a}x$

19. Decide which of the following natural numbers are prime.

(a) 14 (b) 7 (c) 101 (d) 109 (e) 51

20. Decide which of the following polynomials are prime (over the integers).

(a) $5x^2 + 2x + 1$ (d) $x^2 + 9$

(b) $16x^2 - 2x - 3$ (e) $x^2 + x$

(c) $x^2 - 9$

21. Let $p: x \longrightarrow x^3 - 2x^2 - x - 6$

be a real polynomial function.

(a) What is $p(3)$?

(b) Does "$x - 3$" divide "$x^3 - 2x^2 - x - 6$"?

(c) Find a polynomial function q such that for all $x \in R$,

$$x^3 - 2x^2 - x - 6 = (x - 3)\cdot q(x).$$

22. Let p: $x \longrightarrow -x^3 - x^2 - 3x - 10$

be a real polynomial function.

(a) What is $p(2)$?

(b) Does "$x - 2$" divide "$-x^3 - x^2 - 3x - 10$"?

(c) What is $p(-2)$?

(d) Does "$x + 2$" divide "$-x^3 - x^2 - 3x - 10$"?

(Hint: $x + 2 = x - (-2)$; the Factor Theorem may be used.)

(e) Find a polynomial function q such that for all $x \in R$,

$$-x^3 - x^2 - 3x - 10 = (x + 2)\cdot q(x).$$

23. Let p: $x \longrightarrow x^3 - \frac{1}{8}$ be a real polynomial function.

(a) What is $p(\frac{1}{2})$?

(b) Does "$x - \frac{1}{2}$" divide "$x^3 - \frac{1}{8}$"?

(c) Find a polynomial function q such that for all $x \in R$,

$$x^3 - \frac{1}{8} = (x - \tfrac{1}{2})\cdot q(x).$$

24. Let p: $x \longrightarrow x^3 - 7x^2 + 7x + 15$ be a real polynomial function.

(a) Show that "$x - 5$" divides $p(x)$.

(b) Express $p(x)$ as the product of three first degree polynomial factors.

(c) Sketch the graph of the polynomial function p.

25. q: $x \longrightarrow -x^3 + 7x^2 - 7x - 15$ is a real polynomial function.

(a) Sketch the graph of q. (Hint: q is the additive inverse of the function p in Exercise 24.)

(b) Express $q(x)$ as the product of three first degree polynomial factors.

7.13 Quadratic Functions and Equations

The graph of the function

$$f : x \longrightarrow x^2$$

is one we are already familiar with (see Figure 7.1). We have also

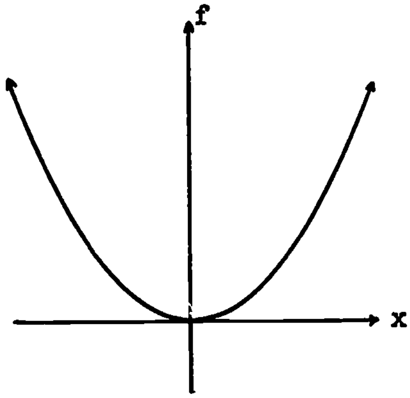


Figure 7.1

f: $x \longrightarrow x^2$

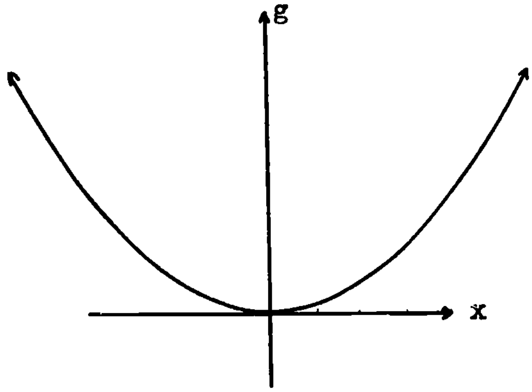


Figure 7.2

g: $x \longrightarrow \frac{1}{2}x^2$

seen the effect of the real number a in the graph of a function

$$t\colon \quad x \longrightarrow ax^2.$$

For instance, the graph of

$$g\colon \quad x \longrightarrow \tfrac{1}{2}x^2$$

appears in Figure 7.2.

Figure 7.3 shows the graph of the function

$$h\colon \quad x \longrightarrow \tfrac{1}{2}(x - 3)^2.$$

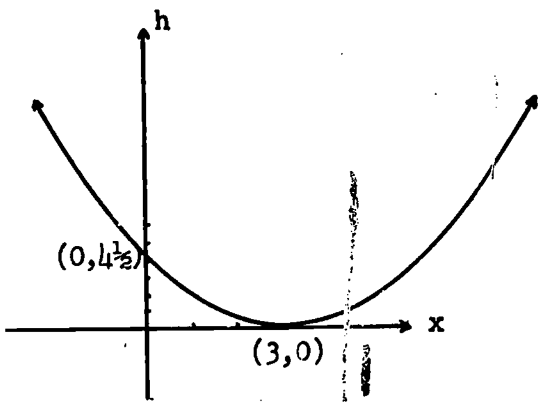


Figure 7.3

h: $x \longrightarrow \frac{1}{2}(x - 3)^2$

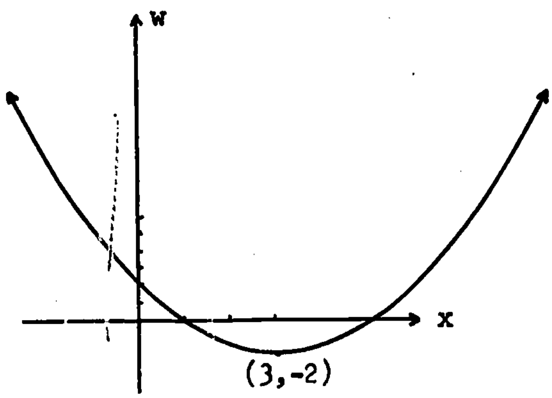


Figure 7.4

w: $x \longrightarrow \frac{1}{2}(x - 3)^2 - 2$

Notice that the "size and shape" of the graph of h is the same as that for g. The two graphs are in fact congruent. Every point on the graph of h: $x \longrightarrow \frac{1}{2}(x - 3)^2$ can be obtained by shifting a point of the graph of g: $x \longrightarrow \frac{1}{2}x^2$ three units to the right. That is, the graph of h: $x \longrightarrow \frac{1}{2}(x - 3)^2$ can be obtained from the graph of g: $x \longrightarrow \frac{1}{2}x^2$ by a translation of three units to the right, or more precisely, by the translation $T_{3,0}$.

Figure 6.4 shows the graph of the function

$$w: \quad x \longrightarrow \tfrac{1}{2}(x - 3)^2 + (-2).$$

It is congruent to the graph of h: $x \longrightarrow \frac{1}{2}(x - 3)^2$ and can be obtained from it by a translation of two units downward, or more precisely, by $T_{0,-2}$.

Comparing Figures 7.2, 7.3, and 7.4, we see that translating the graph of g: $x \longrightarrow \frac{1}{2}x^2$ by $T_{3,-2}$ results in the graph of

$$w: \quad x \longrightarrow \tfrac{1}{2}(x - 3)^2 + (-2)$$

A polynomial function of second degree is called a quadratic function. The function f: $x \longrightarrow x^2$ is the basic quadratic function. And the graph of a quadratic function

$$U: \quad x \longrightarrow a(x - h)^2 + k$$

can be obtained from the graph of

$$V: \quad x \longrightarrow ax^2$$

by the translation $T_{h,k}$.

Example 1. Draw the graph of the quadratic function

$$s: \quad x \longrightarrow 3(x + 5)^2 + 7.$$

Since $3(x + 5)^2 + 7 = 3(x - (-5))^2 + 7$ for all x, the graph of the given function can be obtained

from the graph of r: $x \longrightarrow 3x^2$ by $T_{-5,7}$.
Both graphs are shown in Figure 7.5.

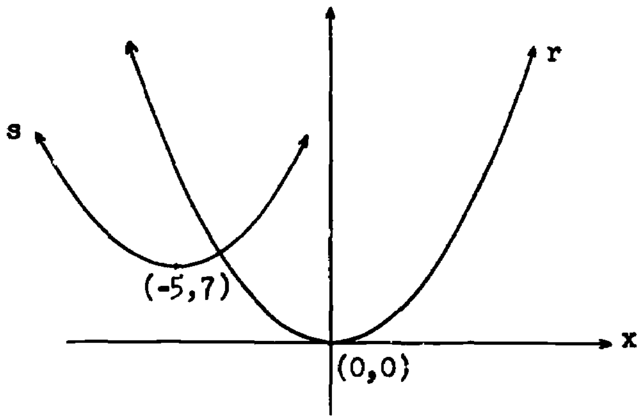


Figure 7.5

<u>Example</u> 2. What are the zeros of the function

$$w: \; x \longrightarrow \tfrac{1}{2}(x - 3)^2 - 2 \; ?$$

The graph of this function is shown in Figure 7.4, and it indicates that there should be two zeros. They may be found as follows:

$$\tfrac{1}{2}(x - 3)^2 - 2 = 0$$
$$\tfrac{1}{2}(x - 3)^2 = 2$$
$$(x - 3)^2 = 4$$
$$x - 3 = \sqrt{4} \text{ or } x - 3 = -\sqrt{4}$$
$$x - 3 = 2 \text{ or } x - 3 = -2$$
$$x = 5 \text{ or } x = 1$$

Therefore the zeros of the given function are 1 and 5.

<u>Example</u> 3. Solve the quadratic equation $2x^2 + 2x - 1 = 0$

This is the same as finding the zeros of the function

$$f: \; x \longrightarrow 2x^2 + 2x - 1.$$

First we try to write the polynomial in the form

$$a(x - h)^2 + k.$$

$$\begin{aligned} 2x^2 + 2x - 1 &= 2(x^2 + x \qquad) - 1 \\ &= 2(x^2 + x + \tfrac{1}{4}) - 1 - \tfrac{1}{2} \\ &= 2(x + \tfrac{1}{2})^2 - \frac{3}{2}. \end{aligned}$$

To find the zeros we proceed as follows:

$$\begin{aligned} 2(x + \tfrac{1}{2})^2 - \frac{3}{2} &= 0 \\ 2(x + \tfrac{1}{2})^2 &= \frac{3}{2} \\ (x + \tfrac{1}{2})^2 &= \frac{3}{4} \\ x + \tfrac{1}{2} = \frac{\sqrt{3}}{\sqrt{4}} &\text{ or } x + \tfrac{1}{2} = -\frac{\sqrt{3}}{\sqrt{4}} \\ x = -\tfrac{1}{2} + \frac{\sqrt{3}}{2} &\text{ or } x = -\tfrac{1}{2} - \frac{\sqrt{3}}{2} \end{aligned}$$

In Example 3, $\frac{1}{4}$ was added to the expression in parentheses so that the resulting expression, "$x^2 + x + \frac{1}{4}$," is a perfect square. (See Exercises 17 and 18 of Section 7.12.) However, we were really adding $\frac{1}{2}$, since the multiplier "2" distributes over the sum in parentheses, and $(2)(\frac{1}{4}) = \frac{1}{2}$. Therefore, in order not to change the given function, we also subtracted $\frac{1}{2}$. The real solutions of any quadratic equation -- provided they exist -- can be found by the process (called "completing the square") used in Example 3.

7.14 Exercises

1. On one set of axes, sketch the graphs of the functions associated with the following quadratic polynomials:

   (a) x^2 (b) $-x^2$ (c) $\frac{1}{2}x^2$ (d) $-\frac{1}{2}x^2$ (e) $3x^2$

   (f) $-3x^2$

2. On one set of axes, sketch the graphs of the functions associated with the following quadratic polynomials:

   (a) $\frac{1}{2}x^2$ (b) $\frac{1}{2}(x - 2)^2$ (c) $\frac{1}{2}(x + 2)^2$

3. On one set of axes, sketch the graphs of the functions associated with the following quadratic polynomials:

   (a) $\frac{1}{2}(x - 2)^2$ (b) $\frac{1}{2}(x - 2)^2 + 3$ (c) $\frac{1}{2}(x - 2)^2 - 3$

4. Given the graph of f: $x \longrightarrow 5x^2$, tell which translation will give the graph of the function associated with each of the following polynomials:

   (a) $5(x + 6)^2 + \frac{3}{4}$ (d) $5(x + 7)^2 - 10$

   (b) $5(x - 2)^2 + 4$ (e) $5x^2 + 2$

   (c) $5(x - \frac{1}{2})^2 - 3$ (f) $5(x + 2)^2$

5. Sketch the graph of each of the following functions.

   (a) f_1: $x \longrightarrow (x - 3)^2 + 2$

   (b) f_2: $x \longrightarrow (x - 3)^2 - 2$

   (c) f_3: $x \longrightarrow 2(x + 4)^2 + 1$

   (d) f_4: $x \longrightarrow 2(x + 4)^2 - 1$

   (e) f_5: $x \longrightarrow 3(x - \frac{1}{2})^2 - \frac{1}{4}$

   (f) f_6: $x \longrightarrow 3(x - \frac{1}{2})^2 + 0$

   (g) f_7: $x \longrightarrow 5(x - \frac{7}{8})^2 - \frac{5}{4}$

6. Sketch the graphs of the functions associated with the following quadratic polynomials, by putting each in the form

$$a(x - h)^2 + k.$$

   (See Example 1 of Section 7.13).

   (a) $x^2 + 5x + 6$

   (b) $x^2 + 3x + 9$

   (c) $2x^2 + 9x - 5$

   (d) $3x^2 - 5x - 12$

   (e) $2x^2 + 7x + 3$

   (f) $2x^2 + 3x + 7$

   (g) $9x^2 + 15x - 14$

   (h) $2x^2 + 5x + 1$

   (i) $x^2 + 3x - 1$

   (j) $x^2 + 3x + 1$

   (k) $3x^2 + 4x - 2$

(ℓ) $ax^2 + bx + c$ (where a, b, c are real numbers, $a \neq 0$).

(The result of Exercise 6(ℓ) is to establish that every quadratic polynomial can be expressed as

$$a(x - h)^2 + k.)$$

7. Solve the following quadratic equations if possible.

(a) $x^2 - x - 12 = 0$

(b) $x^2 + x - 1 = 0$

(c) $7x^2 + 20x - 3 = 0$

(d) $2x^2 + x - 1 = 0$

(e) $x^2 + x + 1 = 0$

(f) $2x^2 + x - 3 = 0$

(g) $3x^2 + 2x - 5 = 0$

(h) $2x^2 - 7x + 3 = 0$

(i) $ax^2 + bx + c = 0$ (where a, b, c are real numbers, $a \neq 0$).

(The result of Exercise 7(i) is a formula, called the <u>quadratic formula</u>, which can be used to find the real number solutions of any quadratic equation, provided they exist.)

8. (a) Sketch a graph of a quadratic function f whose associated function equation $f(x) = 0$ would have exactly two real solutions.

(b) Sketch a graph of a quadratic function g whose associated function equation $g(x) = 0$ would have exactly one real solution.

(c) Sketch a graph of a quadratic function h whose associated function equation $h(x) = 0$ would have no real solutions.

7.15 Rational Functions

$$r: \; x \longrightarrow \frac{5}{x - 1}$$

is a real function provided that the domain does not include the number 1. (Why must 1 be excluded?) The greatest possible subset of the real numbers which may serve as domain of this function is $R\setminus\{1\}$. This real function r may be generated by the functions

$$c_5: \quad x \longrightarrow 5$$
$$j_R: \quad x \longrightarrow x$$
$$c_{-1}: \quad x \longrightarrow -1$$

in the following way:

$$r = [c_5 \div [j_R + c_{-1}]].$$

That is,
$$r(x) = \frac{c_5(x)}{j_R(x) + c_{-1}(x).}$$

Although the function r above is generated by the identity function and constant functions, it is not a polynomial function, since the operation of division of functions is used in the generation. (Review the definition of polynomial functions, in Section 7.1.) r is, however, a real rational function. (In this chapter, all real rational functions have codomain R.)

Definition 4. j_R is a real rational function.
c_a, $a \in R$, are real rational functions.
Any function generated from one or more of the above functions, by using no operations other than addition, multiplication, and division of functions, is a real rational function.

The expression " $\frac{5}{x - 1}$," associated with the function r: $x \longrightarrow \frac{5}{x - 1}$ is called a rational expression.

Example 1. The real function f: $x \longrightarrow \frac{x + 2}{x - 7}$ with domain $\{x: \ x \in R \setminus \{7\}\}$ is a real rational function.
"$\frac{x + 2}{x - 7}$" is a rational expression.

The graph of a rational function is not always an unbroken curve. Shown in Figure 7.6 is the graph of the rational function discussed earlier:

$$r: \quad x \longrightarrow \frac{5}{x - 1}$$

with domain $R \setminus \{1\}$. Note the following points which help in sketching the graph:

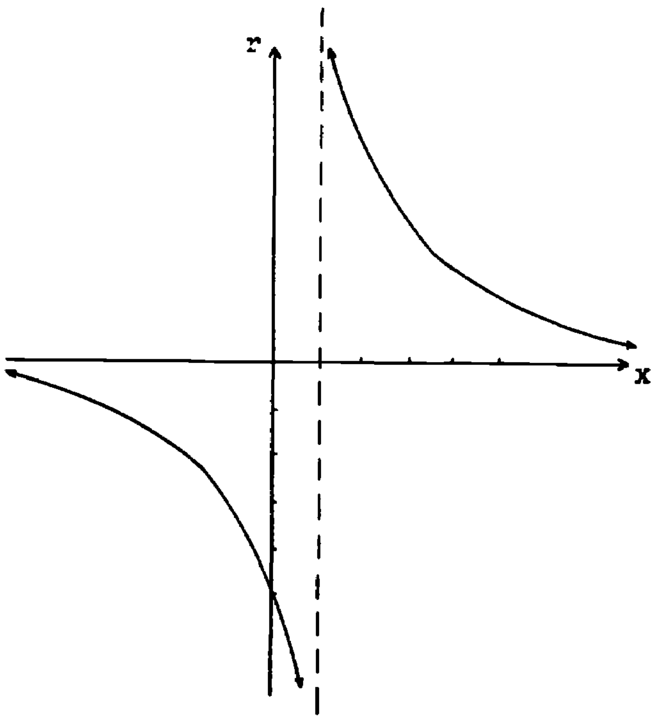


Figure 7.6

$$r: \quad x \longrightarrow \frac{5}{x - 1}$$

(1) The number 1 is not in the domain of r. So the dashed line "x = 1" has been drawn. The graph cannot intersect this line.

(2) $r(0) = -5$. Therefore, the point $(0,-5)$ belongs to the graph.

(3) As $|x|$ becomes very large, $r(x)$ gets "closer and closer" to zero. For instance:

$r(100) = \frac{5}{99}$; $r(1000) = \frac{5}{999}$; $r(-100) = \frac{-5}{101}$; $r(-1000) = \frac{-5}{1001}$.

(4) By taking x close enough to 1, $|r(x)|$ can be made as great as desired. For instance,

$r(1.1) = 50$; $r(1.01) = 500$; $r(.9) = -50$; $r(.99) = -500$

These four points lead us to conclude that the graph of r has two asymptotes. The line "x = 1" is a vertical asymptote, and the line "y = 0" is a horizontal asymptote. Thus, the graph of this rational function is not a smooth, unbroken curve. However, we assume that, except for a break in the neighborhood of x = 1, it is.

The rational function r may also be taken as the quotient of the two polynomial functions:

$$p: x \longrightarrow 5 \text{ and}$$

$$q: x \longrightarrow x - 1.$$

And this suggests the following alternative definition of a real rational function.

<u>Definition</u> 5. If <u>p</u> and <u>q</u> are real polynomial functions, then

$$\left[\frac{p}{q}\right]$$

is a <u>real rational function</u> <u>r</u>.

$r(x) = \frac{p(x)}{q(x)}$ for all $x \in R$, except those for which $q(x) = 0$. Therefore, the domain of <u>r</u> does not include zeroes of <u>q</u>.

<u>Example</u> 2. $t: x \longrightarrow \frac{x^2 + 5x}{x^2 - x - 12}$

is a real rational function. It is the quotient $\frac{p}{q}$ of the following polynomial functions:

$$p: x \longrightarrow x^2 + 5x$$

$$q: x \longrightarrow x^2 - x - 12$$

Domain of t = $R\backslash\{4,-3\}$ since 4 and -3 are zeros of q.

Example 3. Is k: $x \longrightarrow \frac{0}{3 + 2x}$ with domain $R\backslash\{-\frac{3}{2}\}$ a real rational function?

It is, since it is the quotient of the polynomial functions

$$c_0: \quad x \longrightarrow 0$$
$$q: \quad x \longrightarrow 3 + 2x$$

Example 4. Sketch the graph of the real rational function

$$t: \quad x \longrightarrow \frac{2x + 3}{4x - 5}$$

with domain $R\backslash\{\frac{5}{4}\}$. The line "$x = \frac{5}{4}$" is a vertical asymptote. $t(0) = -\frac{3}{5}$, and $t(-\frac{3}{2}) = 0$. So, the points $(0, -\frac{3}{5})$ and $(-\frac{3}{2}, 0)$ are on the graph. Now, for all $x \neq 0$,

$$t(x) = \frac{x(2 + \frac{3}{x})}{x(4 - \frac{5}{x})} = \frac{2 + \frac{3}{x}}{4 - \frac{5}{x}}$$

Therefore, as $|x|$ becomes very large, t(x) gets "closer and closer" to 2/4, or 1/2.

The line "$y = \frac{1}{2}$" is a horizontal asymptote (see Figure 7.7).

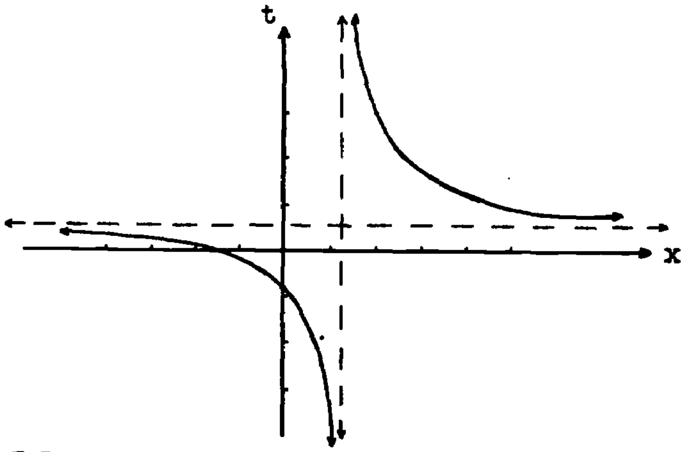


Figure 7.7

7.16 Exercises

1. (a) Is every rational number a real number?
   (b) Is every real number a rational number?
   (c) Is every rectangle a square?
   (d) Is every square a rectangle?
   (e) Is every polynomial function a rational function?
   (f) Is every rational function a polynomial function?

2. Identify each of the following as polynomial function, rational function, both, or neither, by checking the appropriate columns. Specify a domain.

	Polynomial Function	Rational Function
(a) $f_1: x \longrightarrow \frac{1}{2}x$		
(b) $f_2: x \longrightarrow \frac{2}{x}$		
(c) $f_3: x \longrightarrow \frac{x}{x}$		
(d) $f_4: x \longrightarrow \frac{x+2}{x}$		
(e) $f_5: x \longrightarrow \frac{x}{x+2}$		
(f) $f_6: x \longrightarrow \sqrt{x}$		
(g) $f_7: x \longrightarrow x^2$		
(h) $f_8: x \longrightarrow -10$		
(i) $f_9: x \longrightarrow [x]$		
(j) $g_1: x \longrightarrow x^3 + 7x^2 - \frac{1}{2}x + 3$		
(k) $g_2: x \longrightarrow \frac{x^2-3x+5}{x^5+7x-2}$		
(l) $g_3: x \longrightarrow \lvert x \rvert + 2$		

3. Specify the greatest possible subset of the real numbers which may serve as domain of each of the following functions.

(a) $h_1: \quad x \longrightarrow \frac{2}{x}$

(b) $h_2: \quad x \longrightarrow \frac{2}{x - 3}$

(c) $h_3: \quad x \longrightarrow \frac{x - 3}{x + 5}$

(d) $h_4: \quad x \longrightarrow \frac{x + 5}{x + 5}$

(e) $h_5: \quad x \longrightarrow x^2 + 3x + 5$

(f) $h_6: \quad x \longrightarrow \frac{x - 2}{x^2 + 4x - 21}$

(g) $h_7: \quad x \longrightarrow \frac{3x}{(x+2)(x-5)(x+\frac{1}{2})}$

(h) $h_8: \quad x \longrightarrow \frac{x + 7}{2x(x-3)(x+12)(x+\sqrt{2})}$

In Exercises 4 -- 12 sketch the graph of the given function. Be sure to draw asymptotes, and to locate all points where the graph intersects the axes. Specify a domain for each function.

4. $f: x \longrightarrow \frac{1}{x}$

5. $g: \ x \longrightarrow \frac{1}{x - 2}$

6. $h: \ x \longrightarrow \frac{2}{x}$

7. $k: \ x \longrightarrow \frac{1}{x + 2}$

8. $m: \ x \longrightarrow \frac{2}{x - 2}$

*9. $n: \ x \longrightarrow \frac{1}{x^2 + 2x - 15}$

10. $p: \ x \longrightarrow \frac{x}{x - 2}$

11. $q: \ x \longrightarrow \frac{x + 3}{x - 2}$

*12. $r: \ x \longrightarrow \frac{x}{x^2 - x - 6}$

7.17 Operations with Real Rational Functions

Let

$f: \quad x \longrightarrow \frac{2}{x + 3}$ (Domain of $f = R\backslash\{-3\}$)

$g: \quad x \longrightarrow \frac{x}{x - 5}$ (Domain of $g = R\backslash\{5\}$)

be two real rational functions. Then

$$[f + g](x) = f(x) + g(x)$$

$$= \frac{2}{x + 3} + \frac{x}{x - 5}$$

$$= \frac{2}{x + 3}\left(\frac{x - 5}{x - 5}\right) + \frac{x}{x - 5}\left(\frac{x + 3}{x + 3}\right)$$

$$= \frac{2(x - 5)}{(x + 3)(x - 5)} + \frac{x(x + 3)}{(x + 3)(x - 5)}$$

$$= \frac{2x - 10 + x^2 + 3x}{(x + 3)(x - 5)}$$

$$= \frac{x^2 + 5x - 10}{x^2 - 2x - 15}$$

This example suggests that the sum of two real rational functions is a real rational function. Care must be taken about the domain, however. Since $[f + g](x) = f(x) + g(x)$, it is obvious that the domain of $[f + g]$ cannot include numbers that are not in the domain of f or not in the domain of g. Thus, in the above example, the largest possible domain for $[f + g]$ is $R \setminus \{-3,5\}$.

Example 1. $\frac{x}{x + 2} + \frac{-x}{x + 2} = \frac{0}{x + 2}$.

Notice that for all $x \neq -2$,

$$\frac{0}{x + 2} = 0.$$

For this reason, we may think of "$\frac{x}{x + 2}$" and "$\frac{-x}{x + 2}$" as associated with inverse functions under addition. And this gives us a way to interpret subtraction of real rational functions.

Example 2. $\frac{2x + 3}{x^2 + 4x + 4} - \frac{x}{x + 2} = \frac{2x + 3}{x^2 + 4x + 4} + \frac{-x}{x + 2}$

$$= \frac{2x+3}{x^2+4x+4} + \frac{-x}{x+2}\left(\frac{x+2}{x+2}\right)$$

$$= \frac{2x+3}{x^2+4x+4} + \frac{-x^2-2x}{x^2+4x+4}$$

$$= \frac{-x^2 + 3}{x^2 + 4x + 4} \quad (x \neq -2)$$

Example 3. Let $f\colon\ x \longrightarrow \frac{x}{x^2 - 4}$ and

$$g\colon\ x \longrightarrow \frac{x^2 + 5x - 14}{3x^2}$$

with domain of $f = R\backslash\{-2,2\}$ and domain of $g = R\backslash\{0\}$ be two real rational functions. Then

$$\begin{aligned}
[f\cdot g](x) &= f(x) \cdot g(x) \\
&= \frac{x}{x^2 - 4} \cdot \frac{x^2 + 5x - 14}{3x^2} \\
&= \frac{x}{(x + 2)(x - 2)} \cdot \frac{(x + 7)(x - 2)}{3\cdot x\cdot x} \\
&= \frac{x(x + 7)(x - 2)}{3\cdot x\cdot x\cdot (x + 2)(x - 2)} \\
&= \frac{x(x - 2)}{x(x - 2)} \cdot \frac{x + 7}{3x(x + 2)} \\
&= \quad 1 \quad \cdot \frac{x + 7}{3x(x + 2)} \\
&= \frac{x + 7}{3x(x + 2)}
\end{aligned}$$

Thus, the product of the two real rational functions is a real rational function. The greatest possible subset of R which may serve as domain in this case is $R\backslash\{0,-2,2\}$ since both $f(x)$ and $g(x)$ must be defined in order for the above development to have meaning.

Example 4. $$\begin{aligned}
\frac{x^2 + 3}{x - 2} \cdot \frac{x - 2}{x^2 + 3} &= \frac{(x^2 + 3)(x - 2)}{(x - 2)(x^2 + 3)} \\
&= \frac{x^2 + 3}{x^2 + 3} \cdot \frac{x - 2}{x - 2} \\
&= 1 \qquad\qquad x \neq 2.
\end{aligned}$$

From Example 4, we see that the two given functions multiply to give the multiplicative identity function c_1 (except when $x = 2$). For this reason we may consider "$\frac{x^2 + 3}{x - 2}$" and "$\frac{x - 2}{x^2 + 3}$" to be associated with inverse functions under multiplication. And this in turn gives a way to interpret division of real rational functions.

Example 5.

$$\frac{x^2 - 9}{x^2 - 5x} \div \frac{x^2 + 6x + 9}{x^2 - 11x + 30} = \frac{x^2 - 9}{x^2 - 5x} \cdot \frac{x^2 - 11x + 30}{x^2 + 6x + 9}$$

$$= \frac{(x + 3)(x - 3)}{x(x - 5)} \cdot \frac{(x - 6)(x - 5)}{(x + 3)(x + 3)}$$

$$= \frac{(x + 3)(x - 5)}{(x + 3)(x - 5)} \cdot \frac{(x - 3)(x - 6)}{x(x + 3)}$$

$$= 1 \cdot \frac{(x - 3)(x - 6)}{x(x + 3)}$$

$$= \frac{x^2 - 9x + 18}{x^2 + 3x} \qquad x \neq 0,\ 5,\ 6,\ -3$$

Notice especially in Example 5 that, even though the number -3 is in the domain of both of the original functions, it is not in the domain of the quotient function. The reason for this of course is that we multiply by $\frac{x^2 - 11x + 30}{x^2 + 6x + 9}$, and -3 is not in the domain of this function.

We have now seen that real rational functions -- and their associated rational expressions -- can be added, subtracted, multiplied, and divided. At all times however it is important to specify the domain; we shall usually use the greatest possible subset of R. (As in the case of rational numbers, division by the function c_o is prohibited)

7.18 Exercises

1. Each part below consists of two problems, one using rational numbers and one using rational expressions. Express the result as a rational number or expression.

(a) $\frac{2}{3} + \frac{4}{5}$ $\qquad$ $\frac{3}{x + 2} + \frac{5}{x - 5}$

(b) $\frac{3}{35} + \frac{4}{21}$ $\qquad$ $\frac{x}{x^2 - x + 2} + \frac{2x}{x^2 + 4x + 3}$

(c) $\frac{2}{3} \cdot \frac{5}{7}$ $\qquad$ $\frac{x + 2}{x - 12} \cdot \frac{x + 5}{x^3 - x}$

(d) $\frac{3}{7} \div \frac{5}{7}$ $\qquad$ $\frac{x + 2}{x} \div \frac{x^2 - 4}{5x^3}$

2. f: $x \longrightarrow \frac{4}{x - 7}$ $\qquad$ $(x \neq 7)$

g: $x \longrightarrow \frac{x}{x + 3}$ $\qquad$ $(x \neq -3)$

h: $x \longrightarrow \frac{x^2}{x^2 - 4x - 21}$ $\qquad$ $(x \neq 7, -3)$

are real rational functions. Find expressions for each of the following functions; and in each case, state the domain.

(a) $[f + g]$

(b) $[f - g]$

(c) $[f \cdot g]$

(d) $[f + h]$

(e) $[h - g]$

(f) $[\frac{f}{h}]$

(g) $[\frac{[f + g]}{h}]$

3. (a) Draw the graph of the real rational function f: $x \longrightarrow \frac{1}{x}$, with domain $R \setminus \{0\}$.

(b) Explain how the graph of g: $x \longrightarrow \frac{1}{x-5}$, with domain $R \setminus \{5\}$, can be obtained by a translation of the graph of f.

(c) Find a rational expression for $[f \cdot g]$, and state the domain of this function.

(d) True or False: For all $x \in R$, $\frac{x}{x} = 1$.

4. (a) Draw the graph of h: $x \longrightarrow \frac{-1}{x}$ with domain $R \setminus \{0\}$.

(b) Explain how the graph of h can be obtained by a reflection of the graph of f (in exercise 3).

(c) Find a rational expression for $[f + h]$, and state the domain of this function.

(d) True or False: For all $x \in R$, $\frac{0}{x} = 0$.

In Exercises 5 -- 16, add, subtract, multiply, or divide (as indicated) the rational expressions. State all values of the variable which must be excluded (that is, which are not in the domain of the associated rational functions).

5. $\frac{x - 2}{x} + \frac{3x}{x^2 + 5x}$

6. $\frac{2x}{x - 2} - \frac{x^2}{x^2 - 4}$

7. $\frac{x^2 - 9}{x^2 - 4} \cdot \frac{x^2 - 5x + 6}{x^2 - 6x + 9}$

8. $\frac{1}{x - 2} \cdot x - 2$

9. $\frac{x - 2}{x} \div \frac{2 - x}{x}$

10. $\frac{x - 2}{x^2 + 3} \div \frac{x - 2}{x + 3}$

11. $\frac{5}{x} \div \frac{x}{5}$

12. $\frac{x^3 - 8}{3x + 6} \cdot \frac{3}{x - 2}$

13. $\frac{2x}{3x + 1} + \frac{x - 1}{5x - 2}$

14. $\frac{2x}{3x + 1} - \frac{1 - x}{5x - 2}$

15. $\frac{x+2}{x-5} \cdot \frac{x-5}{x^2-49} \cdot \frac{x^2+14x+49}{x^2-4}$

16. $\frac{3}{x+2} + \frac{-7}{x-3} + \frac{x}{x-5}$

7.19 Summary

A polynomial function p is a real function such that

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

for every $x \in R$, where $n \in W$ and $a_i \in R$ $(i = 0,\dots,n)$. Every polynomial function may be generated from the identity function and a finite number of constant functions by using only addition and multiplication of functions.

The expression "$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$" is a polynomial associated with a polynomial function. Assuming $a_n \neq 0$, the degree of the polynomial is n. a_i is the coefficient of x^i $(i = 0,\dots,n)$, a_n is the leading coefficient, and a_0 is the constant term.

If P denotes the set of real polynomial functions, then $(P,+)$ is a commutative group, $(P,\cdot)$ is an operational system but not a group, and $(P,+,\cdot)$ is a commutative ring with unity.

If p and q are not c_o, then $\deg([p + q]) \leq \max(\deg(p), \deg(q))$, and $\deg(p \cdot q) = \deg(p) + \deg(q)$.

Given two real polynomial functions f and p, with $p \neq c_o$, there exist unique polynomial functions q and r, with $r = c_o$ or $\deg(r) < \deg(p)$, such that

$$f = [[p \cdot q] + r].$$

A polynomial function of degree two is called a quadratic function. A quadratic function has at most two real zeros, and a quadratic equation has at most two real solutions.

The graph of "$a(x - h)^2 + k$" may be obtained from the graph of "ax^2" by the plane translation $T_{h,k}$.

A quadratic polynomial "$a_2x^2 + a_1x + a_o$" (a_o, a_1, $a_2 \in Z$) which does not have factors of type $(ax + b)(cx + d)$, a, b, c, d $\in$ Z, is said to be prime over the integers.

If $p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_o$, then $x - r$, $r \in R$, is a factor of $p(x)$ if and only if $p(r) = 0$.

A real rational function is the quotient $[\frac{p}{q}]$ of two real polynomial functions. The domain of a rational function does not include any number x for which $q(x) = 0$. The sum of two rational functions is a rational function as is their difference, product and quotient.

7.20 Review Exercises

1. Identify each of the following as a polynomial, a rational expression, both, or neither.

(a) $x^2 + 3$ (f) $\frac{5}{x - 2}$

(b) $|x| + 3$

(c) $\sqrt{x} + 3$

(d) $\frac{1}{3}x$

(e) $\frac{3}{x}$

(g) $x^3 + 4x^2 - x + \sqrt{7}$

(h) $\frac{x}{x}$

(i) $[x]$

(j) 5

2. $(-3x^2 + 5x + \frac{1}{2}) + (7x^3 + \frac{5}{2}x^2 - 4x - \frac{2}{3}) =$

3. $(x^2 - 3x + 7)(x - 12) =$

4. $(\frac{1}{4}x^3 + 7x + 5) - (4x^2 - 10x + \frac{1}{3}) =$

5. $(x + \sqrt{2})(x - \sqrt{2}) =$

6. $(3x + 7)^2 =$

7. $(2x - 14)(3x^2 + 7) =$

8. $(4x^3 - 8x^2 - 10) + (-4x^3 + 8x^2 + 10) =$

9. $(4x^3 - 8x^2 - 10) - (-4x^3 + 8x^2 + 10) =$

10. $(x - 7)^3 =$

11. For each of the following pairs of polynomials, f and p, find polynomials q and r, with $r = c_0$ or $\deg(r) < \deg(p)$, such that $f = [[p \cdot q] + r]$.

(a) $f(x) = 4x^2 - 7x + 10$, $p(x) = 2x + 5$

(b) $f(x) = x^3 - 8$, $p(x) = x - 2$

(c) $f(x) = x^3 - 8$, $p(x) = x - 1$

(d) $f(x) = x^2 + 2x - 3$, $p(x) = 2x - 5$

12. Find factors, if they exist, of the form $(ax + b)(cx + d)$, $a, b, c, d \in Z$, for each of the following polynomials.

(a) $6x^2 + 17x - 14$ (b) $6x^2 + 9x - 14$ (c) $25x^2 - 20x + 4$

13. Write each of the following quadratic polynomials in the form "$2(x - h)^2 + k$." Then tell how the graph of the associated function can be obtained from the graph of $f: x \longrightarrow 2x^2$.

(a) $2x^2 + 4x$ (b) $2x^2 - 7$ (c) $2x^2 + x + 5$ (d) $2x^2 - 3x + 8$

14. On the same set of axes, sketch the graphs of the following functions.

(a) $f: x \longrightarrow \frac{1}{2}x^2$ (b) $g: x \longrightarrow \frac{1}{2}(x - 3)^2$ (c) $h: x \longrightarrow \frac{1}{2}(x+3)^2$

(d) $k: x \longrightarrow \frac{1}{2}(x + 3)^2 + 2$

15. Find the zeros of the following quadratic functions.

(a) $f: x \longrightarrow x^2 + 7x + 12$ (c) $g: x \longrightarrow x^2 + 2x - 3$

(b) $h: x \longrightarrow x^2 - 2x + 3$ (d) $k: x \longrightarrow 4x^2 - 12x + 9$

16. Solve the following quadratic equations.

(a) $x^2 - 7 = 0$ (c) $x^2 + 7 = 0$ (e) $2x^2 - 7x = 15$

(b) $x^2 + 2x - 2 = 0$ (d) $3x^2 - x = 0$ (f) $2x^2 - 3x - 4 = 0$

17. Sketch a graph of the following function:

$U: x \longrightarrow (x - 7)(x - 2)(x + 5)$ for every $x \in R$.

Also, explain why this is a polynomial function.

18. Sketch the graph of the following real rational functions. In each case, identify the domain of the function.

(a) $r: x \longrightarrow \frac{1}{x}$ (c) $t: x \longrightarrow \frac{1}{x + 4}$

(b) $s: x \longrightarrow \frac{1}{x + 4}$ (d) $f: x \longrightarrow \frac{x + 2}{x - 2}$

CHAPTER 8

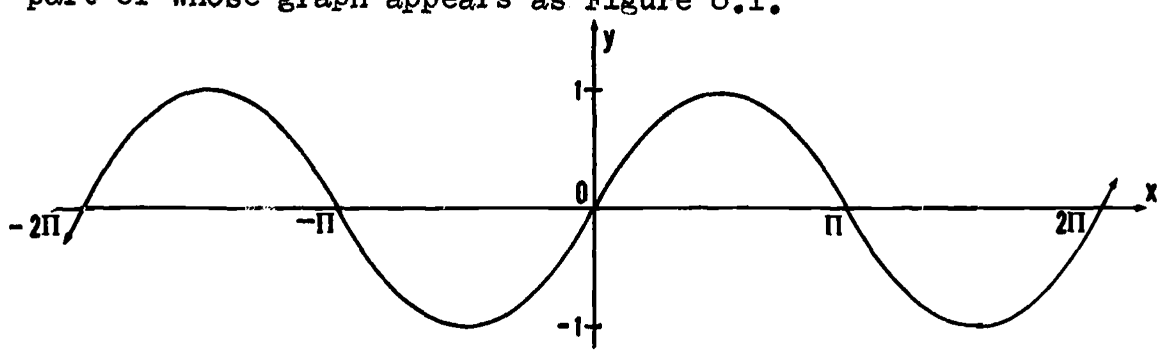
CIRCULAR FUNCTIONS

There are many physical phenomena which are periodic in nature; during some specified time interval they exhibit some behavior which they then "repeat" during subsequent time intervals (i.e., periodically). Think, for instance, of a pendulum which moves back and forth during a certain time period, then retraces its path time and time again. Other examples of periodic phenomena are: a cork floating in choppy water, a point on a vibrating violin string, a point on the tip of a vibrating tuning fork, and a particle of air during passage of a simple sound wave. Even the beat of the human heart is periodic.

Still another physical expression of periodicity is to be found in the study of electricity. For instance, the formula

$$I = a(\sin \omega t)$$

where a and ω are constants, and t is a measure of time, may be used to find the quantity of electric current, I. But what does "sin" represent? It is an abbreviation for the sine function, part of whose graph appears as Figure 8.1.



Graph of $y: x \longrightarrow \sin x$

Figure 8.1

Do you see that this graph suggests periodicity? None of the functions studied so far have graphs with this characteristic. Therefore, in order to study periodic phenomena, new functions -- called the circular, or trigonometric functions -- are needed. The sine function is just one of these.

In this chapter we introduce two of the circular functions. In Course IV we will begin the study of periodicity.

8.1 Sensed Angles

$\overrightarrow{BA}$ and $\overrightarrow{BC}$, in Figure 8.2 are two rays having the same endpoint. The ordered pair of rays $(\overrightarrow{BA},\overrightarrow{BC})$ is a sensed angle, in particular sensed angle ABC. (Compare this with the definition of "angle" in Course I.) The same two rays

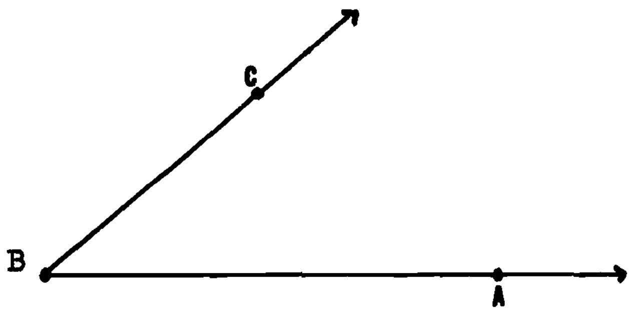


Figure 8.2

determine another sensed angle -- sensed angle CBA, which is the ordered pair of rays $(\overrightarrow{BC},\overrightarrow{BA})$.

$$(\overrightarrow{BA},\overrightarrow{BC}) \neq (\overrightarrow{BC},\overrightarrow{BA}).$$

sensed angle ABC $\neq$ sensed angle CBA.

Definition 1. A sensed angle is an ordered pair of rays having the same endpoint. The ordered pair

$(\overrightarrow{BA}, \overrightarrow{BC})$ is denoted $\vec{\angle}ABC$.

Notice that the definition of sensed angle does not demand that the rays of the pair of rays be non-collinear or even distinct. (See Exercises 9 and 11 of Section 8.2).

In a sensed angle, the first ray of the ordered pair is called the <u>initial side</u> of the angle, the second ray the <u>terminal side</u>.

<u>Example</u> 1. $(\overrightarrow{RS}, \overrightarrow{RT})$ is a sensed angle. Its initial side is $\overrightarrow{RS}$. Its terminal side is $\overrightarrow{RT}$.

In Figure 8.3, is there a direct isometry that maps the initial side of $\vec{\angle}FDE$ onto the initial side of $\vec{\angle}ABC$, and the terminal side of $\vec{\angle}FDE$ onto the terminal side of $\vec{\angle}ABC$?

(In Course II it is implicit that a transformation is a direct isometry if it is the composition of an even number of line reflections; it is an opposite isometry if it is the composition of an odd number.)

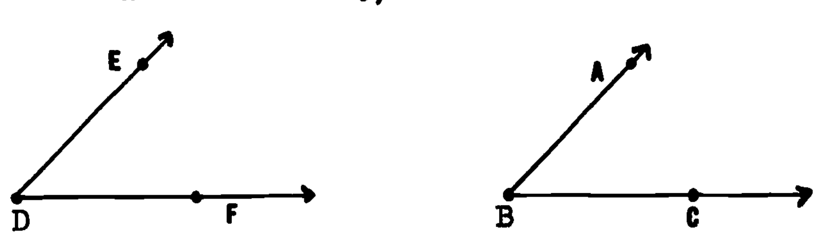


Figure 8.3

The answer -- perhaps surprising -- is "no." If you try to map $\vec{\angle}FDE$ onto $\vec{\angle}ABC$, you might first map D onto B by the reflection R_1 in ℓ_1, the perpendicular bisector of $\overline{DB}$ (Figure 8.4.) We suppose A, C, E, F, are so chosen that BA = BC = DF = DE. R_1 maps F onto F' and E onto E'.

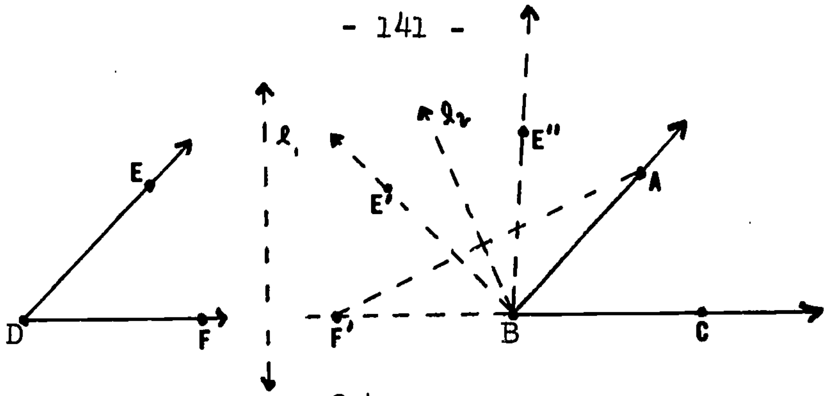


Figure 8.4

Our second mapping is the reflection R_2 in ℓ_2, the perpendicular bisector of $\overline{AF'}$. R_2 maps B onto B (why?), F' onto A, and E' onto E". The composite mapping R_2R_1 is a direct isometry, and it maps $\overrightarrow{DF}$ onto $\overrightarrow{BA}$. But R_2R_1 does not map $\overrightarrow{DE}$ onto $\overrightarrow{BC}$. To map both $\overrightarrow{DF}$ onto $\overrightarrow{BA}$ and $\overrightarrow{DE}$ onto $\overrightarrow{BC}$ requires a third reflection R_3, in the perpendicular bisector of $\overline{CE''}$. Then $R_3R_2R_1$ will map $\overrightarrow{\angle FDE}$ onto $\overrightarrow{\angle ABC}$. But $R_3R_2R_1$ is not a direct isometry, since it is the composition of an odd number of line reflections.

Our interest in this chapter will be with sensed angles that can be mapped onto each other by direct isometries.

<u>Definition</u> 2. $\overrightarrow{\angle RST}$ is <u>congruent</u> to $\overrightarrow{\angle ABC}$ written $\overrightarrow{\angle RST} \cong \overrightarrow{\angle ABC}$ if and only if there is a direct isometry f such that $f(\overrightarrow{SR}) = \overrightarrow{BA}$ and $f(\overrightarrow{ST}) = \overrightarrow{BC}$.

In other words, in order for two sensed angles to be congruent, there must be a direct isometry of the plane that maps initial side onto initial side and terminal side onto terminal side. Note that congruence is being used in a different (extended) way now. That is, congruence of sensed angles is not the same as congruence of ordinary (unsensed) angles.

<u>Example</u> 2.

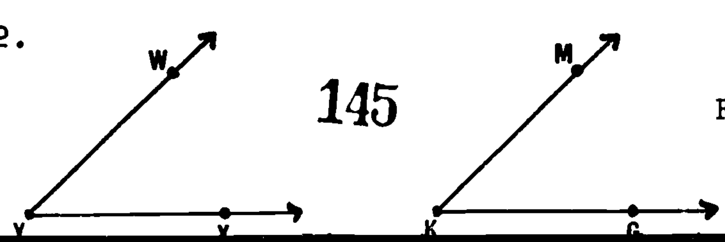


Figure 8.5

In Figure 8.5 $(\overrightarrow{YW},\overrightarrow{YX}) \cong (\overrightarrow{KM},\overrightarrow{KG})$. There is a translation T such that $T(\overrightarrow{YW}) = \overrightarrow{KM}$, and $T(\overrightarrow{YX}) = \overrightarrow{KG}$. And a translation is a direct isometry. Notice however that $(\overrightarrow{YX},\overrightarrow{YW}) \not\cong (\overrightarrow{KM},\overrightarrow{KG})$.

8.2 Exercises

1. Name the initial side and the terminal side of the following sensed angles.

   (a) $\overrightarrow{\angle DRN}$ (b) $\overrightarrow{\angle CXR}$ (c) $\overrightarrow{\angle GLT}$ (d) $\overrightarrow{\angle OEA}$.

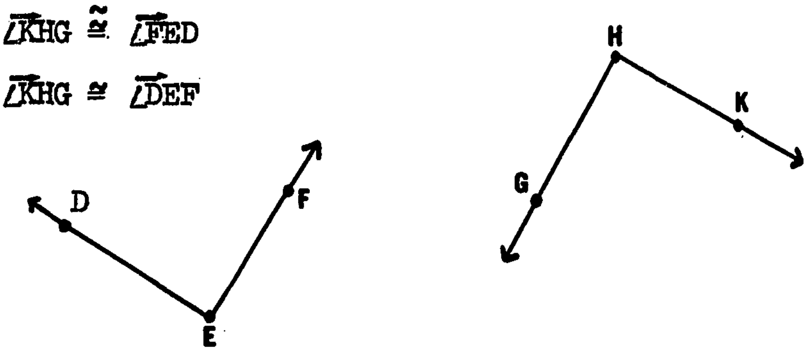
2. Using the figure below, which of the following statements appear to be true?

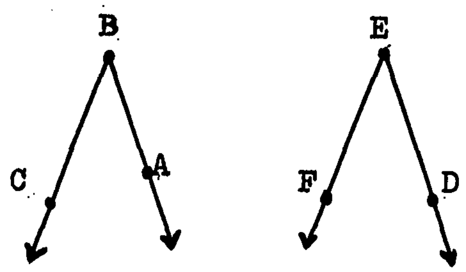
   (a) $(\overrightarrow{ED},\overrightarrow{EF}) \cong (\overrightarrow{HG},\overrightarrow{HK})$

   (b) $(\overrightarrow{ED},\overrightarrow{EF}) \cong (\overrightarrow{HK},\overrightarrow{HG})$

   (c) $\overrightarrow{\angle KHG} \cong \overrightarrow{\angle FED}$

   (d) $\overrightarrow{\angle KHG} \cong \overrightarrow{\angle DEF}$



3. 

   (a) Describe an isometry f such that $f(\overrightarrow{BA}) = \overrightarrow{ED}$, and $f(\overrightarrow{BC}) = \overrightarrow{EF}$.

   (b) Is $\overrightarrow{\angle ABC} \cong \overrightarrow{\angle DEF}$?

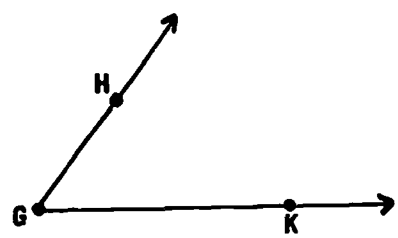
   (c) Describe an isometry g such that $g(\overrightarrow{ED}) = \overrightarrow{BA}$, and $g(\overrightarrow{EF}) = \overrightarrow{BC}$.

(d) Is $\overrightarrow{\angle DEF} \cong \overrightarrow{\angle ABC}$?

(e) Is congruence of sensed angles a symmetric relation? (Can you think of counterexamples?)

4. If $(\overrightarrow{TR}, \overrightarrow{TN}) \cong (\overrightarrow{SA}, \overrightarrow{SD})$ is a true statement, write two other related statements which must be true.

5.



(a) Describe an isometry f such that $f(\overrightarrow{GK}) = \overrightarrow{GK}$, and $f(\overrightarrow{GH}) = \overrightarrow{GH}$.

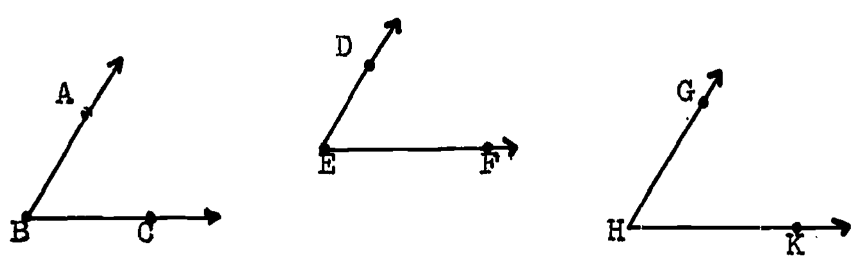
(b) Is $\overrightarrow{\angle KGH} \cong \overrightarrow{\angle KGH}$?

(c) Is $\overrightarrow{\angle HGK} \cong \overrightarrow{\angle HGK}$?

(d) Is congruence of sensed angles a reflexive relation?

(e) Is $\overrightarrow{\angle HGK} \cong \overrightarrow{\angle HGK}$?

6.



(a) Describe an isometry f such that $f(\overrightarrow{BA}) = \overrightarrow{ED}$, and $f(\overrightarrow{BC}) = \overrightarrow{EF}$.

(b) Is $\overrightarrow{\angle ABC} \cong \overrightarrow{\angle DEF}$?

(c) Describe an isometry g such that $g(\overrightarrow{ED}) = \overrightarrow{HG}$, and $g(\overrightarrow{EF}) = \overrightarrow{HK}$.

(d) Is $\overrightarrow{\angle DEF} \cong \overrightarrow{\angle GHK}$?

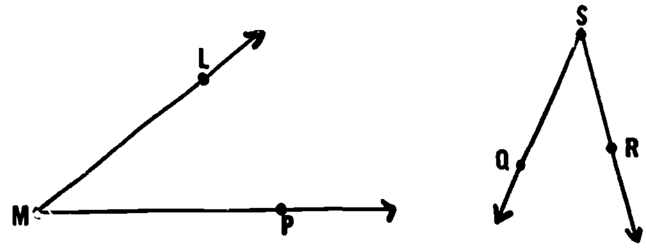
(e) Describe an isometry h such that $h(\overrightarrow{BA}) = \overrightarrow{HG}$, and $h(\overrightarrow{BC}) = \overrightarrow{HK}$.

(f) Is $\overrightarrow{\angle ABC} \cong \overrightarrow{\angle GHK}$?

(g) Is congruence of sensed angles a transitive relation?

7. Is congruence of sensed angles an equivalence relation? (See Exercises 3, 5, and 6 above.)

8.



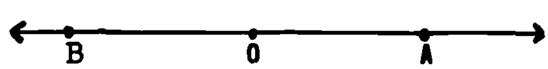
(a) Describe an isometry f such that $f(\overrightarrow{SR}) = \overrightarrow{ML}$ and $f(\overrightarrow{SQ}) = \overrightarrow{MP}$.

(Hint: Consider a translation followed by a rotation.)

(b) Is $\overrightarrow{\angle RSQ} \cong \overrightarrow{\angle LMP}$?

(c) Is $\overrightarrow{\angle RSQ} \cong \overrightarrow{\angle PML}$?

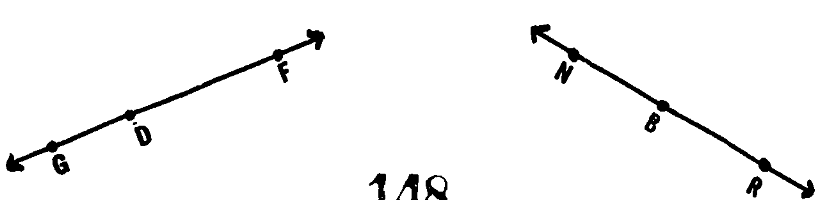
9. Since the definition of sensed angles does not demand non-collinearity of rays, we have for instance $(\overrightarrow{OA},\overrightarrow{OB})$ as a sensed angle, where A, O, and B are collinear points with O between A and B. Such an angle is called a <u>straight</u> <u>angle</u>.



(a) Is $(\overrightarrow{OA},\overrightarrow{OB}) \cong (\overrightarrow{OB},\overrightarrow{OA})$?

(b) If your answer is "no," tell why not. If your answer is "yes," describe a direct isometry f such that $f(\overrightarrow{OA}) = \overrightarrow{OB}$, and $f(\overrightarrow{OB}) = \overrightarrow{OA}$.

10.



(a) Is $\overrightarrow{\angle FDG} \cong \overrightarrow{\angle RBN}$? (If so, describe the direct isometry.)

(b) Is $\overrightarrow{\angle FDG} \cong \overrightarrow{\angle NBR}$? (If so, describe the direct isometry.)

11. Since the definition of congruent sensed angles does not demand distinctness of rays, the ordered pair $(\overrightarrow{OA},\overrightarrow{OA})$ is a sensed angle. Such a sensed angle is called a <u>zero angle</u>. (Recall that an ordered pair of numbers (x,y) permits x and y to be the same number; for instance, (2,2) is a perfectly good ordered pair of numbers. So with an ordered pair of rays we allow the rays of the pair to be the same ray.)

(a) Is $(\overrightarrow{OB},\overrightarrow{OB}) \cong (\overrightarrow{AN},\overrightarrow{AN})$? (If so, describe the isometry.)

(b) Is $(\overrightarrow{OB},\overrightarrow{OB}) \cong (\overrightarrow{XY},\overrightarrow{XY})$, where $\overrightarrow{XY}$ is any other ray in the plane?

12. With ruler and compass, carry out the construction of Figure 8.4.

8.3 Standard Position

The rays of a sensed angle will usually be considered as subsets of the coordinate plane. A sensed angle such as $(\overrightarrow{OA},\overrightarrow{OB})$ in Figure 8.6 is said to be in standard position.

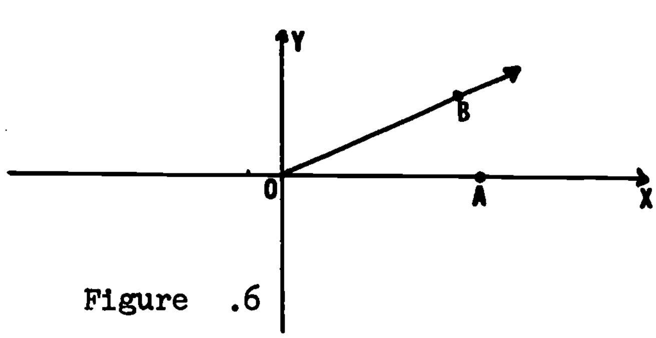


Figure .6

<u>Definition</u> 3. A sensed angle AOB is in <u>standard position</u>

if and only if $\overrightarrow{OA}$ is the positive x-axis. Notice from the definition that it is the initial side of the angle which is the positive x-axis.

Example 1. In Figure 8.7 the sensed angles NOR and AOB are in standard position. The sensed angles RON and BOA are not in standard position.

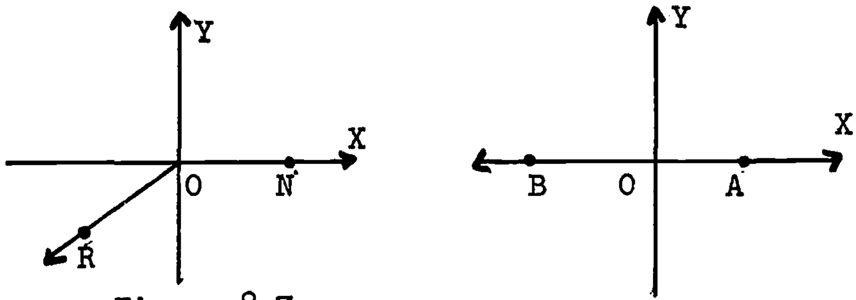


Figure 8.7

Example 2. Figure 8.8 shows $\angle\overrightarrow{AOB}$ in standard position, together with the circle $x^2 + y^2 = r^2$, with center at the origin and radius r. The "darkened piece of the circle" in the figure represents an arc of the circle. In particular, it is the arc intercepted by the rays of the sensed angle. Physically, this arc might be thought of as the "path" covered in moving counterclockwise around the circle from the initial side of the sensed angle to the terminal side.

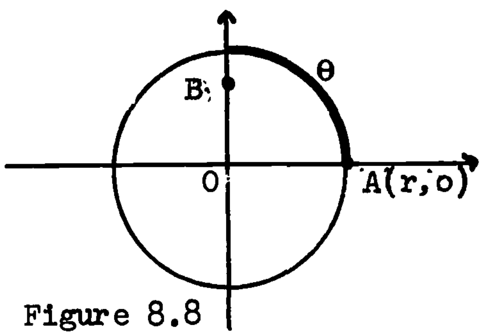


Figure 8.8

Since the circumference of the circle is

$$2\pi r,$$

the arc in this case has a length which is $\frac{1}{4}$ of $2\pi r$, or

$$\frac{1}{2}\pi r.$$

This number, the arc length, is denoted by the symbol "θ." We assume that every arc of a circle has a unique length.

In Figure 8.9 we have $\angle \overrightarrow{F}OD$ in standard position, and two circles:

$$x^2 + y^2 = r_1^2$$
$$x^2 + y^2 = r_2^2 \qquad (r_1 < r_2)$$

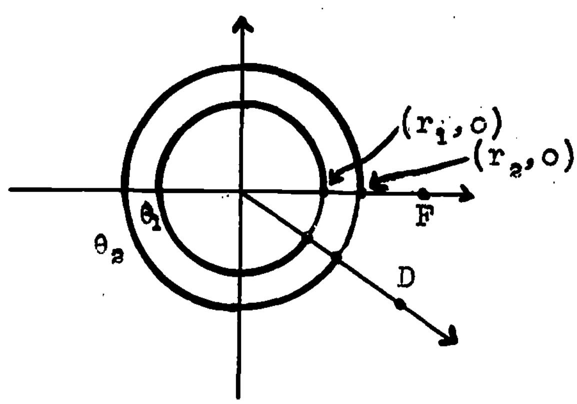


Figure 8.9

The circumference of the larger circle is $2\pi r_2$, and the circumference of the smaller circle is $2\pi r_1$.

$$\frac{2\pi r_2}{2\pi r_1} = \frac{2\pi}{2\pi} \cdot \frac{r_2}{r_1} = \frac{r_2}{r_1}$$

In other words, the ratio of the circumferences is the same as the ratio of the radii. (If, for instance, the radius of one circle is twice that of another circle, then its circumference is twice as great also.) This same equality of ratios holds also for arcs of the circles intercepted by the same standard position angle. Thus in Figure 8.9:

$$\frac{\theta_2}{\theta_1} = \frac{r_2}{r_1}$$

If we want the ratio of arc length to radius, we can write:

$$\frac{\theta_2}{r_2} = \frac{\theta_1}{r_1} \qquad \text{(Why?)}$$

This suggests the following definition.

<u>Definition</u> 4. If AOB is a sensed angle in standard position and it intercepts an arc of length θ (measured counterclockwise from the initial side) on a circle $x^2 + y^2 = r^2$, then we call $\frac{\theta}{r}$ the <u>radian</u> <u>measure</u> of the sensed angle.

Stated informally, to find the radian measure of a standard position angle, divide the length of the arc it intercepts by the radius of the circle. Since this ratio will be the same no matter what circle is used, we shall most often use the <u>unit circle</u>

$$x^2 + y^2 = 1.$$

This is a convenient choice since, for the unit circle, $\frac{\theta}{r} = \frac{\theta}{1} = \theta$. Therefore, the arc length itself is the radian measure of the angle.

<u>Example</u> 3. $\overrightarrow{\angle AOB}$, in standard position, intercepts an arc which is $\frac{1}{8}$ of the circle. (See Figure 8.10.) The circumference of the unit circle is $2\pi(1) = 2\pi$.

$\frac{1}{8}$ of $\pi 2$ is $\frac{\pi}{4}$.

Therefore, the radian measure of $(\overline{OA}, \overline{OB})$ is $\frac{\pi}{4}$.

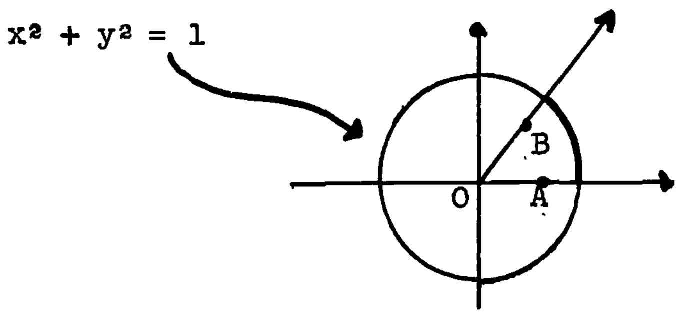


Figure 8.10

The definition of radian measure of a standard position sensed angle sets up a function

$$m : SPSA \longrightarrow [0, 2\pi) \quad ,$$

where SPSA is the set of sensed angles in standard position. Note that the range of m is $\{x : 0 \leq x < 2\pi\}$. Using function notation in Example 3, "$m(\angle \overrightarrow{AOB}) = \frac{\pi}{4}$, " says that the radian measure of $\angle \overrightarrow{AOB}$ is $\frac{\pi}{4}$.

<u>Example</u> 4. In Figure 8.11 (a) $\angle \overrightarrow{TSR}$ is not in standard position. However, $\angle \overrightarrow{TSR} \cong \angle \overrightarrow{AOB}$ of Figure 8.11(b). $m\angle \overrightarrow{AOB} = \frac{1}{4} \times 2\pi = \frac{\pi}{2}$.

Therefore, we say that $\angle \overrightarrow{TSR}$ also has a radian measure $\frac{\pi}{2}$.

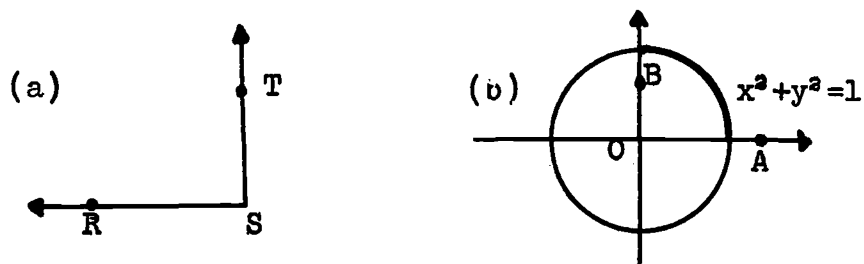


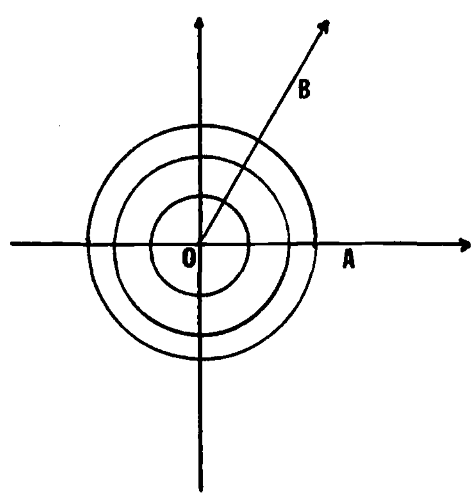
Figure 8.11

Example 4 serves to illustrate

Definition 5. Any sensed angle congruent to $\angle\overrightarrow{AOB}$, where $\angle\overrightarrow{AOB}$ is in standard position, is assigned the same radian measure as $\angle\overrightarrow{AOB}$.

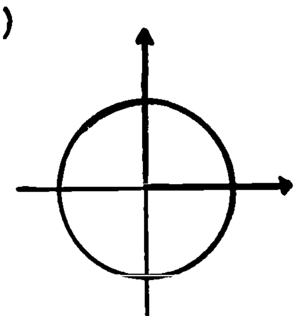
8.4 Exercises

1. The accompanying diagram shows a standard position $\angle\overrightarrow{AOB}$ which intercepts $\frac{1}{6}$ of three different circles. The circles have radii of 1, 2, and 3. For each of the three circles compute θ and $\frac{\theta}{r}$ where θ is the length of the intercepted arc, and r is the radius of the circle.

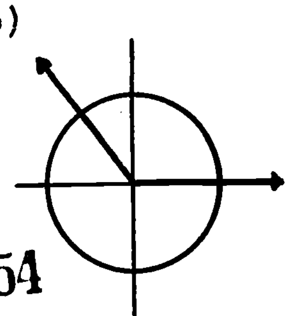


2. In each of the following diagrams, estimate as closely as possible the radian measure of the standard position sensed angle shown.

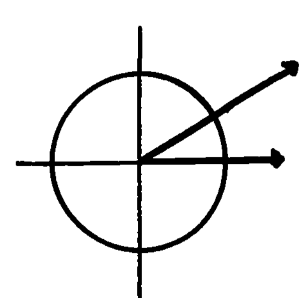
(a)



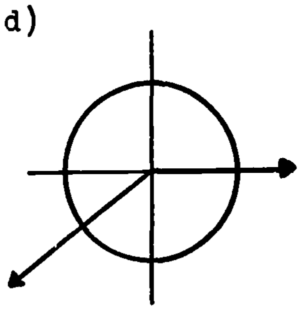
(b)



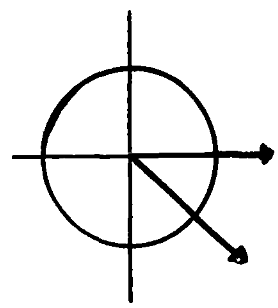
(c)



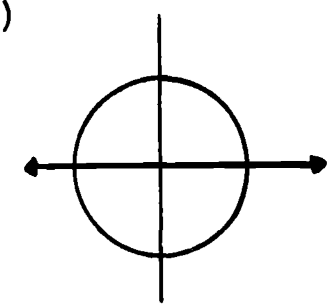
(d)



(e)



(f)



(Hint: Assume the circles are unit circles, and estimate the fraction of the circle which is intercepted by the sensed angle.)

3. Draw a unit circle. Then draw rays so that $\overrightarrow{OA}$ is the positive x-axis and:

(a) $m\angle\overrightarrow{AOB} = \frac{1}{4}\pi$ (b) $m\angle\overrightarrow{AOC} = \frac{1}{2}\pi$

(c) $m\angle\overrightarrow{AOD} = \frac{3}{4}\pi$ (d) $m\angle\overrightarrow{AOE} = \pi$

(e) $m\angle\overrightarrow{AOF} = \frac{5}{4}\pi$ (f) $m\angle\overrightarrow{AOG} = \frac{3}{2}\pi$

(g) $m\angle\overrightarrow{AOH} = \frac{7}{4}\pi$

4. Draw a unit circle. Then draw rays so that $\overrightarrow{OA}$ is the positive x-axis and:

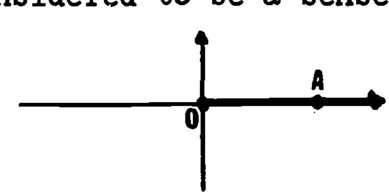
(a) $m\angle\overrightarrow{AOB} = \frac{1}{6}\pi$ (b) $m\angle\overrightarrow{AOC} = \frac{1}{3}\pi$

(c) $m\angle\overrightarrow{AOD} = \frac{2}{3}\pi$ (d) $m\angle\overrightarrow{AOE} = \frac{7}{6}\pi$

(e) $m\angle\overrightarrow{AOF} = \frac{11}{6}\pi$ (f) $m\angle\overrightarrow{AOG} = \frac{4}{3}\pi$

(g) $m\angle\overrightarrow{AOH} = \frac{5}{3}\pi$

5. Recall from Section 8.1 that the ordered pair of rays $(\overrightarrow{OA},\overrightarrow{OA})$ is considered to be a sensed angle.



What is the radian measure of $(\overrightarrow{OA},\overrightarrow{OA})$? Since we may consider this angle to intercept an arc of zero length, we say that the radian measure is 0.

What is the radian measure of $(\overrightarrow{RS}, \overrightarrow{RS})$, where $\overrightarrow{RS}$ is any ray in the plane? Why?

6. The function m assigns to each standard position sensed angle exactly one number as its radian measure.

   (a) Are negative numbers used as assignments?

   (b) Is 0 used as an assignment? (See Exercise 5 above.)

   (c) Explain why 2π is not used as an assignment.

   (d) Are numbers greater than 2π used as assignments?

   (e) What is the range of the function m?

7. If the domain of the function m is taken as the set of sensed angles in standard position, is m a one-to-one function? Is any such sensed angle assigned more than one number? Is any number assigned to more than one such sensed angle?

8. The function $m : SPSA \longrightarrow [0, 2\pi)$ is one-to-one and onto. Hence there is an inverse function

$$m^{-1} : [0, 2\pi) \longrightarrow SPSA.$$

   Draw a unit circle. Then draw appropriate angles for each of the following:

   (a) $m^{-1}(\frac{1}{4}\pi)$ Call it $\overrightarrow{\angle AOB}$.

   (b) $m^{-1}(\frac{3}{4}\pi)$ Call it $\overrightarrow{\angle AOC}$.

   (c) $m^{-1}(\frac{5}{3}\pi)$ Call it $\overrightarrow{\angle AOD}$.

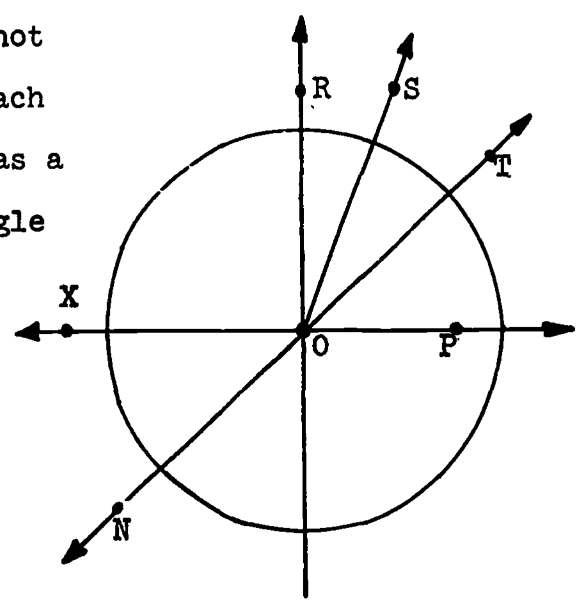
   (d) $m^{-1}(\pi)$ Call it $\overrightarrow{\angle AOE}$.

   (e) $m^{-1}(0)$ Call it by some appropriate name.

(f) $m^{-1}(1)$ Call it $\overrightarrow{\angle AOF}$, estimating as closely as possible.

(g) $m^{-1}(2)$ Call it $\overrightarrow{\angle AOG}$.

9. Using the diagram at the right, complete the following. Notice that the sensed angles are not in standard position, but each will have the same measure as a standard position sensed angle to which it is congruent.



(a) $m\overrightarrow{\angle XOR} =$

(b) $m\overrightarrow{\angle ROT} =$

(c) $m\overrightarrow{\angle XOP} =$

(d) $m\overrightarrow{\angle RON} =$

(e) $m\overleftarrow{\angle SOR} =$

(f) $m\overleftarrow{\angle SOX} =$

(g) $m\overleftarrow{\angle SOP} =$

Be careful! The answer to (a), for example is not $\frac{\pi}{2}$.

8.5 Circular Functions of Angles

Certain important mathematical functions are called circular functions; they may be defined by use of the unit circle $x^2 + y^2 = 1$.

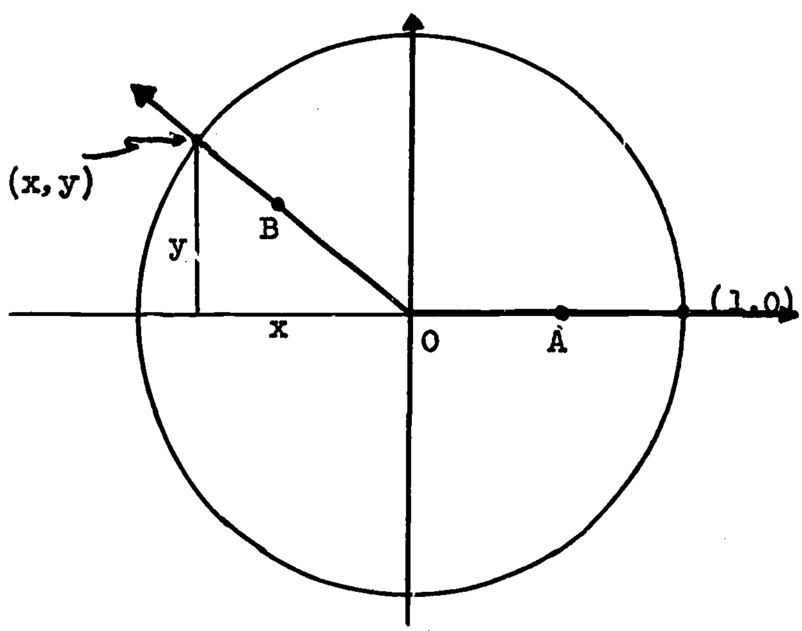


Figure 8.12

<u>Definition</u> 6. Let $\angle \overrightarrow{AOB}$ be a sensed angle in standard position. Then S and C are two functions such that

$$S(\angle \overrightarrow{AOB}) = y, \text{ and}$$
$$C(\angle \overrightarrow{AOB}) = x,$$

where (x, y) is the point of intersection of the unit circle and the terminal side of $\angle \overrightarrow{AOB}$. (See Figure 8.12.) The function S is called the <u>SINE function</u>, and the function C is the <u>COSINE function</u>. The <u>domain</u> of the SINE function and the COSINE function is the set of all sensed angles in standard position. It is important also to consider the range of each of these functions. (See Exercises 3 and 4 in Section 8.6.)

<u>Example</u> 1. In Figure 8.13

$S(\overrightarrow{\angle AOB}) = 1$, or SINE $(\overrightarrow{\angle AOB}) = 1$.

$C(\overrightarrow{\angle AOB}) = 0$, or COSINE $(\overrightarrow{\angle AOB}) = 0$.

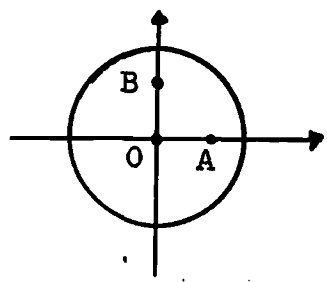


Figure 8.13

<u>Example</u> 2. In Figure 8.14 $(\frac{1}{2}\sqrt{3}, \frac{1}{2})$ is a point on the unit circle. (Why?) Thus:

SINE $(\overrightarrow{\angle AOB}) = \frac{1}{2}$

COSINE $(\overrightarrow{\angle AOB}) = \frac{1}{2}\sqrt{3}$

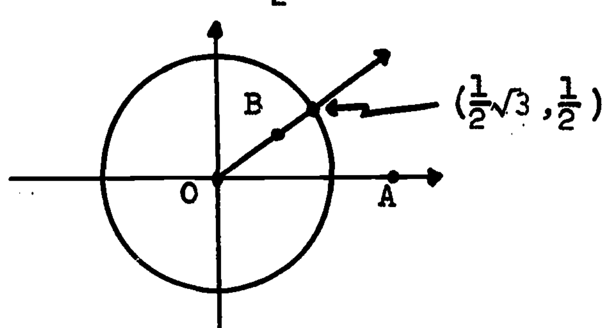
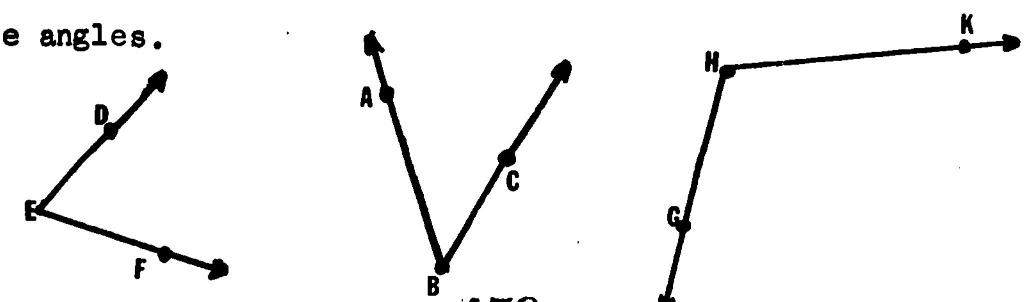


Figure 8.14

Certain pairs of sensed angles are said to have the same sense, while other pairs are said to have opposite sense. We shall not attempt a precise definition of these concepts but shall consider a helpful physical interpretation of them. Look at these angles.



$\vec{\angle}ABC$ and $\vec{\angle}DEF$ have the same sense. If one thinks of moving in a path from A to B to C, the path may be described as counterclockwise; similarly, moving from D to E to F is a counterclockwise path. On the other hand, $\vec{\angle}ABC$ and $\vec{\angle}GHK$ have opposite sense. Whereas the path from A to B to C is counterclockwise, the path from G to H to K is clockwise.

There is an important relation between the SINES of standard position angles and their senses. To illustrate this, look at Figure 8.15. $\vec{\angle}AOB$ and $\vec{\angle}AOC$ have the same sense. Also $\vec{\angle}AOD$ and $\vec{\angle}AOE$ have the same sense. In fact, any two standard position

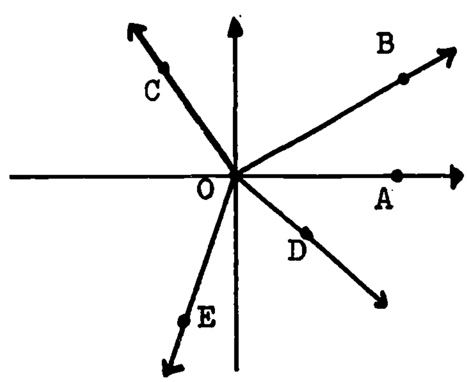


Figure 8.15

sensed angles have the <u>same sense</u> if and only if their terminal sides lie in the same half-plane determined by the x-axis. On the other hand, $\vec{\angle}AOB$ and $\vec{\angle}AOD$ have opposite senses. Any two standard position sensed angles have opposite senses if and only if their terminal sides lie in opposite half-planes determined by the x-axis. Congruent sensed angles (whether or not in standard position) have the same sense.

Speaking informally, then, we can say that two standard position sensed angles have the same sense if their terminal sides are both "above the x-axis" or both "below the x-axis."

However in the first case, the SINES are both positive; in the second, the SINES are both negative. Thus we make the following statement:

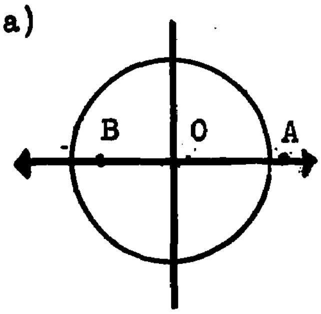
> Two sensed angles in standard position have the same sense if and only if their SINES are both positive, or their SINES are both negative.

The relation between congruence of unsensed angles and congruence of sensed angles can now be stated: $\overrightarrow{\angle ABC} \cong \overrightarrow{\angle DEF}$ iff $\angle ABC \cong \angle DEF$, and $\overrightarrow{\angle ABC}$ and $\overrightarrow{\angle DEF}$ have the same sense.

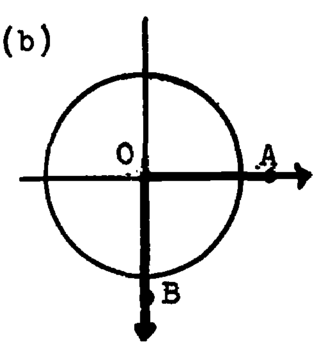
8.6 Exercises

1. Using each of the diagrams below, find SINE($\overrightarrow{\angle AOB}$), and COSINE($\overrightarrow{\angle AOB}$).

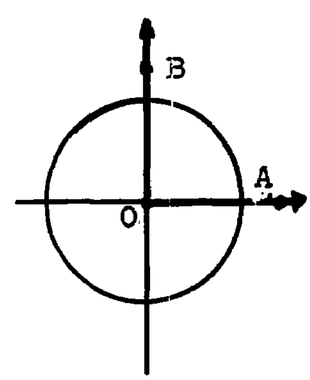
(a)



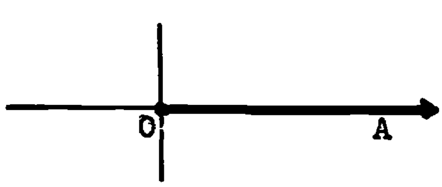
(b)



(c)



2.



Remember that $(\overrightarrow{OA},\overrightarrow{OA})$ is a sensed angle in standard position.

(a) What is SINE($(\overrightarrow{OA},\overrightarrow{OA})$)?

(b) What is COSINE($(\overrightarrow{OA},\overrightarrow{OA})$)?

3. (a) Explain why the SINE function can never assign a number greater than 1 to a sensed angle. (Hint: remem-

ber how the SINE function is defined in terms of the unit circle.)

(b) Explain why the SINE function can never assign a number less than -1 to a sensed angle.

(c) Assuming that the SINE function assigns every real number between -1 and 1 to some sensed angle, what is the range of the SINE function?

(d) Is SINE : SPSA $\longrightarrow$ [-1, 1] a one-to-one function?

4. (a) What is the range of the COSINE function?

(b) Is COSINE : SPSA $\longrightarrow$ [-1, 1] a one-to-one function?

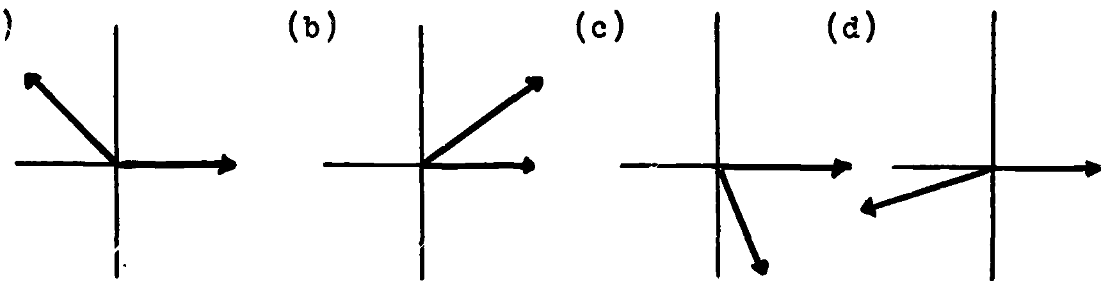
5. Prove that for every $\overrightarrow{\angle AOB}$ in standard position:

$[\text{SINE}(\overrightarrow{\angle AOB})]^2 + [\text{COSINE}(\overrightarrow{\angle AOB})]^2 = 1$

(Hint: Use the unit circle.)

6. For each of the following standard position sensed angles, tell whether the SINE function assigns a positive number, zero, or a negative number.

(a) (b) (c) (d)



7. Answer the same questions for the COSINE function.

8. Draw all sensed angles in standard position to which

(a) the SINE function assigns the number 0.

(b) the SINE function assigns the number 1.

9. Draw all sensed angles in standard position to which

(a) the COSINE function assigns the number 0.

(b) the COSINE function assigns the number 1.

10. Is there a $\angle\overrightarrow{AOB}$ such that SINE($\angle\overrightarrow{AOB}$) = COSINE($\angle\overrightarrow{AOB}$)? How many are there?

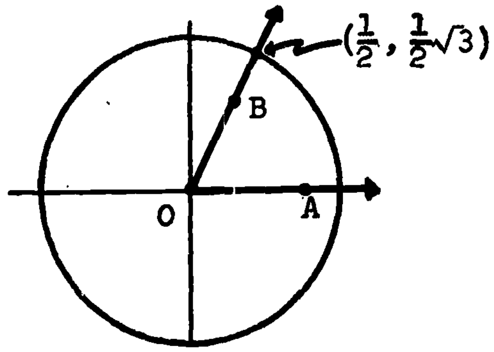
* 11. What number does the SINE function assign to each of the angles in Exercise 10? (Hint: Use Exercise 5 above.)

12. Draw all sensed angles in standard position such that

$$\text{COSINE}(\angle\overrightarrow{AOB}) = -\text{SINE}(\angle\overrightarrow{AOB}).$$

13. Show that $(\frac{1}{2},\frac{1}{2}\sqrt{3})$ is a point on the unit circle, by using the equation of the unit circle.

14.



SINE($\angle\overrightarrow{AOB}$) = $\frac{1}{2}\sqrt{3}$

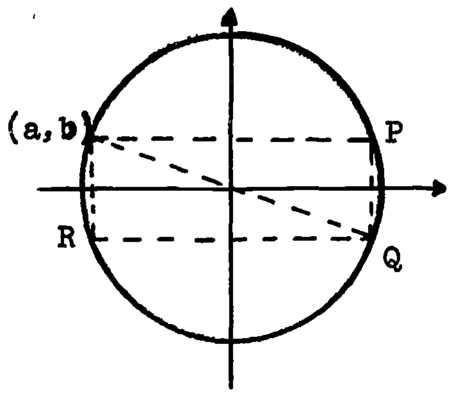
COSINE($\angle\overrightarrow{AOB}$) = $\frac{1}{2}$

(a) Draw $\angle\overrightarrow{AOC}$ (different from $\angle\overrightarrow{AOB}$) such that SINE($\angle\overrightarrow{AOC}$) = $\frac{1}{2}\sqrt{3}$.
(Hint: Use a reflection in the y-axis.)

(b) Draw $\angle\overrightarrow{AOD}$ (different from $\angle\overrightarrow{AOB}$) such that COSINE($\angle\overrightarrow{AOE}$) = $\frac{1}{2}$.
(Hint: Use a line reflection.)

(c) Draw $\angle\overrightarrow{AOE}$ such that SINE($\angle\overrightarrow{AOE}$) = $-\frac{1}{2}\sqrt{3}$, and COSINE($\angle\overrightarrow{AOE}$) = $-\frac{1}{2}$.

15. If (a,b) is the point so labelled on the unit circle, what are the coordinates of the other points shown?



16. Draw a picture showing that two standard position angles do not necessarily have the same sense if their COSINES are both positive or both negative.

8.7 Circular Functions of Real Numbers

The mapping m assigns a real number θ, $0 \leq \theta < 2\pi$, to every sensed angle in standard position. Thus the domain of m is the set SPSA (standard position sensed angles), and the range is the set $\{\theta : 0 \leq \theta < 2\pi\}$, as indicated below:

$$\text{SPSA} \xrightarrow{m} \{\theta : 0 \leq \theta < 2\pi\}.$$

Since this mapping is one-to-one and onto, there is an inverse mapping m^{-1}, represented by

$$\{\theta : 0 \leq \theta < 2\pi\} \xrightarrow{m^{-1}} \text{SPSA}$$

Note that the range of m^{-1} is SPSA; and we have previously defined a mapping

$$\text{SPSA} \xrightarrow{\text{SINE}} \{x : -1 \leq x \leq 1\}$$

whose domain is SPSA. Therefore these two functions may be composed as shown below resulting in a new (composite) function.

$$\begin{tikzcd}
\{\theta : 0 \leq \theta < 2\pi\} \arrow[r, "m^{-1}"] \arrow[rr, bend right, "\mathrm{SINE} \circ m^{-1}"'] & \mathrm{SPSA} \arrow[r, "\mathrm{SINE}"] & \{x : -1 \leq x \leq 1\}
\end{tikzcd}$$

The domain of this composite function is $\{\theta : 0 \leq \theta < 2\pi\}$, and the range is $\{x : -1 \leq x \leq 1\}$. This function is called the <u>sine</u> (abbreviated "sin") <u>function</u>, to distinguish it from the SINE function, whose domain is SPSA. Thus while the SINE function assigns a real number to each standard position sensed angle, the sine function assigns a real number to every real number between 0 and 2π (not including 2π). The sine function is formally defined as follows.

<u>Definition</u> 7. The function

$\underline{\text{sine}}$: $\{\theta : 0 \leq \theta < 2\pi\} \longrightarrow \{x : -1 \leq x < 1\}$

is defined by

$$\text{sine } \theta = \text{SINE } (\angle \overrightarrow{COD}),$$

where $\angle \overrightarrow{COD}$ is the unique standard position sensed angle with radian measure θ.

<u>Example</u> 1. Suppose $\angle \overrightarrow{AOB}$ is a standard position sensed angle such that

$$m(\angle \overrightarrow{AOB}) = \frac{\pi}{2}$$

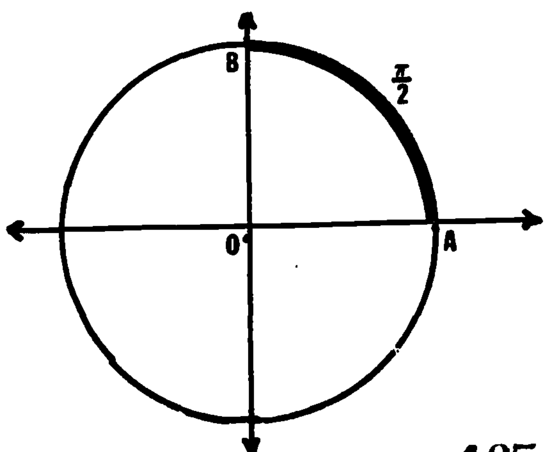


Figure 8.16

Then $m^{-1}(\frac{\pi}{2}) = \angle\vec{A}OB$.

$\text{SINE}(\angle\vec{A}OB) = 1$

$(\text{SINE} \circ m^{-1})(\frac{\pi}{2}) = \text{sine}(\frac{\pi}{2}) = 1$

(See Figure 8.17.)

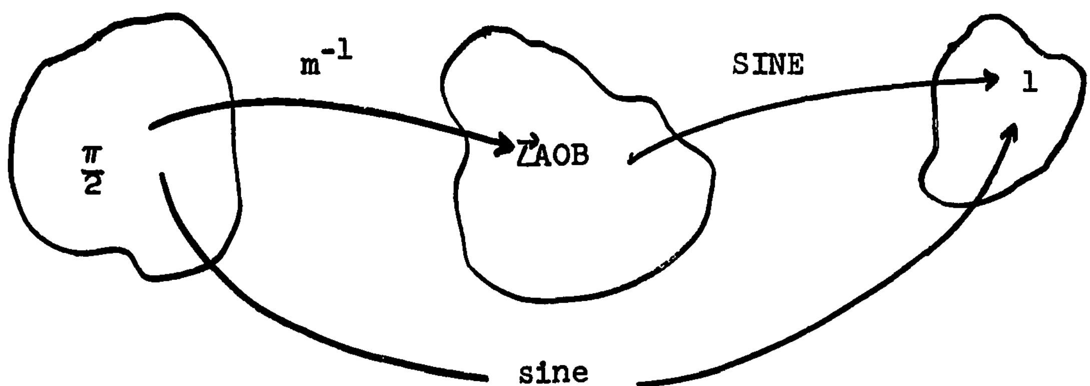


Figure 8.17

<u>Example</u> 2. $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ is a point on the unit circle.

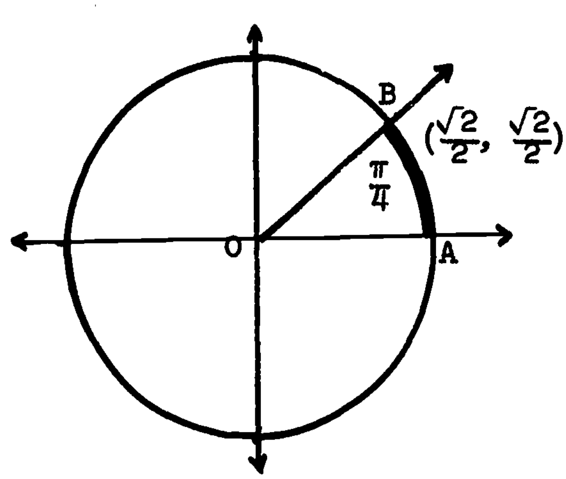


Figure 8.18

$m(\angle\vec{A}OB) = \frac{\pi}{4}$

$m^{-1}(\frac{\pi}{4}) = \angle\vec{A}OB$

$\text{SINE}(\angle\vec{A}OB) = \frac{\sqrt{2}}{2}$

Therefore sine $\frac{\pi}{2} = \frac{\sqrt{2}}{2}$

That is, sine $(\frac{\pi}{4}) = \text{SINE}(m^{-1}(\frac{\pi}{4})\ )$

A <u>cosine function</u> is defined in a way comparable to that in which the sine function was defined.

<u>Definition</u> 8. The function

$\underline{\text{cosine}}: \{\theta : 0 \leq \theta < 2\pi\} \longrightarrow \{x : -1 \leq x \leq 1\}$

is defined by

$$\text{cosine } \theta = \text{COSINE}(\overrightarrow{\angle COD}),$$

where $\overrightarrow{\angle COD}$ is the unique standard position sensed angle whose radian measure is θ.

$$\begin{tikzcd}
\{\theta : 0 \leq \theta < 2\pi\} \arrow[r, "m^{-1}"] \arrow[rr, bend right, "\text{cosine}"'] & \text{SPSA} \arrow[r, "\text{COSINE}"] & \{x : -1 \leq x \leq 1\}
\end{tikzcd}$$

<u>Example</u> 3. Suppose $\overrightarrow{\angle AOB}$ is a standard position sensed angle such that

$$m(\overrightarrow{\angle AOE}) = \frac{\pi}{2}.$$

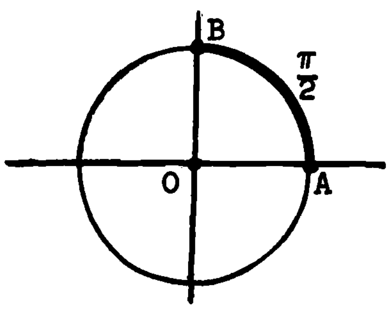


Figure 8.19

Then $m^{-1}(\frac{\pi}{2}) = \overrightarrow{\angle AOB}$.

$\text{COS}(\overrightarrow{\angle AOB}) = 0$

Therefore $\cos \frac{\pi}{2} = 0$.

That is, $\cos \frac{\pi}{2} = \text{COSINE}(m^{-1}(\frac{\pi}{2}))$.

(See Figure 8.20.)

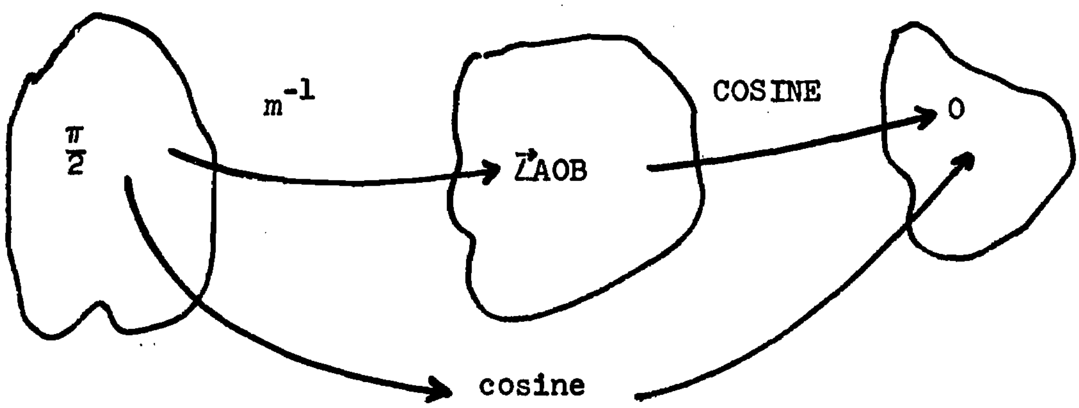


Figure 8.20

Although the domain of the SINE and COSINE functions has b been specifed as the set of sensed angles in standard position, we do not want to restrict ourselves to speaking of the SINE and COSINE of only those angles which are in standard position. Acco: dingly:

<u>Definition</u> 9. Let $\overrightarrow{\angle RST}$ be a sensed angle.

Then $\mathrm{SIN}(\overrightarrow{\angle RST}) = \mathrm{SIN}(\overrightarrow{\angle AOB})$

and $\mathrm{COS}(\overrightarrow{\angle RST}) = \mathrm{COS}(\overrightarrow{\angle AOB})$

if and only if

$\overrightarrow{\angle RST} \cong \overrightarrow{\angle AOB}$, and $\overrightarrow{\angle AOB}$ is in standard position.

<u>Example</u> 4. In Figure 8.21 $\overrightarrow{\angle AOB}$ is in standard position.

$\mathrm{SIN}(\overrightarrow{\angle AOB}) = 1.$

$\overrightarrow{\angle COD} \cong \overrightarrow{\angle AOB}$. So

$\mathrm{SIN}(\overrightarrow{\angle COD}) = 1.$

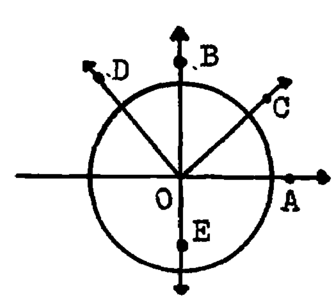


Figure 8.21

On the other hand,

$\mathrm{SIN}(\overrightarrow{\angle DOC}) = \mathrm{SIN}(\overrightarrow{\angle BOA}) = \mathrm{SIN}(\overrightarrow{\angle AOE}) = -1,$

$\overrightarrow{\angle AOE}$ being a standard position angle.

8.8 <u>Exercises</u>

1. Show that $(-\frac{1}{2}\sqrt{3}, \frac{1}{2})$ is a point on the unit circle.

2. Using the diagram, complete statements (a) -- (f).

(a) $m(\angle \vec{A}OB) =$

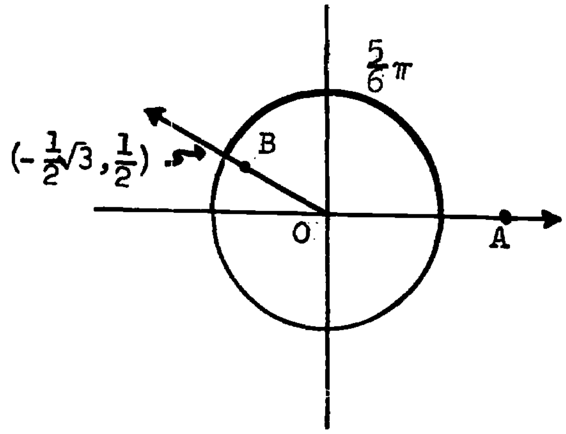
(b) $m^{-1}(\frac{5}{6}\pi) =$

(c) $\text{SINE}(\angle \vec{A}OB) =$

(d) sine $\frac{5}{6}\pi =$

(e) $\text{COSINE}(\angle \vec{A}OB) =$

(f) cosine $\frac{5}{6}\pi =$



3. Show that $(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$ is a point on the unit circle.

4. Using the diagram, complete statement (a) - (h).

(a) $m(\angle \vec{A}OB) =$

(b) $m^{-1}(\frac{7}{4}\pi) =$

(c) $\text{SIN}(\angle \vec{A}OB) =$

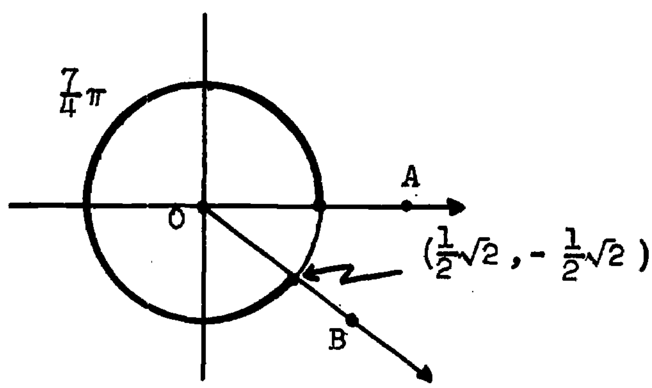
(d) $\text{SIN}(m^{-1}(\frac{7}{4}\pi)) =$

(e) $\sin\frac{7}{4}\pi =$

(f) $\text{COS}(\angle \vec{A}OB) =$

(g) $\text{COS}(m^{-1}(\frac{7}{4}\pi)) =$

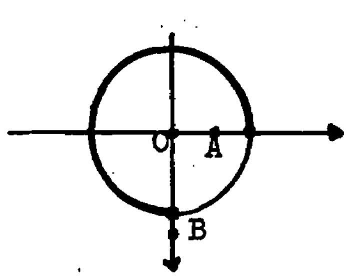
(h) $\cos \frac{7}{4}\pi =$



5. Use the diagram to complete statements (a) and (b).

(a) $[\text{SIN} \circ m^{-1}](\frac{3}{2}\pi) =$

(b) $[\text{COS} \circ m^{-1}](\frac{3}{2}\pi) =$



6. Complete the following:

(a) $\sin \frac{\pi}{2} =$

(b) $\sin \frac{3}{2}\pi =$

(c) $\sin \pi =$

(d) $\sin 0 =$

(e) $\cos \frac{\pi}{2} =$

(f) $\cos \frac{3}{2}\pi =$

(g) $\cos \pi =$

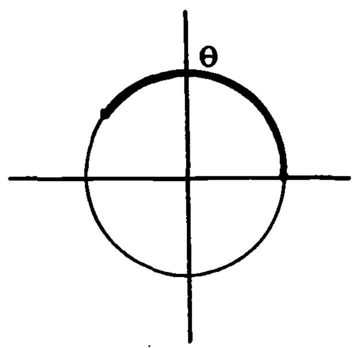
(h) $\cos 0 =$

7. For each of the following diagrams, decide which of the following statements are true:

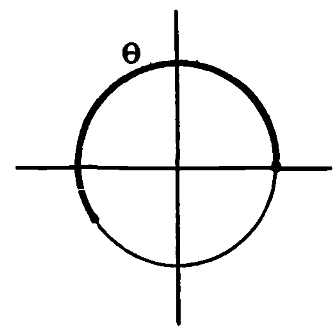
$\sin \theta < 0$; $\sin \theta = 0$; $\sin \theta > 0$

$\cos \theta < 0$; $\cos \theta = 0$; $\cos \theta > 0$.

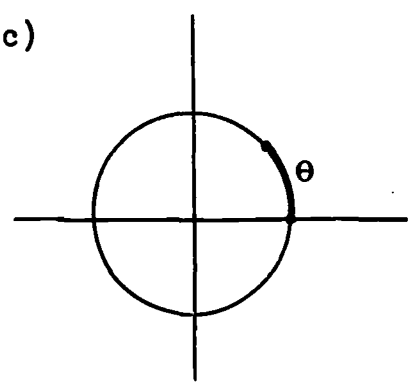
(a)



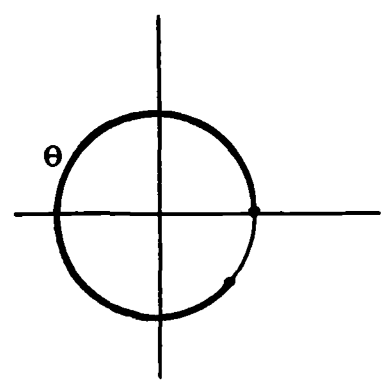
(b)



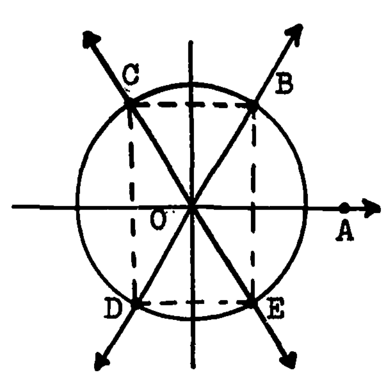
(c)



(d)



8. In the diagram $m(\angle \overrightarrow{AOB}) = \theta_1$, $m(\angle \overrightarrow{AOC}) = \theta_2$, $m(\angle \overrightarrow{AOD}) = \theta_3$, $m(\angle \overrightarrow{AOE}) = \theta_4$. Which of the following are true? Which are false



(a) $\sin \theta_1 = \sin \theta_2$ (b) $\cos \theta_1 = \cos \theta_2$

(c) $\sin \theta_1 = \sin \theta_3$ (d) $\cos \theta_1 = \cos \theta_3$

(e) $\sin\theta_1 = \sin\theta_4$ (f) $\cos\theta_1 = \cos\theta_4$

(g) $\sin\theta_1 = -\sin\theta_2$ (h) $\sin\theta_1 = -\sin\theta_3$

(i) $\sin\theta_1 = -\sin\theta_4$ (j) $\cos\theta_1 = -\cos\theta_2$

(k) $\cos\theta_1 = -\cos\theta_3$ (l) $\cos\theta_1 = -\cos\theta_4$

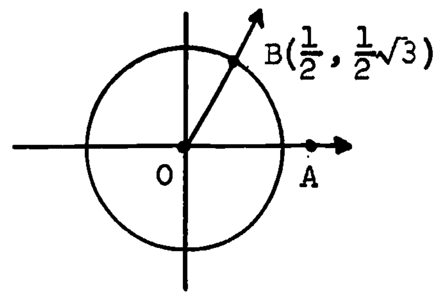
9. Given that $\angle\vec{A}OB$, in the diagram, intercepts an arc which is 1/6 of the circle, complete statements (a) -- (f).

(a) $m(\angle\vec{A}OB) =$

(b) $m^{-1}(\quad) = \angle\vec{A}OB$

(c) $\text{SINE}(\angle\vec{A}OB) =$

(d) $\text{SINE}(m^{-1}(\frac{\pi}{3})) =$



(e) $\sin\frac{\pi}{3} =$

(f) $\cos\frac{\pi}{3} =$

10. Given that $\angle\vec{A}OB$, in the diagram, intercepts an arc which is 3/8 of the unit circle, complete statements (a) -- (f).

(a) $m(\angle\vec{A}OB) =$

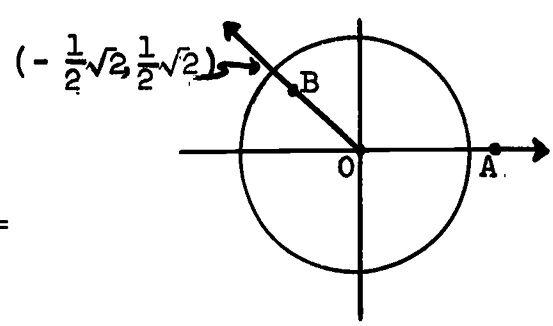
(b) $m^{-1}(\quad) = \angle\vec{A}OB$

(c) $\text{SIN}(\angle\vec{A}OB) =$

(d) $\text{SIN}(m^{-1}(\frac{3}{4}\pi)) =$

(e) $\sin\frac{3}{4}\pi =$

(f) $\cos\frac{3}{4}\pi =$



11. (a) What is the domain of the SINE function?

(b) Is the SINE function one-to-one?

(c) What is the domain of the sine function?

(d) Is the sine function one-to-one?

(e) What is the domain of the COSINE function?

(f) Is the COSINE function one-to-one?

(g) What is the domain of the cosine function?

(h) Is the cosine function one-to-one?

8.9 Degree Measure, Radian Measure, and Angle Addition

The protractor was used in Course I as an instrument for measuring unsensed angles. Such an instrument is based on the assumption that each unsensed angle is assigned a unique number called its degree measure. Thus, in Figure 8.22, $\angle ABC$ has a measure of 30° (read 30 degrees.)

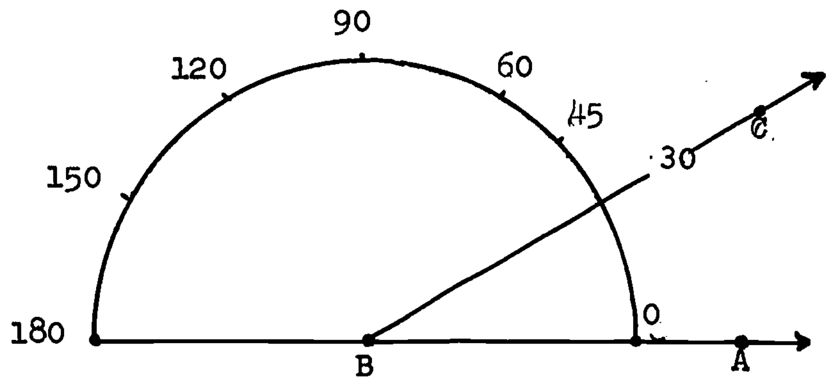


Figure 8.22

Degrees are also used to measure sensed angles. In Figure 8.23, $\overrightarrow{\angle AOB}$ is in standard position. It intercepts an arc of length $\frac{\pi}{2}$ on the unit circle. And so $m(\angle\overrightarrow{AOB}) = \frac{\pi}{2}$; that is, the radian measure of the sensed angle is $\frac{\pi}{2}$. However, $\overrightarrow{\angle AOB}$ is a

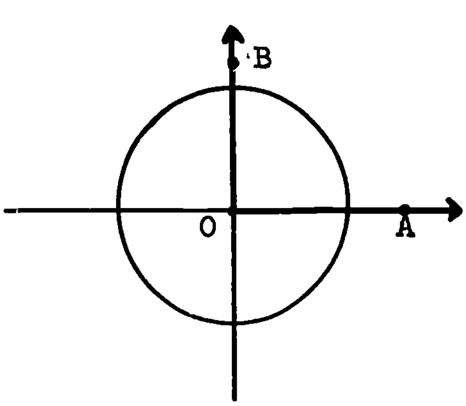


Figure 8.23

right angle, which has a degree measure of 90. Since $\overrightarrow{\angle AOB}$ is in standard position, we assign 90 as the degree measure of $\overrightarrow{\angle AOB}$ (but not of $\overrightarrow{\angle BOA}$); and

$$\frac{\pi}{2} \text{ radians} = 90^\circ$$

is a short way of stating that a sensed angle with degree measure 90 is assigned the number $\frac{\pi}{2}$ if radian measure is used. Similarly,

$$\pi \text{ radians} = 180^\circ,$$

and this equality is the basis for changing from one unit of angle measurement to the other, as are the following:

$$\pi \text{ radians} = 180^\circ \qquad \pi \text{ radians} = 180^\circ$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ \qquad \frac{\pi}{180} \text{ radians} = 1^\circ$$

Example 1. If a sensed angle has degree measure 105, what is its radian measure?

$$\begin{aligned} 180^\circ &= \pi \text{ radians} \\ 1^\circ &= \frac{\pi}{180} \text{ radians} \\ 105^\circ &= \left(105 \times \frac{\pi}{180}\right) \text{ radians} \\ &= \frac{7}{12}\pi \text{ radians} \end{aligned}$$

Example 2. If $m(\overrightarrow{\angle AOB}) = \frac{\pi}{5}$, what is the degree measure of $\overrightarrow{\angle AOB}$?

$$\begin{aligned} \pi \text{ radians} &= 180^\circ \\ \frac{\pi}{5} \text{ radians} &= \left(\frac{180}{5}\right)^\circ \\ &= 36^\circ \end{aligned}$$

Example 3. Given $\sin \frac{\pi}{2} = 1$. Then since $\frac{\pi}{2}$ radians $= 90^\circ$, it follows that $\sin 90^\circ = 1$.

Notice that the domain of the sine function is $\{x : 0 \leq x < 2\pi\}$. We will, when convenient, replace radian measure by equivalent degree measure and write $\sin \frac{\pi}{2} = \sin 90^\circ$, $\sin \frac{\pi}{3} = \sin 60^\circ$, etc.

If $m(\angle \overrightarrow{AOB}) = \frac{3}{2}\pi$, what is its degree measure? Figure 8.24 suggests a simple way of extending the protractor in order to find the degree measure of any sensed angle. Thus, $\frac{3}{2}\pi$ radians = 270^o.

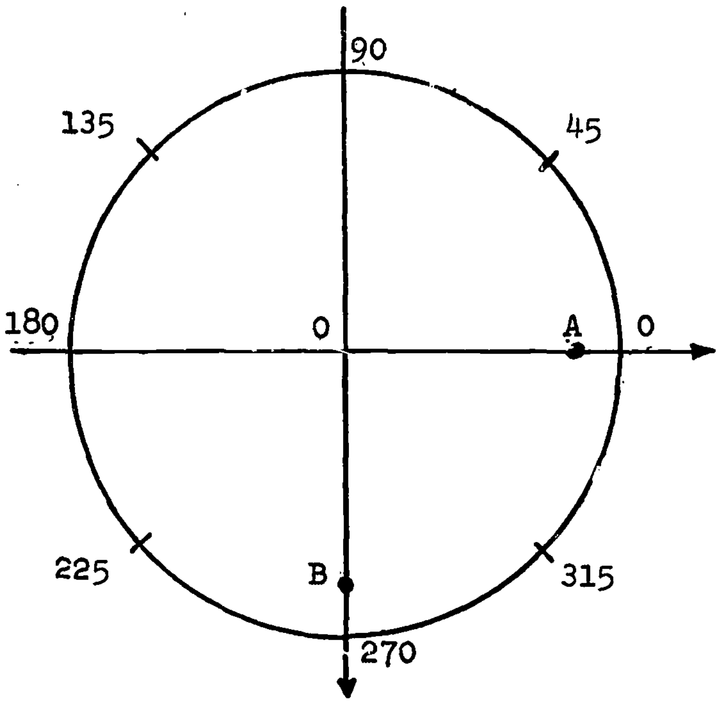


Figure 8.24

Note that if a sensed angle in standard position has its terminal side in the first quadrant, the angle has degree measure between 0 and 90; if in the second quadrant, between 90 and 180; if in the third quadrant, between 180 and 270; and if in the fourth quadrant, between 270 and 360.

Example 4. What is the radian measure of a sensed angle whose degree measure is 330?

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians}$$
$$330^o = (330 \times \frac{\pi}{180}) \text{ radians}$$
$$= \frac{11}{6}\pi \text{ radians.}$$

We turn now to some principles which will be useful in later work. In Figure 8.25, $\overrightarrow{\angle AOB}$ is a sensed angle intercepting an arc of length θ on the unit circle; thus, $m(\overrightarrow{\angle AOB}) = \theta$ (radians.)

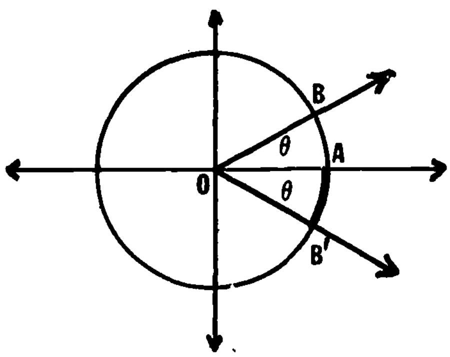


Figure 8.25

$\overrightarrow{\angle AOB'}$ is the reflection of $\overrightarrow{\angle AOB}$ in the x-axis. It seems reasonable to assume that $\overrightarrow{\angle AOB'}$ also intercepts an arc of length θ - that is, to assume that isometries (a line reflection in this case) preserve arc lengths just as they preserve lengths of segments. And since the circumference of the circle is 2π, this means that the arc associated with $\overrightarrow{\angle AOB'}$ has length $2\pi - \theta$; that is, $m(\overrightarrow{\angle AOB'}) = 2\pi - \theta$. This illustration suggests the general principle stated below.

If $\overrightarrow{\angle AOB}$ is a standard position sensed angle with $m(\overrightarrow{\angle AOB}) = \theta$, and $\overrightarrow{\angle AOB'}$ is the reflection of $\overrightarrow{\angle AOB}$ in the x-axis, then $m(\overrightarrow{\angle AOB'}) = 2\pi - \theta$.

Of course the principle is applicable also when degree measure is used, the degree measure of the reflected angle then being $360 - \theta$.

<u>Example</u> 5. Suppose $\vec{\angle}AOB$ is a standard position sensed angle with degree measure 225.

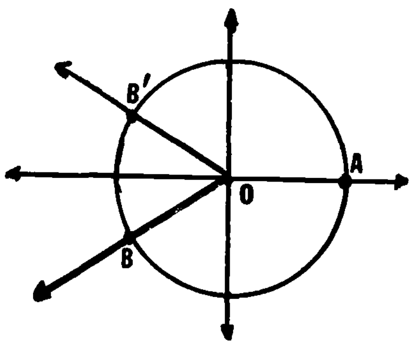


Figure 8.26

Let $\vec{\angle}AOB'$ be the reflection of $\vec{\angle}AOB$ in the x-axis. Then the degree measure of $\vec{\angle}AOB'$ is 360 - 225, or 135.

A second principle is closely related to the one above. Thus, in Figure 8.27, $\vec{\angle}AOB'$ is the image of $\vec{\angle}AOB$ under reflection in the x-axis

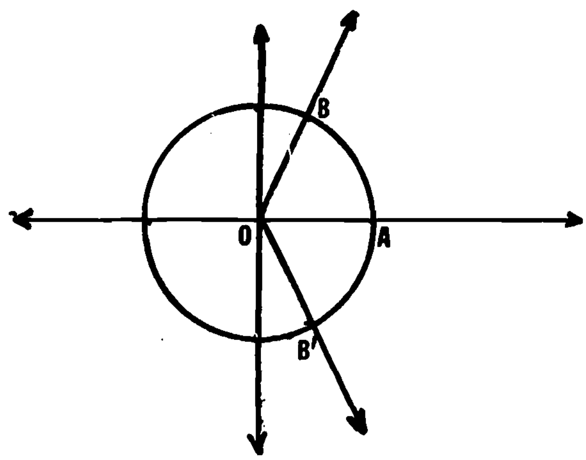


Figure 8.27

If $m(\vec{\angle}AOB) = \theta$, then $m(\vec{\angle}AOB') = 2\pi - \theta$, by the earlier principle. However, $\vec{\angle}BOA \cong \vec{\angle}AOB'$, since there is a direct isometry (rotation) mapping initial side onto initial side and terminal side onto terminal side. Therefore, since congruent sensed angles are assigned the same measure, $m(\overrightarrow{\angle BOA}) = 2\pi - \theta$, whereas $m(\overrightarrow{\angle AOB}) = \theta$. In other words, if θ is the measure of a sensed angle, then $2\pi - \theta$ is the measure of the angle obtained by "interchanging" the initial and terminal sides.

> If $\overrightarrow{\angle AOB}$ is a sensed angle, and $m(\overrightarrow{\angle AOB}) = \theta$,
> then $m(\overrightarrow{\angle BOA}) = 2\pi - \theta$.

The principle applies equally well in case degree measure is used, with 360 replacing 2π.

<u>Example</u> 6. Suppose $\overrightarrow{\angle AOR}$ -- that is, the ordered pair $(\overrightarrow{OA}, \overrightarrow{OR})$ -- has a degree measure of 120.

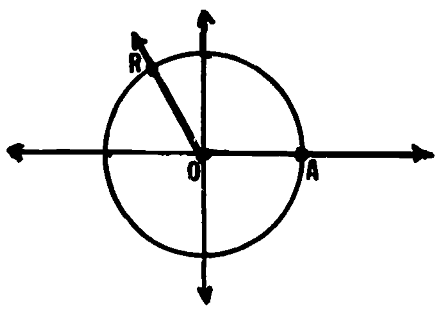


Figure 8.28

Then $\overrightarrow{\angle ROA}$ -- the ordered pair $(\overrightarrow{OR}, \overrightarrow{OA})$-- has degree measure 240.

Finally in this section we introduce the concept of angle addition. By <u>angle addition</u> is meant a binary operation on the set of sensed angles. Thus we must have a way of assigning a unique sensed angle to every ordered pair of sensed angles. Thus let $\overrightarrow{\angle ABC}$ and $\overrightarrow{\angle DEF}$ be two sensed angles. (See Figure 8.29(a)). Then

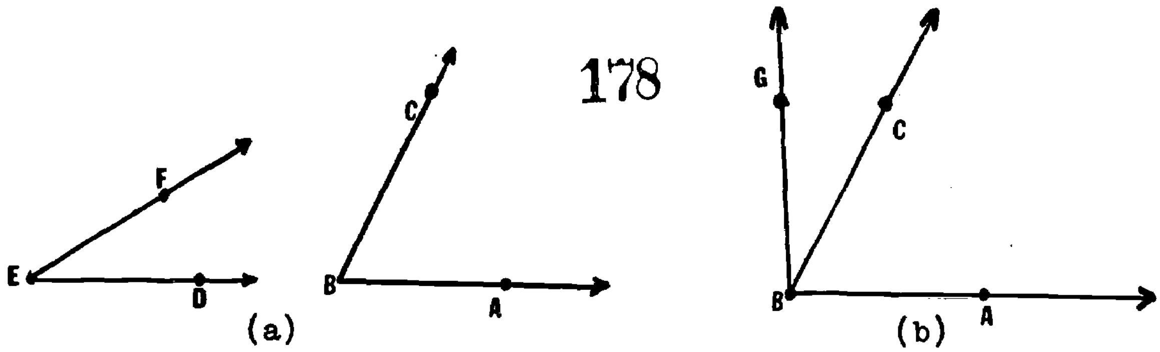


Figure 8.29

we shall assume the existence of a unique ray $\overrightarrow{BG}$ such that $\overrightarrow{\angle CBG} \cong \overrightarrow{\angle DEF}$. (See Figure 8.29(b)). $\overrightarrow{\angle ABG}$ -- the ordered pair $(\overrightarrow{BA}, \overrightarrow{BG})$ -- is defined to be the sum $\overrightarrow{\angle ABC} + \overrightarrow{\angle DEF}$.

<u>Definition 10</u>. If $\overrightarrow{\angle ABC}$ and $\overrightarrow{\angle DEF}$ are two sensed angles, and $\overrightarrow{\angle CBG} \cong \overrightarrow{\angle DEF}$, then $\overrightarrow{\angle ABC} + \overrightarrow{\angle DEF} = \overrightarrow{\angle ABG}$.

8.10 <u>Exercises</u>

In Exercises 1 -- 21, angle measurements are listed. If the measurement is in radians, write the equivalent degree measurement; if the given measurement is in degrees, write the equivalent radian measurement.

1. 0°
2. 30°
3. 45°
4. 60°
5. 90°
6. π radians
7. $\frac{2}{3}\pi$ radians
8. $\frac{5}{6}\pi$ radians
9. $\frac{3}{4}\pi$ radians
10. 315°
11. 330°
12. 300°
13. 270°
14. $\frac{5}{4}\pi$ radians
15. $\frac{4}{3}\pi$ radians
16. $\frac{7}{6}\pi$ radians
17. 15°
18. 2 radians
19. 2°
20. r radians
21. d degrees

22. In the accompanying figure, B' and C' are reflections of B and C, respectively, in the x-axis. If $m(\overrightarrow{\angle AOB}) = 55^\circ$, and $m(\overrightarrow{\angle AOC}) = 200^\circ$, determine the degree measurement of the following:

   (a) $\overrightarrow{\angle AOB'}$

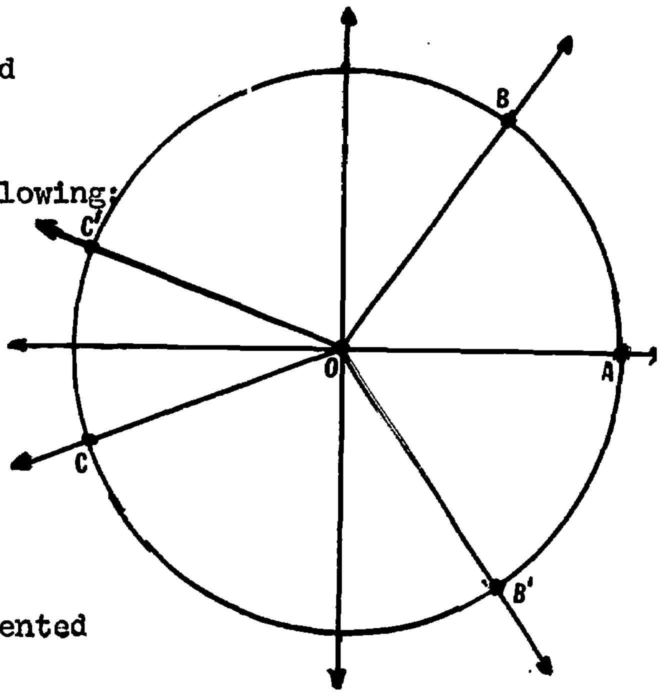
   (b) $\overrightarrow{\angle AOC'}$

   (c) $\overrightarrow{\angle BOA}$

   (d) $\overrightarrow{\angle COA}$

   (e) $\overrightarrow{\angle B'OA}$

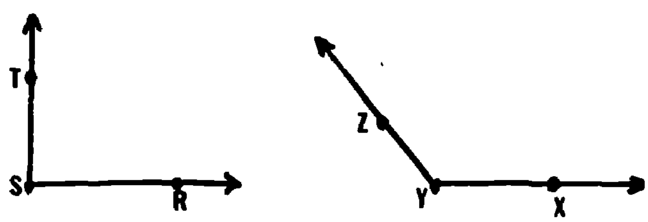
   (f) $\overrightarrow{\angle C'OA}$



23. Copy the sensed angles represented at the right. Then draw

   (a) the angle which is the sum $\overrightarrow{\angle RST} + \overrightarrow{\angle XYZ}$

   (b) the angle which is the sum $\overrightarrow{\angle XYZ} + \overrightarrow{\angle RST}$.



24. What is the sum $\overrightarrow{\angle AOB} + \overrightarrow{\angle AOB}$ if $\overrightarrow{\angle AOB}$ is in standard position and $\overrightarrow{OB}$ contains:

   (a) the positive x-axis

   (b) the positive y-axis

   (c) the negative x-axis

   (d) the negative y-axis

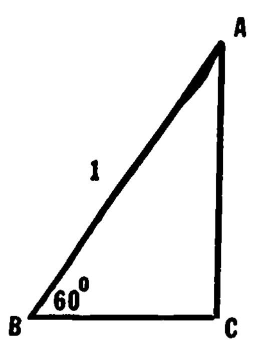
25. If $\overrightarrow{\angle AOB}$ is any sensed angle, what is $\overrightarrow{\angle AOB}$ + zero angle? zero angle + $\overrightarrow{\angle AOB}$?

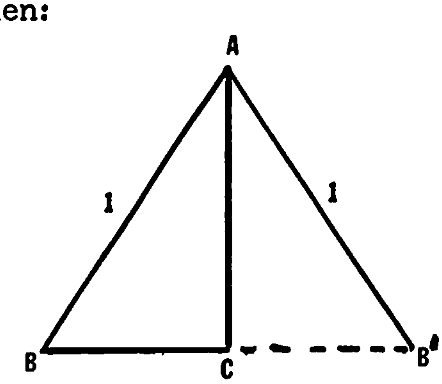
. If $\overrightarrow{\angle AOB}$ is any sensed angle, what is $\overrightarrow{\angle AOB} + \overrightarrow{\angle BOA}$?

8.11 Some Special Angles

The SINE and COSINE of certain angles -- called "special angles" in the title of this section -- arise frequently enough in applications of the circular functions to merit attention. Furthermore, determining the SINE and COSINE of these angles emphasizes some interesting relationships among the circular functions and some geometric principles studied earlier.

Consider first a right triangle such as that illustrated in Figure 8.30(a), with an acute angle measuring 60°, and a hypotenuse of unit length (i.e., length one). Reflect in line $\overleftrightarrow{AC}$. (See Figure 8.30(b).) Then:





(1) B, C, and B'are collinear, since $\overline{BC} \perp \overline{AC}$, and each of two perpendicular lines is its own image under reflection in the other. Thus ABB' is a triangle, with point C contained in side $\overline{BB'}$.

(2) $\angle B'$ has a degree measurement of 60°, since isometries of preserve angle measure.

(3) We now know triangle ABB' is equiangular. (Why?) Therefore it is equilateral, and BB' = 1. But BC = CB', since isometries preserve distance, and so $BC = \frac{1}{2}$.

(4) Using the Pythagorean principle in right triangle ABC, $(AC)^2 = 1^2 - (\frac{1}{2})^2 = 1 - \frac{1}{4} = \frac{3}{4}$. Therefore, $AC = \frac{1}{2}\sqrt{3}$.

So, *in any right triangle having unit hypotenuse and a* 60° angle, the leg "opposite the 60° angle" measures $\frac{1}{2}\sqrt{3}$, and the other leg measures $\frac{1}{2}$. Thus, in Figure 8.31(a) $\angle \overrightarrow{A}OB$ is in standard position, and has a degree measure of 60. Since $BC = \frac{1}{2}\sqrt{3}$ and $OC = \frac{1}{2}$, the coordinates of B are $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$. And we have

$$\sin 60^\circ = \frac{1}{2}\sqrt{3}$$

$$\cos 60^\circ = \frac{1}{2} \quad .$$

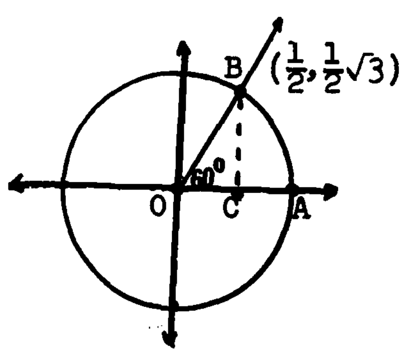


Figure 8.31a

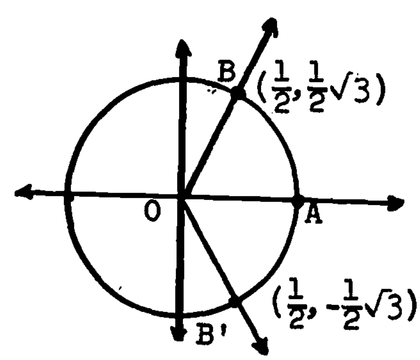


Figure 8.31b

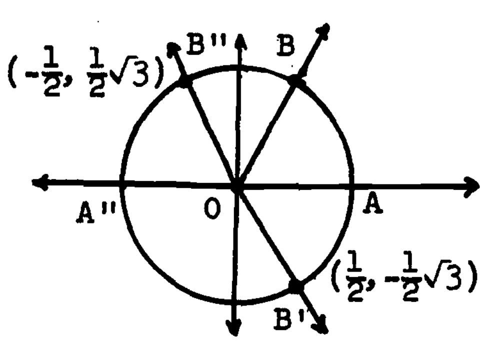


Figure 8.31c

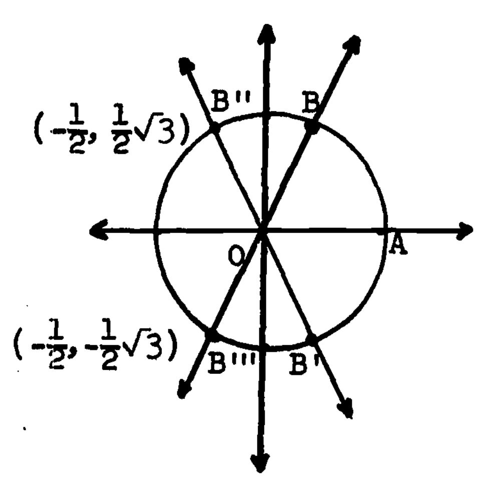


Figure 8.31d

Reflecting in the x-axis (see Figure 8.31(b)), the image of B is $B'(\frac{1}{2}, -\frac{1}{2}\sqrt{3})$. The degree measure of $\overrightarrow{\angle AOB'}$ is 360 - 60 = 300°. Consequently,

$$\sin 300^\circ = -\frac{1}{2}\sqrt{3}$$

$$\cos 300^\circ = \frac{1}{2}$$

Now reflecting in the origin, the image of B' is $B''(-\frac{1}{2}, \frac{1}{2}\sqrt{3})$. (See Figure 8.31(c)). Also the image of A is A''. $\overrightarrow{\angle A''OB''}$ has degree measure 300° since it is the image of $\overrightarrow{\angle AOB'}$. Therefore $\overrightarrow{\angle B''OA''}$ has degree measure 360 - 300 = 60°. And since $\overrightarrow{\angle AOB''} + \overrightarrow{\angle B''OA''} = \overrightarrow{\angle AOA''}$ which has degree measure 180, $\angle AOB''$ has degree measure 120. Thus,

$$\sin 120^\circ = \frac{1}{2}\sqrt{3}$$

$$\cos 120^\circ = -\frac{1}{2}$$

Finally, reflecting in the x-axis again, the image of B'' is $B'''(-\frac{1}{2}, -\frac{1}{2}\sqrt{3})$. (See Figure 8.31(d).) Since $\overrightarrow{\angle AOB''}$ has a degree measure of 120°, the degree measure of $\angle AOB'''$ is 360 - 120 = 240°. Therefore,

$$\sin 240^\circ = -\frac{1}{2}\sqrt{3}$$

$$\cos 240^\circ = -\frac{1}{2}$$

The results above are summarized in the table below:

	60°	120°	240°	300°
sin	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$
cos	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

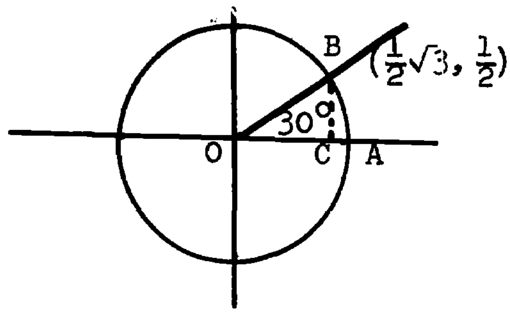


Figure 8.32

In Figure 8.32, $\angle AOB$ is a standard position sensed angle with a dgree measure of 30. Since BOC is a right triangle with hypotenuse of length one, and $\angle OBC$ is a 60° angle, we know $OC = \frac{1}{2}\sqrt{3}$, and $BC = \frac{1}{2}$. Thus the coordinates of B are $(\frac{1}{2}\sqrt{3}, \frac{1}{2})$, and we have:

$$\sin 30^\circ = \frac{1}{2} \qquad \cos 30^\circ = \frac{1}{2}\sqrt{3}$$

By reflecting in the x-axis, then in the origin, then again in the x-axis, the sine and cosine of 330°, 150°, and 210° may be determined. This is left for the exercises. (See Exercise 1 of Section 8.12.)

Consider next an isosceles right triangle whose hypotenuse in Figure 8.33, with hypotenuse of unit length. Let a denote the length of each of the two legs of the triangle. Then, by the Pythagorean principle,

$$a^2 + a^2 = 1^2$$

$$2a^2 = 1$$

$$a^2 = \frac{1}{2} \quad \text{or} \quad a = \frac{1}{2}\sqrt{2}$$

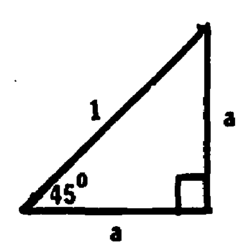


Figure 8.33

In effect, in any isosceles right triangle whose hypotenuse is assigned length one, each of the two legs measures $\frac{1}{2}\sqrt{2}$.

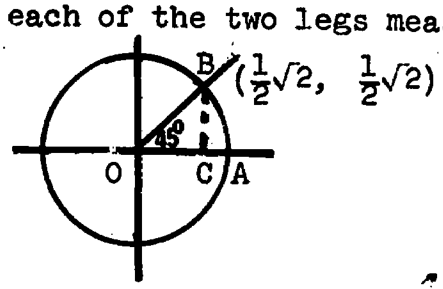


Figure 8.34

In Figure 8.34, $\angle AOB$ is a standard position sensed angle with degree measure 45°. Thus, ABC is an isoceles right triangle whose hypotenuse measures one. By the work above, we know $OC = BC = \frac{1}{2}\sqrt{2}$; as a result, the coordinates of B are $(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$. Therefore,

$$\sin 45^\circ = \frac{1}{2}\sqrt{2}$$
$$\cos 45^\circ = \frac{1}{2}\sqrt{2}$$

From these values, the sine and cosine of 315°, 135°, and 225° can be determined by reflecting in the x-axis, then reflecting in the origin, then reflecting again in the x-axis. Exercise 2 of Section 8.12 is concerned with this.

8.12 Exercises

1. In Section 8.11 it was established that

$$\sin 30^\circ = \frac{1}{2} \quad \text{and} \quad \cos 30^\circ = \frac{1}{2}\sqrt{3} \quad .$$

By reflecting in the x-axis, then in the origin, then in the x-axis, determine the following:

(a) $\sin 330^\circ$ (d) $\cos 150^\circ$
(b) $\cos 330^\circ$ (e) $\sin 210^\circ$
(c) $\sin 150^\circ$ (f) $\cos 210^\circ$

2. Also in Section 8.11, it was demonstrated that

$$\sin 45^\circ = \frac{1}{2}\sqrt{2} \quad \text{and} \cos 45^\circ = \frac{1}{2}\sqrt{2} \quad .$$

Use these values to determine the following:

(a) $\sin 315^\circ$ (d) $\cos 135^\circ$
(b) $\cos 315^\circ$ (e) $\sin 225^\circ$
(c) $\sin 135^\circ$ (f) $\cos 225^\circ$

3. Copy and complete the following table:

	sine	cosine
0°		
30°		
45°		
60°		
90°		
120°		
135°		
150°		
180°		
210°		
225°		
240°		
270°		
300°		
315°		
330°		

4. Complete the following:

(a) $\sin \frac{\pi}{4} =$

(b) $\cos \frac{\pi}{4} =$

(c) $\sin \frac{3}{4}\pi =$

(d) $\cos \frac{3}{4}\pi =$

(e) $\sin \frac{5}{4}\pi =$

(f) $\cos \frac{5}{4}\pi =$

(g) $\sin \frac{7}{4}\pi =$

(h) $\cos \frac{7}{4}\pi =$

(i) $\sin \frac{\pi}{6} =$

(j) $\cos \frac{5}{6}\pi =$

(k) $\sin \frac{11}{6}\pi =$

(l) $\cos \frac{7}{6}\pi =$

(m) $\sin \frac{\pi}{3} =$

(n) $\cos \frac{2}{3}\pi =$

(o) $\sin \frac{4}{3}\pi =$

(p) $\cos \frac{5}{3}\pi =$

5. For each of the following, give all values of θ (in degrees) which make the equation true.

(a) $\sin\theta = \frac{1}{2}$ (b) $\sin\theta = -\frac{1}{2}$

(c) $\sin\theta = \cos\theta$ (d) $\cos\theta = -\sin\theta$

6. For each of the following, give all real numbers (which may be interpreted as radian measures of angles) which make the equation true.

(a) $\sin\theta = 0$ (b) $\cos\theta = \frac{1}{2}$

(c) $\sin\theta = -\frac{1}{2}\sqrt{2}$ (d) $(\sin\theta)(\cos\theta) = 0$

7. Complete the following:

(a) $\sin 30^{\circ} =$

(b) $\cos 30^{\circ} =$

(c) $(\sin 30^{\circ})^2 + (\cos 30^{\circ})^2 =$

(d) $\sin 45^{\circ} =$

(e) $\cos 45^{\circ} =$

(f) $(\sin 45^{\circ})^2 + (\cos 45^{\circ})^2 =$

(g) $\sin \frac{5}{6}\pi =$

(h) $\cos \frac{5}{6}\pi =$

(i) $(\sin \frac{5}{6}\pi)^2 + (\cos \frac{5}{6}\pi)^2 =$

8. Prove the following:

$$(\sin\theta)^2 + (\cos\theta)^2 = 1,$$

where θ is the measure of a sensed angle (in either degrees or radians.) Refer to the definition of the sine and cosine functions; see also Exercise 5 of Section 8.6. (Note: $(\sin \theta)^n$ is written $\sin^n\theta$ and $(\cos \theta)^n$ is written $\cos^n\theta$. In this notation the above equality is:

$$\sin^2 \theta + \cos^2 \theta = 1.)$$

9. Complete the following:

   (a) $\sin 60^\circ =$

   (b) $\sin 30^\circ =$

   (c) True or false: $\sin 60^\circ = 2 \sin 30^\circ$

   (d) $\sin 90^\circ =$

   (e) $\sin 45^\circ$

   (f) $\sin 150^\circ$

   (g) True or false: $\sin 150^\circ = \sin 90^\circ + \sin 60^\circ$

10. Find a rational approximation of $\sqrt{2}$, correct to three decimal places.

    Then give a rational approximation for $\sin 45^\circ$ and $\cos 45^\circ$, correct to three decimal places.

11. Find a rational approximation of $\sqrt{3}$, correct to three decimal places.

    Then give a rational approximation for $\cos 30^\circ$ and $\sin 60^\circ$, correct to three decimal places.

12. Complete the following table, giving sine and cosine assignments correct to three decimal places.

Degree Measure of Angle	Radian Measure (θ) of Angle	sin θ	cos θ
0°			
30°			
45°			
60°			
90°			
120°			
135°			
150°			
180°			
210°			
225°			
240°			
270°			
300°			
315°			
330°			

8.13 Graphs of Circular Functions

Graphs are often important in the study of circular functions, just as in the case of polynomial and rational functions in Chapter 7. Let s be the function with rule $s(x) = \sin x$, $0 \leq x < 2\pi$. (In earlier sections, "$\sin \theta$" has been used. However one variable is as good as another; and so, to be consistent with earlier graphs in the coordinate plane, the variable "x" is used.) Table 8.1 lists some of the values used to plot some points of this graph in Figure 8.35.

x	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
sin x	0	.50	.71	.87	1.00	.87	.71	.50	0

x	$\frac{7}{6}\pi$	$\frac{5}{4}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{7}{4}\pi$	$\frac{11}{6}\pi$
sin x	-.50	-.71	-.87	-1.0	-.87	-.71	-.50

Table 8.1

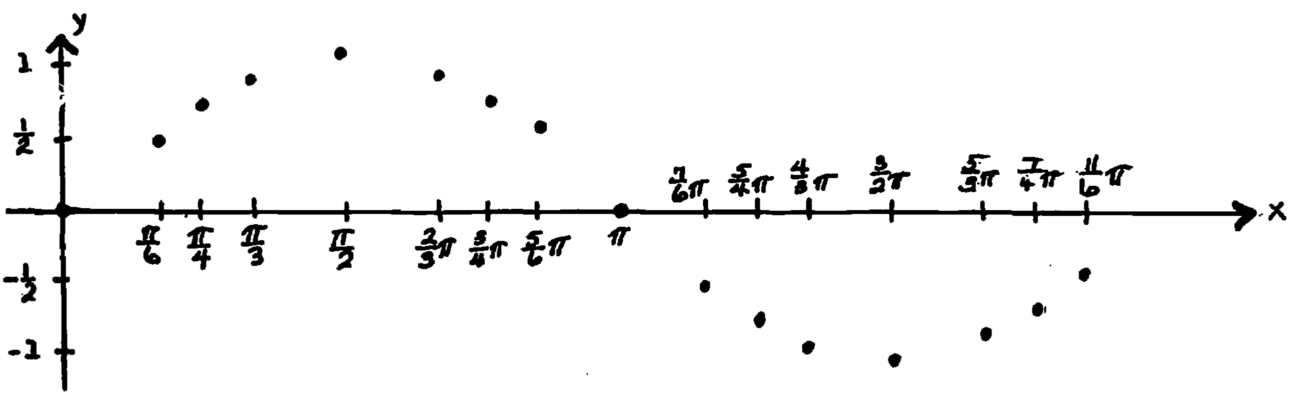


Figure 8.35

If the graph of the sine function is assumed to be a smooth curve, then the points may be connected as in Figure 8.16.

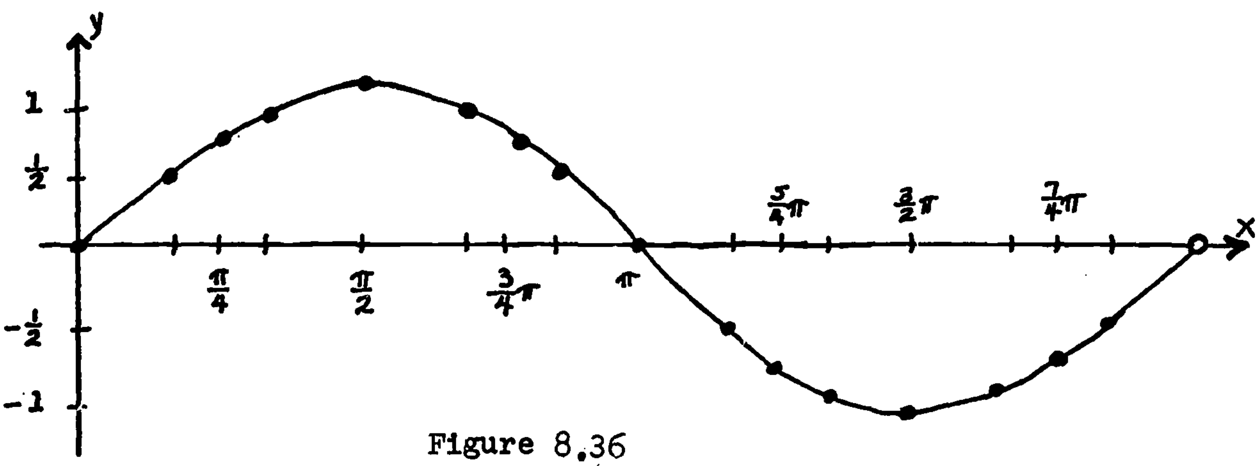


Figure 8.36

Sometimes the graph of the sine function is drawn using degree measures of angles for x in the rule $s(x) = \sin x$. This does not change the basic characteristics of the graph. (See

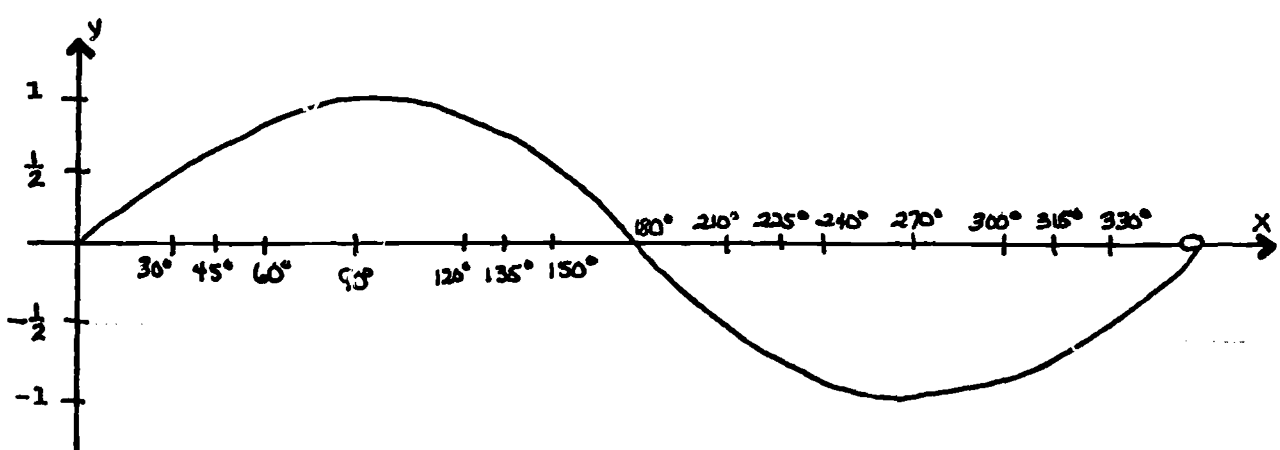


Figure 8.37

The exercises in Section 8.14 deal with the graphs of the sine and cosine functions and functions obtained from them. As an aid in sketching these graphs, Table 8.2 listing values for the sine and cosine functions is included at this time. Table 8.2 will also be needed for the exercises in Section 8.16.

Angle Measure (degrees)	sine	cosine	Angle Measure (degrees)	sine	cosine
1	0.017	1.000	46	0.719	0.695
2	0.035	0.999	47	0.731	0.682
3	0.052	0.999	48	0.743	0.669
4	0.070	0.998	49	0.755	0.656
5	0.087	0.996	50	0.766	0.643
6	0.105	0.995	51	0.777	0.629
7	0.122	0.993	52	0.788	0.616
8	0.139	0.990	53	0.799	0.602
9	0.156	0.988	54	0.809	0.588
10	0.174	0.985	55	0.819	0.574
11	0.191	0.982	56	0.829	0.559
12	0.208	0.978	57	0.839	0.545
13	0.225	0.974	58	0.848	0.530
14	0.242	0.970	59	0.857	0.515
15	0.259	0.966	60	0.866	0.500
16	0.276	0.961	61	0.875	0.485
17	0.292	0.956	62	0.883	0.469
18	0.309	0.951	63	0.891	0.454
19	0.326	0.946	64	0.899	0.438
20	0.342	0.940	65	0.906	0.423
21	0.358	0.934	66	0.914	0.407
22	0.375	0.927	67	0.921	0.391
23	0.391	0.921	68	0.927	0.375
24	0.407	0.914	69	0.934	0.358
25	0.423	0.906	70	0.940	0.342
26	0.438	0.899	71	0.946	0.326
27	0.454	0.891	72	0.951	0.309
28	0.469	0.883	73	0.956	0.292
29	0.485	0.875	74	0.961	0.276
30	0.500	0.866	75	0.966	0.259
31	0.515	0.857	76	0.970	0.242
32	0.530	0.848	77	0.974	0.225
33	0.545	0.839	78	0.978	0.208
34	0.559	0.829	79	0.982	0.191
35	0.574	0.819	80	0.985	0.174
36	0.588	0.809	81	0.988	0.156
37	0.602	0.799	82	0.990	0.139
38	0.616	0.788	83	0.993	0.122
39	0.629	0.777	84	0.995	0.105
40	0.643	0.766	85	0.996	0.087
41	0.656	0.755	86	0.998	0.070
42	0.669	0.743	87	0.999	0.052
43	0.682	0.731	88	0.999	0.035
44	0.695	0.719	89	1.000	0.017
45	0.707	0.707			

Table 8.2

8.14 Exercises

1. From Table 8.2 find the following:

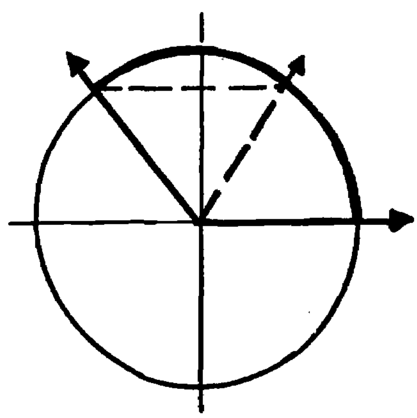
   (a) $\sin 27^\circ$ (b) $\cos 27^\circ$

   (c) $\sin 40^\circ$ (d) $\cos 50^\circ$

   (e) $\sin 10^\circ$ (f) $\cos 80^\circ$

   (g) $\sin 48^\circ$ (h) $\cos 42^\circ$

2. Use Table 8.2 to find $\sin 130^\circ$. (Hint: The figure below, suggesting a reflection, suggests a way in which "$\sin 130^\circ$" may be read from the table, even though "130°" is not listed there.)



3. Use Table 8.2 to find the following (see Exercise 2):

   (a) $\cos 130^\circ$ (b) $\sin 250^\circ$

   (c) $\cos 200^\circ$ (d) $\sin 290^\circ$

   (e) $\cos 290^\circ$ (f) $\sin 179^\circ$

   (g) $\cos 269^\circ$ (h) $\sin 359^\circ$

4. Draw graphs for the following functions on the same set of axes:

   (a) Function f such that $f(x) = \sin x$, $0 \leq x < 2\pi$.

   (b) Function g such that $g(x) = -\sin x$, $0 \leq x < 2\pi$.

   What transformation of the plane may be used to relate these two graphs?

5. Draw graphs of the following functions on the same set of axes.

   (a) Function f such that $f(x) = \sin x$, $0 \leq x < 2\pi$.

   (b) Function g such that $g(x) = 2 \cdot \sin x$, $0 \leq x < 2\pi$.

   (c) Function h such that $h(x) = -2 \cdot \sin x$, $0 \leq x < 2\pi$.

   (d) Give the range for each of the functions f, g, and h.

6. Draw graphs of the following functions on the same set of axes.

   (a) $f\colon\ x \longrightarrow \sin x$, $0 \leq x < 2\pi$.

   (b) $g\colon\ x \longrightarrow (\sin x) + 2$, $0 \leq x < 2\pi$.

   (c) $h\colon\ x \longrightarrow (\sin x) - 2$, $0 \leq x < 2\pi$.

   (d) What plane transformation relates the graphs of g and h?

7. Draw the graph of the function c with rule $c(x) = \cos x$, $0 \leq x < 2\pi$.

8. Draw the graph of the cosine function, using degree measure of angle on the x-axis.

9. Draw the graphs of the following functions on the same set of axes.

   (a) Function f such that $f(x) = \sin x$, $0 \leq x < 2\pi$.

   (b) Function g such that $g(x) = \cos x$, $0 \leq x < 2\pi$.

   (c) Function [f+g] such that $[f+g](x) = \sin x + \cos x$, $0 \leq x < 2\pi$.

10. Draw the graphs of the following functions on the same set of axes:

   (a) $f\colon\ x \longrightarrow \cos x$, $0 \leq x < 2\pi$.

   (b) $g\colon\ x \longrightarrow 3(\cos x)$, $0 \leq x < 2\pi$.

(c) h: $x \longrightarrow -3(\cos x)$, $0 \leq x < 2\pi$.

(d) k: $x \longrightarrow (\cos x) + 3$, $0 \leq x < 2\pi$.

*11. Draw the graph of the function f such that $f(x) = \sin \frac{1}{2}x$, $0 \leq x < 4\pi$.

*12. Draw the graph of the function g such that $g(x) = \cos \frac{1}{2}x$, $0 \leq x < 4\pi$.

13. (a) What is $\sin \frac{3}{4}\pi + \sin \frac{5}{4}\pi$?

(b) Explain how the above can be predicted from the graph of the sine function.

14. (a) What is $\sin \frac{1}{6}\pi + \sin \frac{7}{6}\pi$?

(b) What is $\sin 120^\circ + \sin 240^\circ$?

(c) Complete the following: $\sin 170^\circ + \sin$ ______ = 0.

15. (a) What is $\cos \frac{1}{3}\pi + \cos \frac{2}{3}\pi$?

(b) Explain how the above can be predicted from the graph of the cosine function.

16. (a) What is $\cos \frac{1}{4}\pi + \cos \frac{3}{4}\pi$?

(b) What is $\cos 30^\circ + \cos 150^\circ$?

(c) Complete the following: $\cos 43^\circ + \cos$ _____ = 0.

8.15 Law of Cosines and Law of Sines

One of the many applications of circular functions is that of finding unknown lengths of sides and measures of angles in a triangle.

Example 1. A surveyor wants to find the distance across a marsh, from A to B. He can find a point C for which he can measure directly BC, AC, and $\angle C$. For the data shown in Figure 8.38 find AB (to the nearest yard).

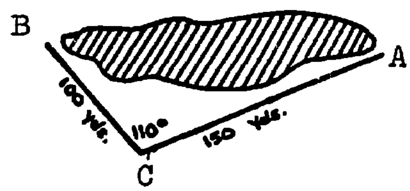


Figure 8.38

The surveyor begins the solution of the problem like this:

$(AB)^2 = 100^2 + 150^2 - 2(100)(150)(\cos 110^\circ)$.

Why is the surveyor's method in Example 1 correct -- or is it? Instead of working with the particular triangle of that example, look at triangle ACB in Figure 8.39 In this triangle, let AB = c, AC = b, and BC = a. (Thus, the side "opposite" angle C has length c, etc.)

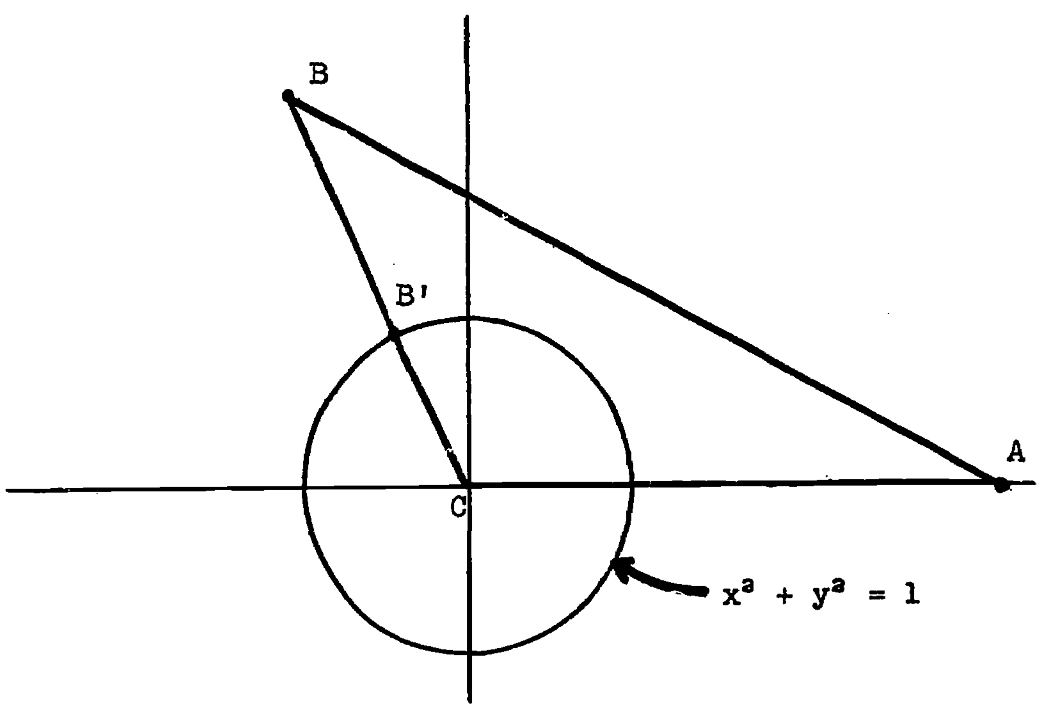


Figure 8.39

The following conclusions can now be drawn:

(1) The coordinates of A are (b,0). (Remember that b is the length of segment $\overline{AC}$.)

Also of course the coordinates of C are (0,0).

(2) If B' is the point at which $\overrightarrow{CB}$ intersects the unit circle, then the coordinates of B' are (COSINE C, SINE C).

We shall write these coordinates as (cos C, sin C), to mean the sine and cosine functions of the measure of C.

(3) B is the point of intersection of $\overrightarrow{CB}$ with a circle having center C and radius a. (Remember that a is the length of segment $\overline{CB}$.)

So B is the image of B' under the dilation, with center (0,0) and scale factor a.

Therefore, the coordinates of B are (a·cos C, a·sin C).

Figure 8.40 shows triangle ACB with the coordinates of all vertices labeled.

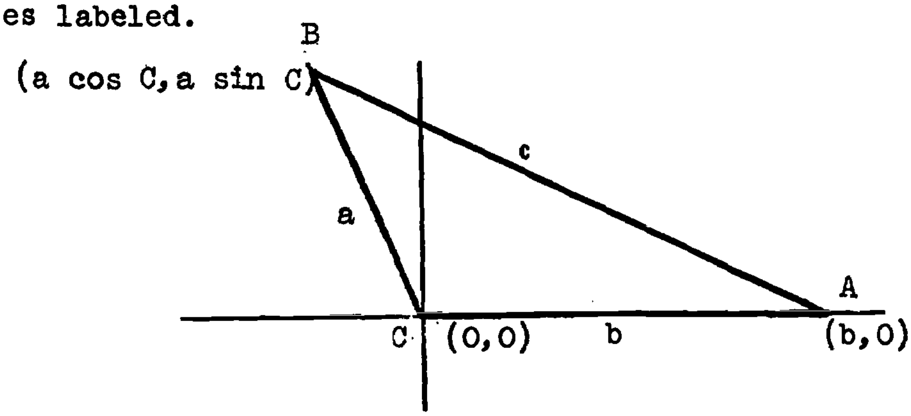


Figure 8.40

Applying the distance formula to segment $\overline{AB}$;

$$c^2 = (a \cdot \cos C - b)^2 + (a \cdot \sin C - 0)^2$$
$$= a^2 \cdot \cos^2 C - 2ab \cdot \cos C + b^2 + a^2 \cdot \sin^2 C$$
$$= a^2(\cos^2 C + \sin^2 C) + b^2 - 2ab \cdot \cos C$$
$$= a^2(1) + b^2 - 2ab \cdot \cos C$$
$$= a^2 + b^2 - 2ab \cdot \cos C.$$

It should now be clear why the surveyor's method in Example 1 is correct.

<u>Example</u> 2. Find the length c (where c = AB) in the triangle in Figure 8.41.

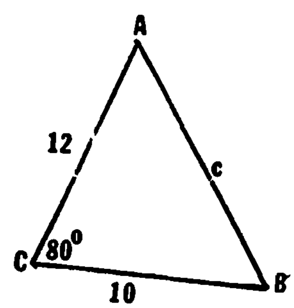


Figure 8.41

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$
$$= 10^2 + 12^2 - 2(10)(12) \cdot \cos 80^\circ$$
$$= 100 + 144 - (240)(.174)$$
$$= 244 - 41.76$$
$$= 202.24.$$

Therefore $c = \sqrt{202.24}$

≈ 14.2 (to the nearest tenth).

The formula "$c^2 = a^2 + b^2 - 2ab \cdot \cos C$" is a form of the <u>Law of Cosines</u>. The "unknown" side need not be called c; it may be either a or b (or indeed some other variable). However, the "pattern" of the Law of Cosines remains the same.

Example 3. In triangle ABC (Figure 8.42), $m\angle A = 45^\circ$ and $m\angle B = 60^\circ$. If $BC = a = 10$, what is the length of side AC?

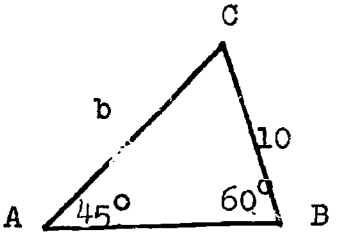
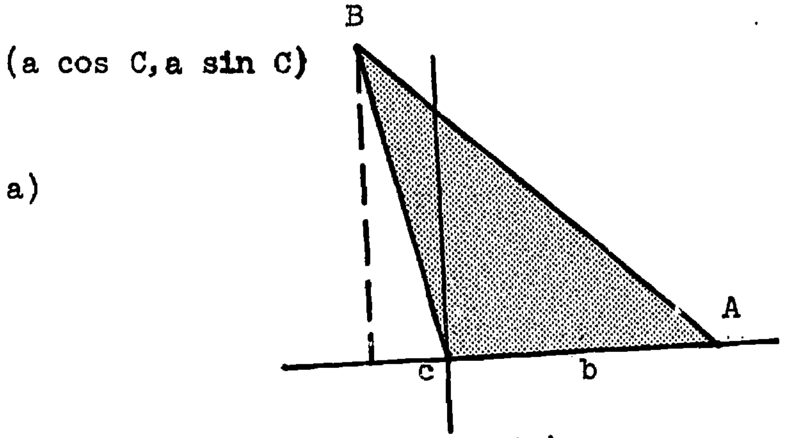


Figure 8.42

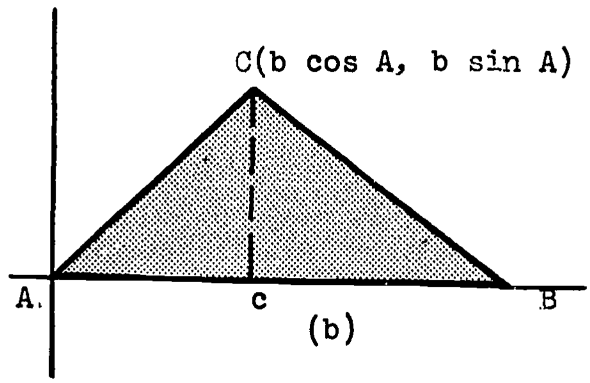
The problem in Example 3 is somewhat like that in Example 1; an "unknown part" of a triangle is to be found. And yet the problem is different. It cannot easily be solved by using the Law of Cosines (try it.) The development below results in a formula which may be used to solve Example 3.

We introduce the coordinate system in three different ways with respect to the same triangle ABC.

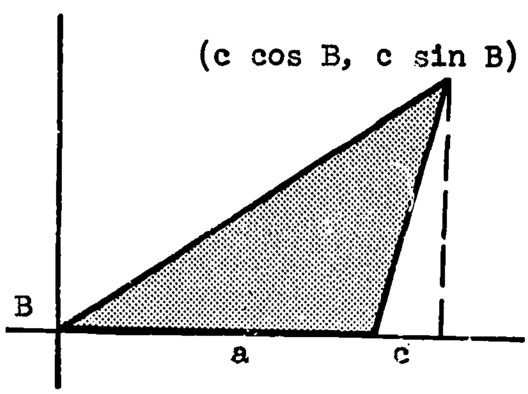
(a)



(a)



(b)



(c)

Figure 8.43

In Figure 8.43(a) the coordinates of B are (a·cos C, a·sin C) and AC = b. If we let K denote the area of triangle ABC, then

$$K = \frac{1}{2} \cdot b \cdot a \cdot \sin C.$$

The triangle ABC in Figure 8.43(b) is congruent to the triangle in Figure 8.43(a). Now however, the coordinates of A are (0,0), and the coordinates of C are (b·cos A, b·sin A). And AB = c. Therefore,

$$K = \frac{1}{2} \cdot c \cdot b \cdot \sin A.$$

Once again, the triangle ABC in Figure 8.43(c) is congruent to the other triangles. Now however, the coordinates of B are (0,0), the coordinates of A are ($c \cdot \cos B$, $c \cdot \sin B$), and BC = a. Therefore,

$$K = \frac{1}{2} \cdot a \cdot c \cdot \sin B.$$

These are expressions for the measures of area of congruent triangles. Since we assign equal measures to areas of congruent triangles we have:

$$\frac{1}{2}(b)(a)(\sin C) = \frac{1}{2}(c)(b)(\sin A) = \frac{1}{2}(a)(c)(\sin B).$$

From $\frac{1}{2}(b)(a)(\sin C) = \frac{1}{2}(c)(b)(\sin A)$, we get

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

From $\frac{1}{2}(c)(b)(\sin A) = \frac{1}{2}(a)(c)(\sin B)$, we get

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

Thus, by commutativity and transitivity of equality, we have:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This formula is called the <u>Law of Sines.</u> In words, it says that for a given triangle the ratio of the length of a side to the sine of the opposite angle is the same, regardless of which side is chosen. This formula may be used to solve the problem in Example 3, as follows (see Figure 8.42):

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{b}{\sin 60^\circ} = \frac{10}{\sin 45^\circ}$$

$$\frac{b}{.866} = \frac{10}{.707}$$

$$b = \frac{8.66}{.707}$$

$$= 12.2 \text{ (to the nearest tenth).}$$

8.16 Exercises

1. In a certain triangle ABC, $a = 20$, $b = 10$, and $m(\angle C) = 20^\circ$. Use the Law of Cosines to find c.

   (Note: The same notation is being used here as in Section 8.15. That is, $AB = c$, the side opposite $\angle C$; $BC = a$, the side opposite $\angle A$; $AC = b$, the side opposite $\angle B$.)

2. In triangle ABC, $a = 5$, $b = 12$, and $m(\angle C) = 90^\circ$. Use the Law of Cosines to find c.

3. In Section 8.15 the Law of Cosines was given in the form

   $$c^2 = a^2 + b^2 - 2ab(\cos C),$$

   in which c is considered as the "unknown side."

   (a) Write a form of the Law of Cosines in which a is considered as the "unknown side." Thus, the formula should begin

   $$a^2 =$$

   (b) Give a form of the Law of Cosines in which b is treated as the "unknown side."

4. If the Law of Cosines is used to find the length of a side of a triangle, what other parts of the triangle must be known?

5. In triangle ABC, $b = 12$, $c = 6$, and $\angle A$ has measure 52°. Use the Law of Cosines to find a.

6. In triangle ABC, $a = 12$, $c = 6$, and $\angle B$ has measure 128°. Use the Law of Cosines to find b.

7. Suppose the Law of Cosines, $c^2 = a^2 + b^2 - 2ab(\cos C)$, is applied to a triangle ABC in which $\angle C$ has measure 90°.

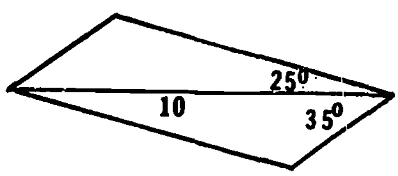
   (a) What is cos C?

   (b) What is the product 2ab(cos C)?

   (c) What already familiar property of a right triangle results?

8. In triangle ABC, $b = 10$, $c = 12$, and $\angle C$ has measure 60°. Use the Law of Sines to find the degree measure of $\angle B$.

9. The longer diagonal of a parallelogram is 10 inches long. At one end the diagonal makes angles of 35° and 25° with the sides of the parallelogram. Find the lengths of the sides of the parallelogram. (Hint: Use the Law of Sines.)

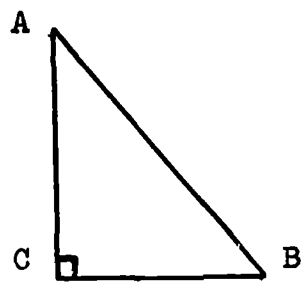


10. Find the angles of a triangle if its sides measure 3, 4, and 5.

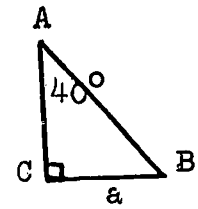
11. In triangle ABC, $\angle C$ has measure 90°.

   (a) Use $\frac{a}{\sin A} = \frac{c}{\sin C}$ to show that $\sin A = \frac{a}{c}$.

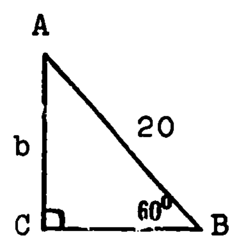
   (b) Also use the Law of Sines to show that $\sin B = \frac{b}{c}$.



12. Use the result of Exercise 11(a) to find a in the figure.



13. Use the result of Exercise 11(b) to find b in the figure.



14. If ABC is a triangle, is it possible to find all the other sides and angles of the triangle if:

   * (a) $\angle A$, a, and b are known?
   (b) $\angle A$, $\angle B$, and $\angle C$ are known?
   (c) a, b, and c are known?
   (d) $\angle A$, $\angle B$ and c are known?
   (e) b, a, and $\angle C$ are known?

8.17 Summary

$\overrightarrow{\angle AOB}$ is an ordered pair of rays $(\overrightarrow{OA}, \overrightarrow{OB})$, with $\overrightarrow{OA}$ called the initial side, and $\overrightarrow{OB}$ the terminal side. $\overrightarrow{\angle AOB}$ is in standard position if OA is the positive x-axis in a plane rectangular coordinate system.

If $\overrightarrow{\angle AOB}$ is in standard position and intercepts an arc of length θ on the circle $x^2 + y^2 = r^2$, then the real number $\frac{\theta}{r}$ is assigned as the radian measure of the angle. In the case of the unit circle, $\frac{\theta}{r} = \frac{\theta}{1} = \theta$.

The function SPSA $\xrightarrow{m}$ R, which assigns to each standard position sensed angle its radian measure, has $\{x : 0 \leq x < 2\pi\}$ as range.

Two sensed angles in the plane are congruent if there is a direct isometry that maps initial side onto initial side, and terminal side onto terminal side.

If $\overrightarrow{\angle AOB}$ is in standard position, and its terminal side intersects the unit circle at the point (x,y), then SINE$(\overrightarrow{\angle AOB}) = y$, and COSINE$(\overrightarrow{\angle AOB}) = x$. In this way the circular functions SINE and COSINE are defined, each with the set of standard position sensed angles as domain and $\{x: -1 \leq x \leq 1\}$ as range.

If SINE$(\overrightarrow{\angle AOB}) = y$, and $m\overrightarrow{\angle AOB} = \theta$, then $\sin \theta = y$. Thus, sine = [SINE $\circ$ m^{-1}]

If COSINE$(\overrightarrow{\angle AOB}) = x$, and $m\overrightarrow{\angle AOB} = \theta$, then $\cos \theta = x$. Thus, cosine = [COSINE $\circ$ m^{-1}]

In this way, the sine and cosine functions are defined, each with $\{x: 0 \leq x < 2\pi\}$ as domain, and $\{x: -1 \leq x \leq 1\}$ as range.

Degrees as well as radians may be used to measure sensed angles. π radians $= 180^\circ$.

If ABC is a triangle, with AB = c, AC = b, and BC = a, then the relation

$$c^2 = a^2 + b^2 - 2ab\cdot\cos C$$

always holds, and is called the Law of Cosines.

Also, in any triangle ABC, the relation

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

always holds, and is called the Law of Sines.

8.18 Review Exercises

1. (a) What is the initial side of $\overrightarrow{\angle RST}$?

   (b) What is the terminal side of $\overrightarrow{\angle RST}$?

2. Name the two sensed angles determined by $\overrightarrow{CD}$ and $\overrightarrow{CF}$?

3. Define a sensed angle in standard position.

4. (a) If a standard position sensed angle intercepts an arc of length $\frac{\pi}{4}$ on the circle $x^2 + y^2 = 4$, what is the radian measure of the sensed angle?

   (b) If a standard position sensed angle intercepts an arc of length $\frac{\pi}{4}$ on the unit circle, what is the radian measure of the sensed angle?

5. Draw the unit circle and sensed angles in standard position so that the following statements are true:

   (a) $m(\overrightarrow{\angle AOB}) = \frac{\pi}{2}$ (b) $m(\overrightarrow{\angle AOC}) = \frac{\pi}{4}$

   (c) $m(\overrightarrow{\angle AOD}) = \frac{3}{2}\pi$ (d) $m(\overrightarrow{\angle AOE}) = \frac{3}{4}\pi$

   (e) $m(\overrightarrow{\angle AOF}) = \frac{7}{6}\pi$ (f) $m(\overrightarrow{\angle AOG}) = \frac{7}{4}\pi$

6. The terminal side of $\overrightarrow{\angle AOB}$, in standard position, intersects the unit circle at $(\frac{1}{4}, \frac{1}{4}\sqrt{15})$.

   (a) What is SINE$(\overrightarrow{\angle AOB})$?

   (b) What is COSINE$(\overrightarrow{\angle AOB})$?

7. (a) If SINE$(\overrightarrow{\angle AOB}) = \frac{1}{3}$, what are the possible values of COSINE$(\overrightarrow{\angle AOB})$?

(b) If COSINE($\overrightarrow{\angle AOB}$) = a, what are the possible values of SINE($\overrightarrow{\angle AOB}$)?

8. $\overrightarrow{\angle AOB}$ is in standard position, and intersects the unit circle at $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

(a) What is SINE($\overrightarrow{\angle AOB}$)?

(b) What is the length of the arc which $\overrightarrow{\angle AOB}$ intercepts on the unit circle?

(c) What is m($\overrightarrow{\angle AOB}$)?

(d) What is $\sin \frac{3}{4}\pi$?

9. (a) Give the domain and range of the SINE function for sensed angles in standard position.

(b) Give the domain and range of the sine function.

10. Use the identity $\sin^2\theta + \cos^2\theta = 1$ to show that $\cos\theta$ cannot be greater than 1.

11. Complete the following:

(a) 30° = _______ radians (b) π radians = _______ $^\circ$

(c) $\frac{3}{2}\pi$ radians = _______ $^\circ$ (d) $\frac{1}{3}\pi$ radians = _______ $^\circ$

(e) 330° = _______ radians (f) 150° = _______ radians

12. Complete the following:

(a) $\sin 135^\circ$ = ___________

(b) $\cos 315^\circ$ = ___________

(c) $\sin 120^\circ$ = ___________

(d) $\cos 210^\circ$ = ___________

13. If $\frac{\pi}{2} < \theta < \pi$, then which of the following is true?

(a) $\sin\theta > 0$ and $\cos\theta > 0$

(b) $\sin \theta > 0$ and $\cos \theta < 0$

(c) $\sin \theta < 0$ and $\cos \theta < 0$

(d) $\sin \theta < 0$ and $\cos \theta > 0$

14. In the same set of axes, draw the graphs of two functions f and g with rules $f(x) = 2 \cdot \sin x$ and $g(x) = 2 \cdot \cos x$, each with domain $\{x: 0 \leq x < 2\pi\}$.

15. Consider the graph of the function s with rule $s(x) = \sin x$, $0 \leq x < 2\pi$.

(a) Does the graph of x have point symmetry? (What is the image of (0,0)?)

(b) Does the graph of x have line symmetry?

16. In $\triangle ABC$, $\angle A$ has measure 50°, $\angle B$ has measure 60°, and BC = 4.

(a) What is the degree measure of $\angle C$?

(b) Use the Law of Sines to find AB and AC.

17. In $\triangle ABC$, $\angle A$ has measure 30°, $\angle C$ has measure 30°, and BC = 10.

(a) What is the degree measure of $\angle B$?

(b) Use the Law of Sines to find AC and AB.

Chapter 9

INFORMAL SPACE GEOMETRY

9.1 Space Geometry and Plane Geometry

Although we live in a three-dimensional space, most of our previous study of geometry was limited to two-dimensional sets of points. "Lattice Points in the Plane," "Segments, Angles, and Isometries," "Affine Plane Geometry," "Coordinate Geometry," and "Transformations in the Plane" were all investigations of geometrical figures in a single plane.

This restriction to planar sets of points has two justifications. First, a wide range of practical geometry problems involve only two-dimensional figures such as parallel and perpendicular lines, angles, rectangles, and so on. Second, many of the properties established for planes and subsets of planes lead to analogous properties of space and subsets of space. This chapter generalizes the notions of incidence, parallelism, perpendicularity, and cocrdinate system to three dimensions.

9.2 Planes in Space

We think of a plane as being flat, and extending without boundary. In earlier chapters we studied certain subsets of a plane (lines, rays, segments, angles, polygons, etc.) These sets are also subsets of space as are planes themselves.

The surface of a table is often suggested as an illustration of a plane. A table top is not a plane because it does not extend without bound, but it usually is flat. If you place a straight ruler on such a table in any position whatever, each point of the ruler will be in contact with the table. If you perform the same test on a warped table or the surface of a corrugated roof, the ruler will not touch the surface at all points. This carpenter's test for flatness can be formalized to give a mathematical description of a plane.

<u>Observation</u> 1. A plane is a set of points with the property that whenever two points are in the set, the line containing them is in the set.

Notice that the line joining any two points of the surface must lie entirely in the surface. This phrasing avoids surfaces with holes, such as a slice of Swiss cheese (see Figure 9.1),

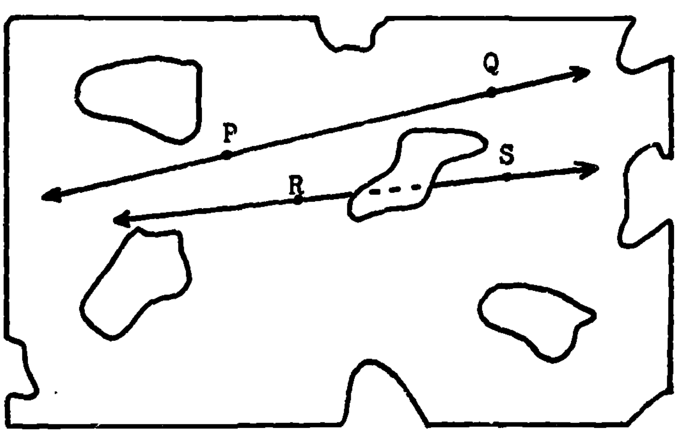


Figure 9.1

(Line PQ lies in the surface, but RS does not.)

surfaces that are curved, such as a waste paper basket (see Figure 9.2),

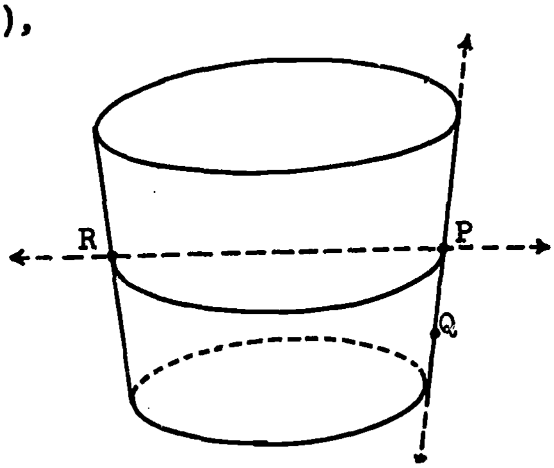


Figure 9.2

(The line joining P and Q lies on the surface of the basket, but the line joining P and R does not. The line joining any two points on the surface must lie in the surface.)

and surfaces that are bounded (see Figure 9.3).

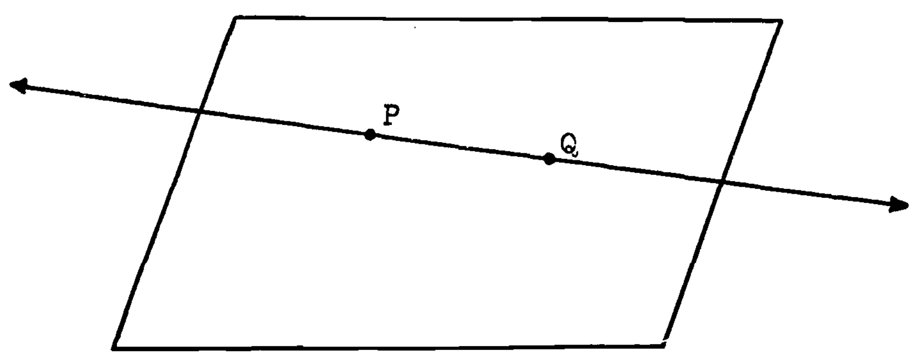


Figure 9.3

(The line PQ extends without bound.)

<u>Activity</u> 1. Materials Needed: one piece of cardboard (any shape) the size of this book or larger and three sharpened pencils of the same length.

First, hold a pencil vertically with the pencil point up. Place the cardboard so that it touches the pencil point. In how many positions can you hold the cardboard? Do the different positions all represent different planes?

Second, hold two pencils vertically with points up. Place the cardboard so that it touches both pencil points. In how many positions can you hold the cardboard? Do the different positions represent different planes?

Third, hold three pencils vertically with points up, placing them in a line, and place the cardboard so that it touches the three pencil points. In how many positions can you place the cardboard? Do the different positions represent different planes?

Finally, hold three pencils so that they are not in a line (see Figure 9.4). Hold the cardboard so that it touches the three pencil points. In how many positions can you place the cardboard?

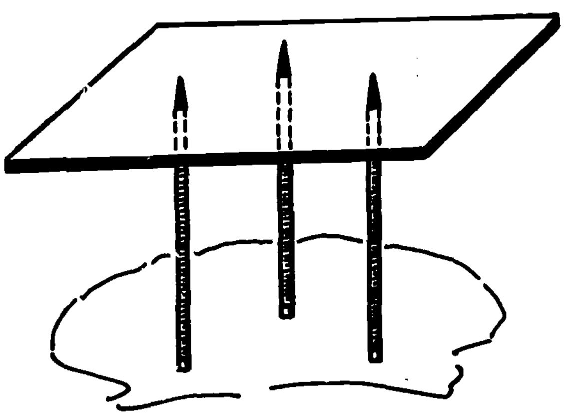


Figure 9.4

Now think of the pencil points as mathematical points and of the cardboard as a plane. Answer the following questions on the basis of your observations in Activity 1:

(1) How many planes are there that contain one given point?

(2) How many planes are there that contain two given points?

(3) How many planes are there that contain three given points if the points are collinear? if the points are non-collinear?

Your experience in this activity has probably led you to conclude that there are many planes that contain any given single point in space, or any given pair of points in space, or any given triple of collinear points in space. However, the situation is different with three given non-collinear points.

Observation 2. Given three non-collinear points there is one and only one plane that contains them.

A quick glance at your own classroom - whose walls, floor, and ceiling represent planes - makes one other fact of space obvious.

Observation 3. Not all points lie in the same plane.

As convincing justification for this observation, try to imagine a single flat surface that contains the points at the (1) front left bottom, (2) front right bottom, (3) front right top, and (4) rear right bottom corners of the room!

The exercises that follow present other combinations of points that may or may not lie in a plane.

9.3 Exercises

1. How many planes are there containing
   (a) 3 given collinear points?
   (b) 4 given collinear points?
   (c) a given line?
2. If two distinct lines intersect in a point, how many planes are there that contain both lines? Describe a physical situation that illustrates your answer.
3. If m is a line and P a point not on m, how many planes are there that contain P and m? Describe a physical situation which illustrates your answer.
4. If m and n are distinct parallel lines, how many planes are there that contain both lines? Describe a physical situation that illustrates your answer.
5. Three lines meet in the point determined by the lower right front corner of your classroom: the lines of intersection of the front wall and the floor, the right side wall and the floor, and the front and right side walls.
   (a) Is there a plane containing all three lines?
   (b) How many planes are there that contain at least two of these lines?
   (c) Imagine a diagonal line running from the point described above to the upper left rear corner of the room. Is there a single plane containing all four lines now under consideration? How many planes are there that contain at least three of the four lines? at least two of the four lines?

6. Let P, Q, R, S be distinct points, no three of which lie on the same line.
   (a) For (every, some, no) choice of such points, there is a plane containing all four points. Describe physical situations illustrating the answer you chose.
   (b) How many planes are there that contain at least three of the four points? Does this number depend on the location of the points? If so, how?
7. If P, Q, R, S are the points determined at the front right bottom, front right top, front left bottom, and back right bottom corners of a room,
   (a) how many planes are there that contain all four points?
   (b) how many planes are there that contain at least three of the points?
   (c) how many planes are there that contain at least two of the points?
8. Given two lines m and n is there always a plane that contains them? Illustrate your answer by describing appropriate physical situations.
9. Three legged stools are very common while two legged stools are as scarce as hen's teeth. Use the ideas discussed in Section 9.2 to explain this phenomenon.
10. Given a four legged table suffering from wobbles because of uneven legs, what is the minimum number of legs needing shortening to steady the table?

11. Describe several physical situations representing two intersecting planes.
    (a) What geometric figure is determined by the set of points common to the two planes?
    (b) If two planes have points P and Q in common, is line $\overleftrightarrow{PQ}$ also common to both planes? Do any observations about the properties of planes justify your answer?
12. Which of the following physical objects can serve as models of planar surfaces? In each case explain your answer.
    (a) The floor of your classroom.
    (b) The roof of the U. N. General Assembly Building.
    (c) A basketball.
    (d) A bath sponge (with rectangular faces).
    (e) The surface of Lake Placid (on February 1).

9.4 Parallel Lines and Planes in Space

In plane geometry parallelism is an important relation between lines. In what follows we shall use the Greek letter "π" to denote a plane. Lines will be denoted by lower case letters such as "m", "n", etc.

Definition 1. Lines m and n in plane π are parallel if and only if $m = n$ or $m \cap n = \emptyset$.

A line is considered parallel to itself, and two lines in a plane are parallel if they have no points in common.

You can probably find many objects that suggest models of parallel lines: the lines of notebook paper, railroad tracks, the lines where ceiling and floor meet a single wall of a room, and many others. What about the lines formed by a river and road passing over a bridge of that river? (See Figure 9.5.)

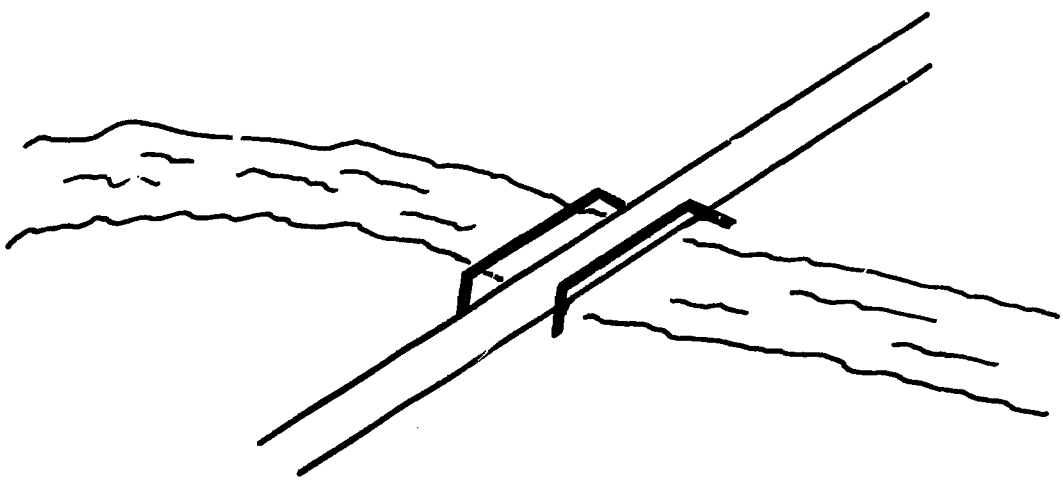


Figure 9.5

What about the lines in a room formed where the ceiling and front wall meet, and where the floor and a side wall meet? (See Figure 9.6.)

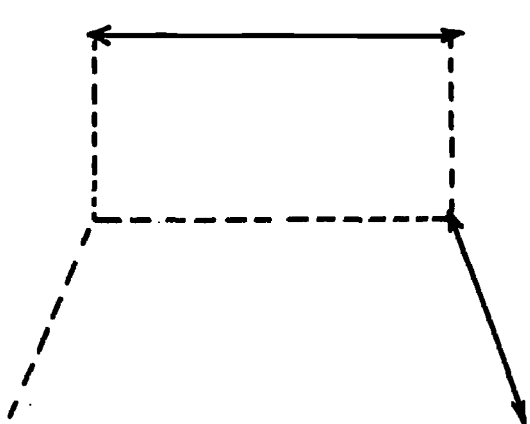


Figure 9.6

Do those lines intersect? Are they parallel? They don't seem to be related in the same way as parallel lines in a plane.

As a simple experiment, have a friend hold two pencils simulating the positions of the lines described in Figures 9.5 and 9.6. Is there a single plane that contains both of these lines? Use a piece of cardboard and try to fit it along both pencils. Remember, of course, that the pencils and cardboard are bounded in size and therefore might lead you to a false conclusion.

Next try to find a single plane containing two lines which are parallel. Experiment with the pencils and cardboard again and compare your findings with those above when the pencils were positioned differently.

The preceding experiments should make the following definitions clear:

Definition 2. (a) If two lines lie in the same plane, they are called coplanar.

(b) Two lines in space which are not coplanar are called skew.

Therefore, the pairs of lines described in Figures 9.5 and 9.6 are skew. The existence of skew lines in space emphasizes the importance of the phrase "in plane π" in the earlier definition of parallel lines. There are pairs of lines in space that do not intersect and are not parallel.

Although it is difficult to sketch space figures and relationships on a flat sheet of paper, the following techniques are generally used:

(1) A plane is represented by a parallelogram--with the understanding that the edges of the parallelogram do not indicate boundaries of the plane.

(2) Parallel lines are usually shown in a plane, as in Figure 9.7.

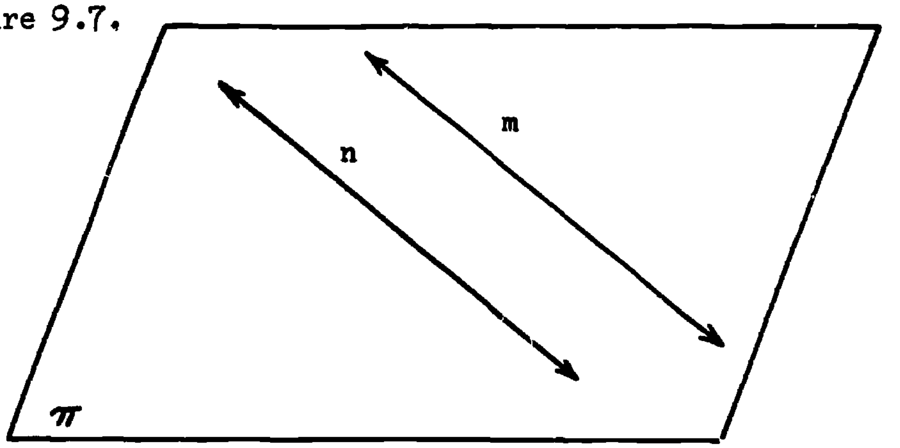


Figure 9.7

(3) Skew lines are represented as non-intersecting lines. In Figure 9.8 the drawing is intended to show that line n "passes under" line m.

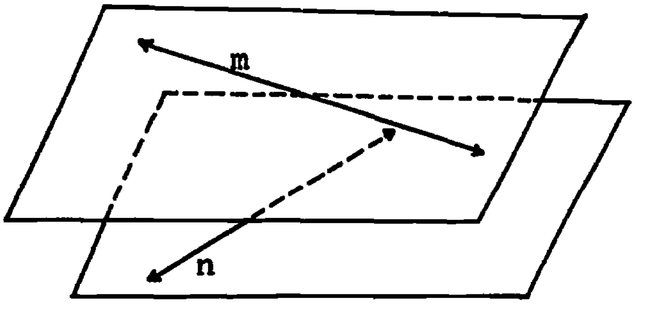


Figure 9.8

Lines in space are either intersecting, parallel, or skew. In what ways can a line and a plane be related? If line m intersects plane π, this is usually sketched as in Figure 9.9.

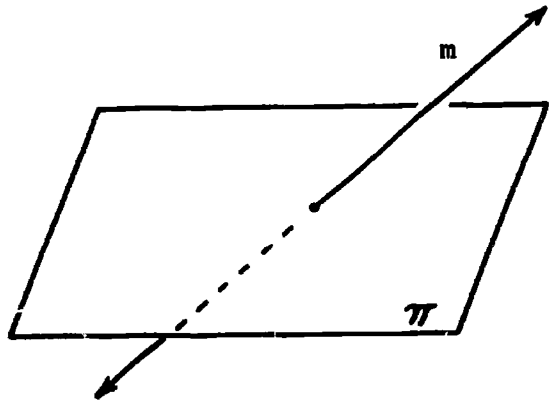


Figure 9.9

Question. What if line m intersects plane π in more than one point?

Activity 2. What are the possibilities if line m does not intersect plane π? Using a yardstick as a model of a line, and a table top as a model of a plane, place the "line" so that it is everywhere equidistant from the "plane"; that is, so that each point on the yeardstick is the same height above the table top. Next place the yardstick line so that the distance from a point P on the stick to the table top is greater than the distance from another point Q on the stick to the table top (but keep the yardstick from touching the table). You will probably agree that although in both cases the intersection of the physical models for the line and the plane are empty, the first case represents the natural meaning of "a line parallel to a plane." In the second situation the line will eventually intersect the plane because the line and the plane represented actually extend without bound. The above, and the definition of parallel lines, suggest the following definition:

Definition 3. Line m is parallel to plane π (or π is parallel to m) if and only if m is in π or $m \cap \pi = \emptyset$.

A line parallel to a plane is usually drawn as in Figure 9.10.

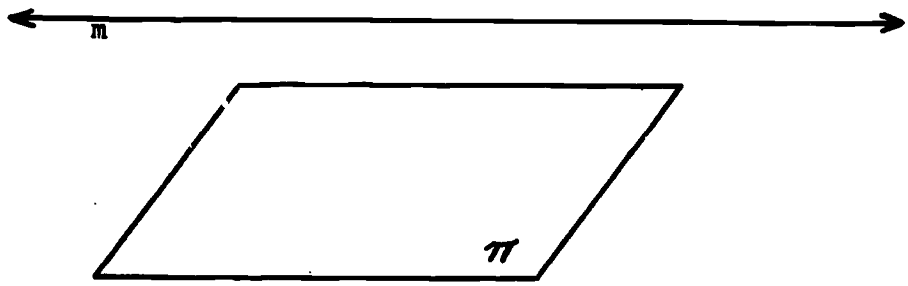


Figure 9.10

There is no such thing as a line "skew" to a plane in three-dimensional space since any line that is even slightly inclined to a plane will intersect that plane.

Again using a yardstick as a model of a line, a table top as a model of a plane, and a pencil point as a model of a point not on the plane, try to hold the yardstick in several different positions--each of which represents a line through the pencil point parallel to the given plane. How many lines are there through a point P that are parallel to plane π? How would you describe the figure formed by the lines through P that are parallel to π?

What possible relationships can exist between two planes in space? It seems reasonable to say that two planes intersect if they have at least one point in common; in fact, if you look at the examples of intersecting planes around you, it appears that any two planes that intersect at all must have an entire line in common.

Experimenting with pieces of cardboard to represent planes might lead you to conjecture that some planes have only one point in common.(see Figure 9.11),

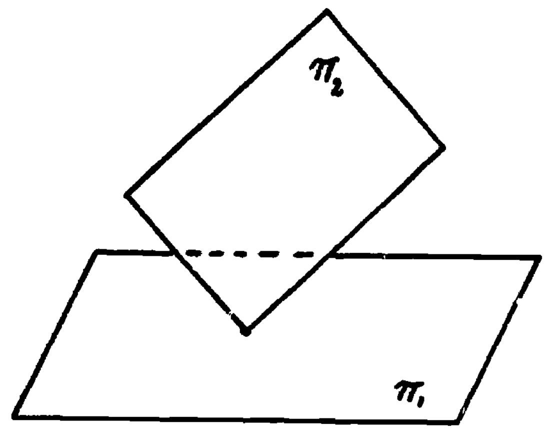


Figure 9.11

but remember that planes extend without bound. (see Figure 9.12).

Observation 4. If two planes have a point in common, they have a line in common.

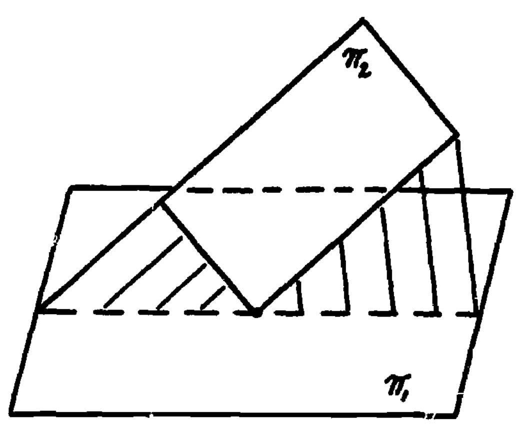


Figure 9.12

The definition of parallel lines and the definition of a line parallel to a plane suggest the following similar definition for parallel planes.

Definition 4. **Planes π_1 and π_2 are parallel if and only if they are the same plane or $\pi_1 \cap \pi_2 = \emptyset$.**

Notice that it is impossible to have two disjoint planes that are not parallel.

It is easy to find familiar objects that suggest models of parallel planes: for example, the floor and ceiling of the classroom, the opposite walls of the classroom, or the shelves of a bookcase. Parallel planes are usually drawn as parallel parallelograms (see Figure 9.13), where again, the edges of the parallelograms do not represent boundaries of the planes.

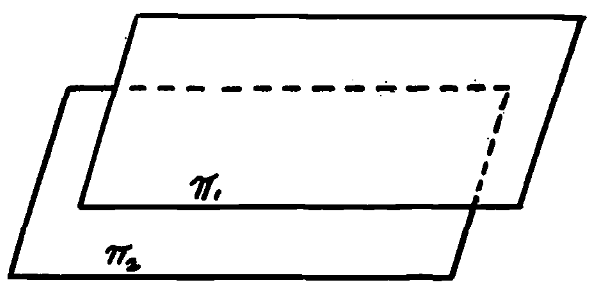


Figure 9.13

(The dashed lines indicate the part of π_2 hidden by π_1.) Although some objects that suggest planes may appear to be neither intersecting nor parallel, remember that planes extend without boun... Thus if two planes are not everywhere equidistant, they must intersect.

In an earlier experiment you found that if P is a point, there are an infinite number of lines that contain P and are parallel to π. The infinite set of lines forms a plane containing P and parallel to π. This suggests a generalization of the parallel postulate stated earlier for lines in a single plane; namely, through a point there is one and only one plane parallel to π.

If you do some further experimenting with pieces of cardboard it may seem that you can find other planes through P that do not meet plane π_1. (See Figure 9.14.)

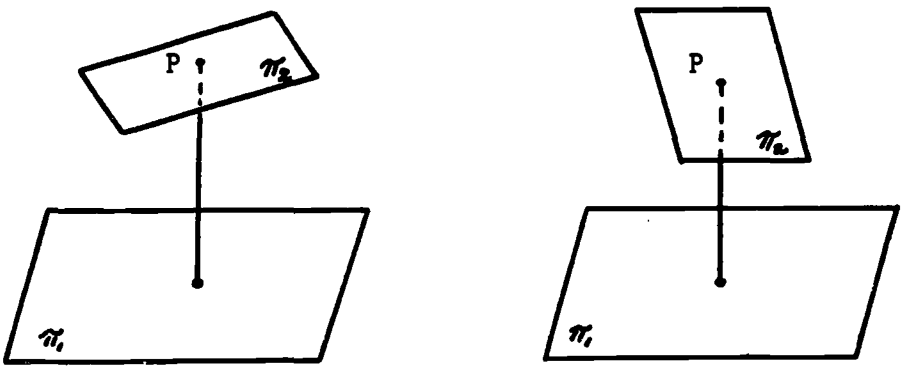


Figure 9.14

This experimentation, which may lead you to doubt the parallel postulate for planes in three-dimensional space, illustrates again the limitations of physical models for geometric objects. Although the pieces of cardboard used to represent planes are in some cases very helpful, they have one basic feature that makes them inadequate---the cardboard models of planes are bounded and mathematical planes are not. Thus, despite the fact that "cardboard planes" can be placed to seem neither parallel nor intersecting, the planes that these pieces of cardboard represent will meet because they extend without bound.

<u>Observation</u> 5. If P is a point and π_1 is a plane, there is exactly one plane π_2 containing P and parallel to π_1.

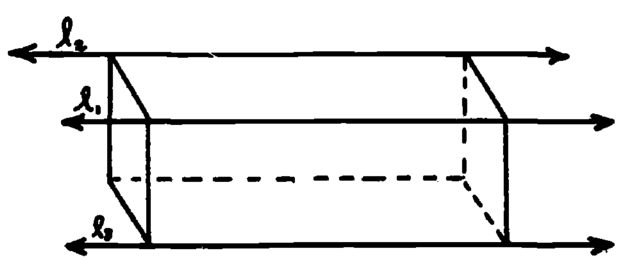
The following exercises explore many more possible relations of planes and lines in space.

9.5 Exercises

In Exercises 1 -- 15 determine whether the given statement is true or false. Then describe a physical situation or make a drawing that supports your answer. Remember, a true statement must be true without exception. (In these sentences "ℓ" represents a line, "π" a plane, and "P" a point.)

1. If line ℓ_1 is parallel to line ℓ_3 and line ℓ_2 is parallel to line ℓ_3, then ℓ_1 is parallel to ℓ_2.

   Answer (sample): True--For example, the line ℓ_1 formed where the ceiling of a room meets the right side wall is parallel to the line ℓ_3 formed where that wall meets the floor, and the line ℓ_2 formed where the ceiling meets the left side wall is also parallel to the line ℓ_3 formed where the right side wall meets the floor. The lines ℓ_1 and ℓ_2 where the ceiling meets the right and left side walls are, of course, parallel.



$\ell_1 || \ell_3$ and $\ell_2 || \ell_3$ implies $\ell_1 || \ell_2$.

2. If $\ell_1 || \pi$ and $\ell_2 || \pi$, then $\ell_1 || \ell_2$.
3. If $\pi_1 || \pi_3$ and $\pi_2 || \pi_3$, then $\pi_1 || \pi_2$.
4. If $\pi_1 || \ell$ and $\pi_2 || \ell$, then $\pi_1 || \pi_2$.
5. If $\ell_1 || \pi$ and $\ell_2 || \pi$, then ℓ_1 and ℓ_2 are skew.
6. If $\ell_1 || \ell_2$ and $\ell_2 || \ell_3$, then ℓ_1 and ℓ_3 are skew.

7. If $\ell || \pi_1$ and $\pi_1 || \pi_2$, then $\ell || \pi_2$.

8. If $\ell_1 || \ell_2$ and $\pi || \ell_1$, then $\pi || \ell_2$.

9. If $\ell_1 || \pi$ and ℓ_2 skew to ℓ_1, then ℓ_2 intersects π.

10. If $\ell_1 || \ell_2$ and ℓ_1 intersects π in a single point, then ℓ_2 intersects π in a single point.

11. If $\pi_1 || \pi_2$ and ℓ intersects π_1 in a single point, then ℓ intersects π_2 in a single point.

12. If $\pi_1 || \pi_2$ and $\pi_1 \cap \pi_3 = \ell_1$, then π_2 intersects π_3 in a line ℓ_2, with $\ell_1 || \ell_2$.

13. If $\ell_1 || \pi$ and $\ell_1 \cap \ell_2 = \{P\}$, then $\ell_2 || \pi$.

14. Find several physical situations that illustrate each of the following properties of lines and planes that were observed in the preceding section:

    (a) Skew lines do not intersect.

    (b) A line parallel to a plane is everywhere equidistant from the plane.

    (c) Two parallel planes are everywhere equidistant from each other.

    (d) If P is a point, there are an infinite number of lines that contain P and are parallel to plane π.

15. Make drawings to indicate the following:

    (a) Two intersecting lines both parallel to a plane.

    (b) Two parallel planes, both intersected by a line.

    (c) Two intersecting planes.

    (d) Two parallel planes intersected by a third plane.

9.6 Deductive Approach to Geometry in 3-Space

The exploratory activities of Sections 9.2 and 9.4 have provided you with some useful notions about points, lines, and planes in 3-space. You have observed that:

(Observation 1) A plane is a set of points with the property that whenever two points are in the set, the line containing them is in the set.

(Observation 2) Given three non-collinear points, there is one and only one plane that contains them.

(Observation 3) Not all points lie in the same plane.

(Observation 4) If two planes have a point in common, they have a line in common.

(Observation 5) If P is a point and π_1 is a plane, there is exactly one plane π_2 containing P and parallel to π_1.

Accepting these given observations as reasonable descriptions of reality and remembering those notions about points and lines in a single plane which you studied in previous courses, you are in a good position to deduce additional statements about points, lines and planes in 3-space.

We could refine these observations and state them as "axioms" for a 3-dimensional affine geometry. We could then define precisely some of the terms we have used, and proceed to deduce various statements which we would then call theorems.

In fact, this is what we did in our study of plane geometry.

However, we do not intend to develop a formal axiomatic system such as the one studied for 2-space; rather, we want to demonstrate how logic can be used to build on a set of accepted notions to increase our understanding of geometry. In a set of exercises which follow this section, you will have an opportunity to try out your deductive skills.

Example 1. Suppose we have a line m and a point P which is not in m. How many planes are there that contain both P and m? (See Figure 9.15.)

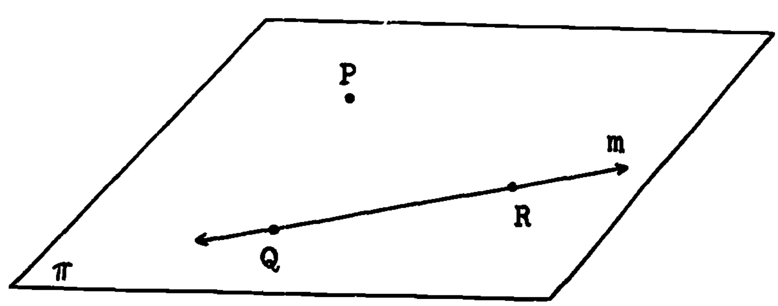


Figure 9.15

Recalling from our study of lines in a single plane that a line contains at least two points, we know that m contains two points which we call Q and R. Since P is not in line m, while Q and R are in m, we recognize that P, Q, and R are three non-collinear points.

By Observation 2, three non-collinear points (P, Q and R) are contained in one and only one plane, say π. But Q, R in π implies m = $\overleftrightarrow{QR}$

is in π, by Observation 1. Thus π is a plane containing both P and m. Could there be two such planes? Suppose π' is a plane containing m and P. Since $m \subset \pi'$, and Q, R $\in$ m, it follows that Q, R $\in \pi'$, and so P, Q, R $\in \pi'$. By Observation 2, $\pi' = \pi$. We therefore see that:

A line and a point not in the line are contained in exactly one plane.

Example 2. Let m and n be distinct lines which have a single point of intersection P. (See Figure 9.16.) How many planes contain both lines m and n?

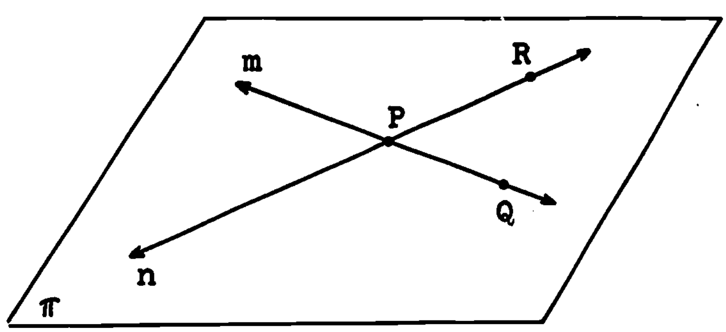


Figure 9.16

We know that there is a second point, call it Q, on line m. Since P is the only point of m which is also in n, we have line n and a point Q which is not in n. In Example 1 we showed that Q and n are contained in exactly one plane, say π. Since P is in n, P is in π, and by Observation 1, $m = \overleftrightarrow{PQ}$ is in π. Thus π is a plane containing m and n. As in Example 1, suppose π' is a plane containing m and n. Then π' contains Q and n (why?), and we conclude $\pi' = \pi$, or:

> Two distinct lines which have a single point of intersection are contained in exactly one plane.

Example 3. Let m and n be distinct lines that are parallel. Definition 1 assures us that there is a plane that contains m and n. Can there be another such plane? Try to imitate the type of reasoning used in Examples 1 and 2 to prove:

> If m and n are distinct parallel lines, then they are contained in exactly one plane.

9.7 Exercises

Using the ideas about lines in a single plane studied in previous courses, your observations made in Section 9.4 and listed for your convenience in Section 9.6, and the results of Examples 1-3 in Section 9.6, try to demonstrate that the following statements follow logically. In proving a new statement, you may make use of any statement already proved. Diagrams are required.

1. Any pair of distinct parallel lines lie in exactly one plane. (See Example 3 of Section 9.6.)
2. If a plane intersects one of two parallel planes, then it intersects the other.
3. If a plane intersects two parallel planes, then the intersections are parallel lines.
4. If a plane intersects one of two parallel lines in a point, then it intersects the other line in a point.
5. If a line intersects one of two parallel planes in a point, it intersects the other in a point.
6. If lines ℓ and m are parallel, then any plane that contains line ℓ is parallel to line m.
7. For the set of all planes in 3-space, "is parallel to" is an equivalence relation.

*8. For the set of all lines in 3-space, "is parallel to" is an equivalence relation.

9.8 Coordinate Systems in 3-Space

In a previous course, you studied how coordinate systems could be introduced into affine plane geometry. You may recall that in order to do this it was necessary to add further axioms to those for the affine plane. The new axioms served to introduce the real numbers as coordinates for points on a line.

We will now see how coordinate systems can be introduced into space geometry. Our development will be based on our previous study of plane geometry, and the material in Sections

9.6 and 9.7. You may be surprised to learn that the axioms that we used previously for coordinatizing the affine plane will serve equally well for coordinatizing affine 3-space.

You will recall that a coordinate system for a line is an assignment that matches each point of the line with a unique real number. The assignment is completely determined by the choice of two points $\mathcal{O}$ and I, $\mathcal{O}$ to be assigned coordinate 0, I to be assigned coordinate 1.

A coordinate system for a plane is an assignment that matches each point of the plane with an ordered pair of real numbers. The assignment is completely determined by the choice of three non-collinear points $\mathcal{O}$, I and J, $\mathcal{O}$ to be assigned (0, 0), I to be assigned (1, 0), and J to be assigned (0, 1) (see Figure 9.17). The choice of these three points allows us to assign an ordered pair of real numbers to every point T in the plane. Lines $\overleftrightarrow{\mathcal{O}I}$ and $\overleftrightarrow{\mathcal{O}J}$ are coordinatized ($\mathcal{O} \longrightarrow 0$, $I \longrightarrow 1$ and $\mathcal{O} \longrightarrow 0$, $J \longrightarrow 1$). The coordinates of a point T are determined by locating T with respect to the coordinatized lines $\overleftrightarrow{\mathcal{O}I}$ and $\overleftrightarrow{\mathcal{O}J}$.

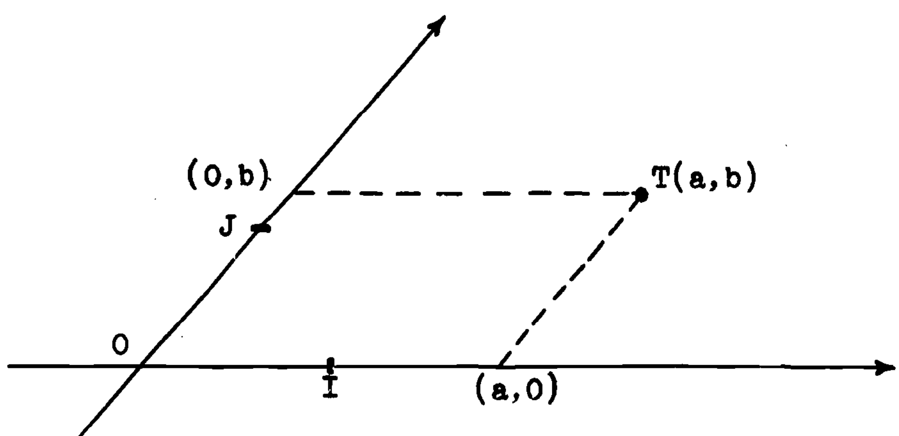


Figure 9.17

The existence and uniqueness of coordinates for any point T in a plane is guaranteed by the parallel postulate:

(1) There is a unique line through T parallel to $\overleftrightarrow{OJ}$ (and it intersects $\overleftrightarrow{OI}$ in a point with O,I-coordinate a).

(2) There is a unique line through T parallel to $\overleftrightarrow{OI}$ (and it intersects $\overleftrightarrow{OJ}$ in a point with O,J-coordinate b).

T is assigned coordinates (a,b).

We are now ready to tackle the task of introducing a coordinate system into affine 3-space. We start by choosing four non-coplanar points in space

O, I, J, and K,

no three of which are collinear. We know that four such points exist because of Observation 3. We may refer to the quadruple

(O, I, J, K)

as the base for a space coordinate system which we are about to introduce.

Let us call the line $\overleftrightarrow{OI}$ the x-axis, the line $\overleftrightarrow{OJ}$ the y-axis, and the line $\overleftrightarrow{OK}$ the z-axis. Next, let us introduce a line coordinate system on each of the three axes, using respectively the bases (O,I) (O,J) and (O,K). The point O is therefore a common origin for all three axes (coordinate system), and the points I, J, K are respectively unit points for the x-, the y- and the z-axes. We may diagram our axes as in Figure 9.18.

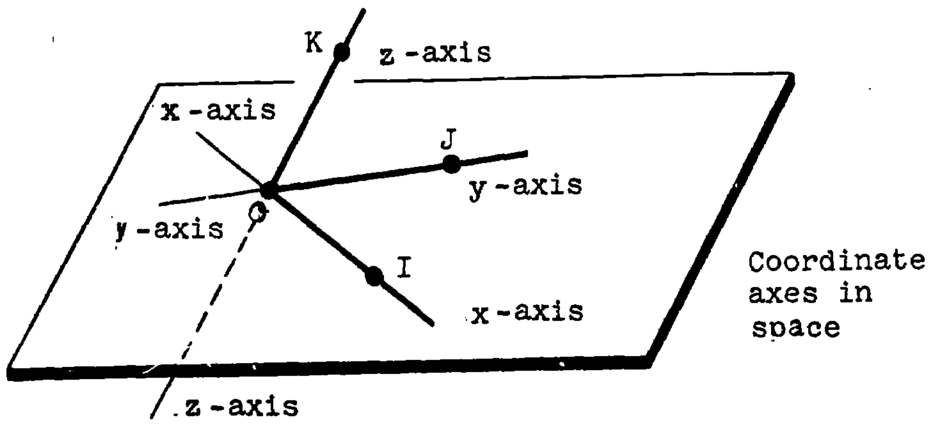


Figure 9.18

Each pair of coordinate axes determines a unique plane containing that pair of axes. (Why?) We call each of these planes a <u>coordinate plane</u>. There are clearly three coordinate planes. They are conveniently called the xy -plane, the yz-plane and the xz -plane. (Which of these three coordinate planes is depicted in Figure 9.18?)

Let us first consider any point X on the x -axis. This point will have a unique O,I-coordinate which we shall call the x-coordinate of point X. Similarly any point on the y-axis will have a unique O,J-coordinate which we shall call the y-coordinate of that point. Finally, every point on the z-axis has a unique O,K-coordinate which we shall refer to as its z-coordinate.

Now let us consider any point P in space. By Observation 5 there exists a unique plane π_1 which contains the point P and is parallel to the yz-plane. We know that the x-axis intersects the yz-plane in the unique point O. It follows that the x-axis must intersect the plane π_1 in some unique point X. (See Ex. 10, Section 9.7.) This state of affairs [is] diagrammed in Figure 9.19.

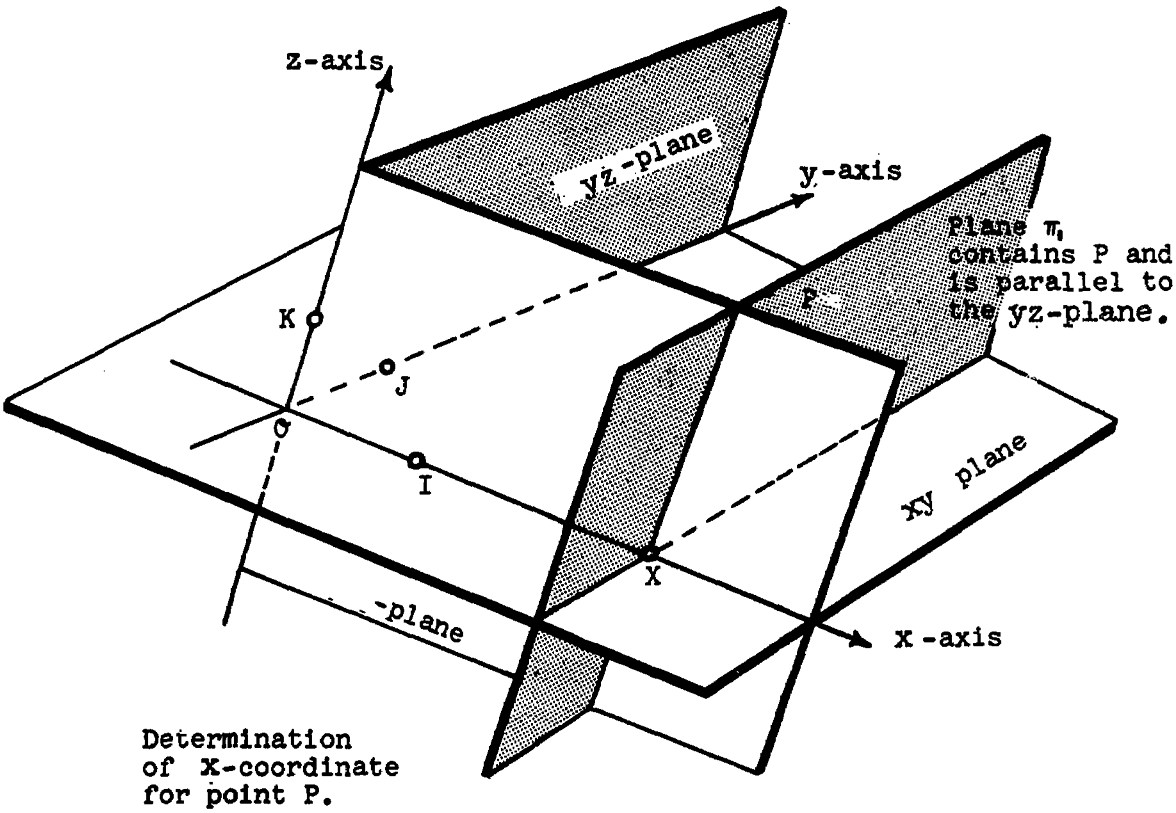


Figure 9.19

The point X has a unique x-coordinate namely its O,I-coordinate as described above. We shall agree to assign this value to point P. We shall call it the x-coordinate of P. We shall also assign an y-coordinate and a z-coordinate to point P. To assign an y-coordinate to point P we use Observation 5 once again to obtain a unique plane π_2 which contains P and is parallel to the xz-plane. (See Figure 9.20.)

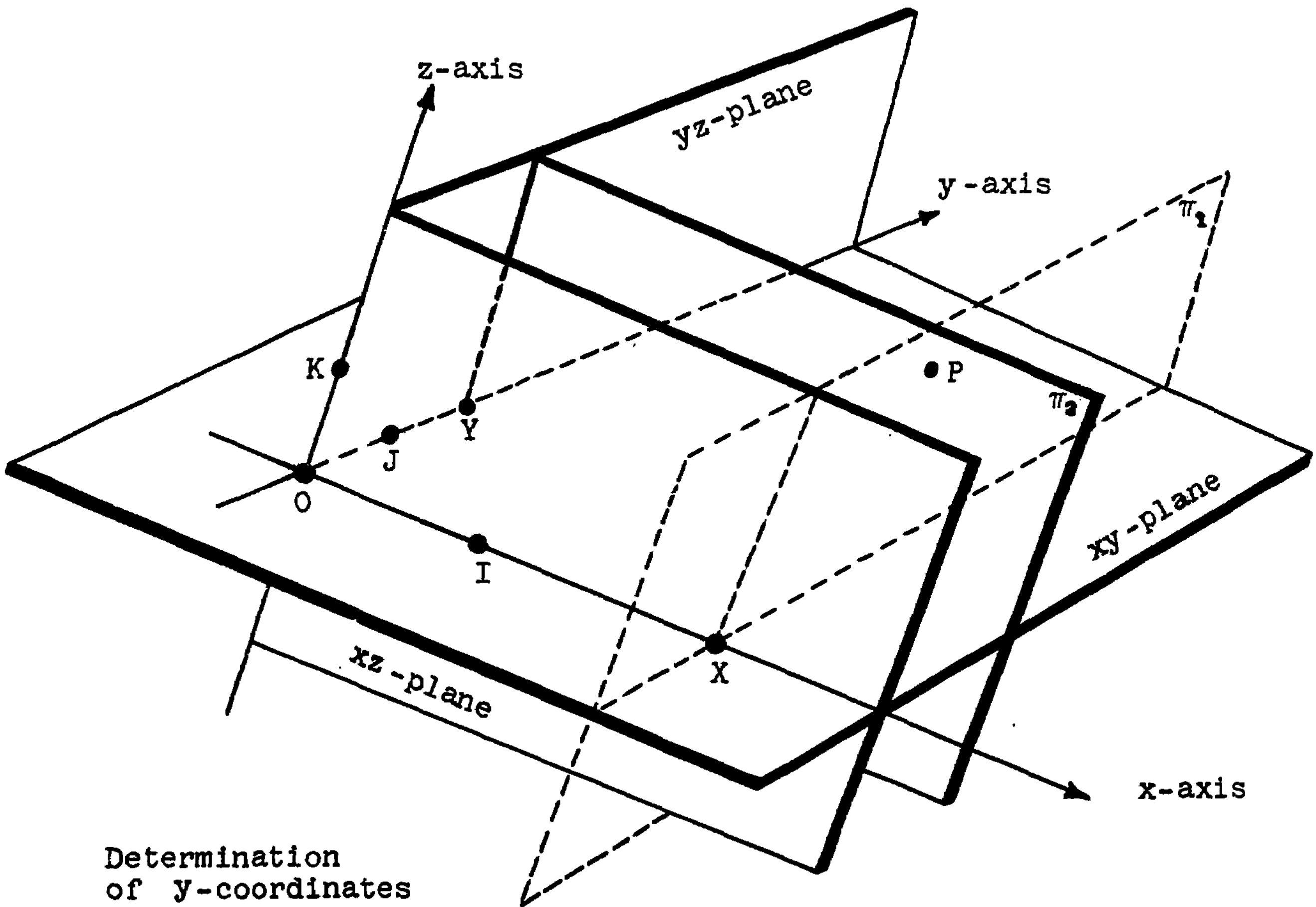


Figure 9.20

This plane π_2 must intersect the y-axis in a unique point Y. (See Ex. 10 of Sec. 9.7.) The point Y has a unique x_2-coordinate namely its O,J-coordinate. We assign this value to point P as the y-coordinate of point P. We leave it to the student to describe in a similar fashion how we assign a z-coordinate to point P. (See Exercise 2 in Section 9.9). The three coordinates which are thus assigned to the point P are assembled into an ordered triple (x, y, z), called the coordinate triple for point P.

We have shown above that for each point P in 3-space, there is a unique coordinate triple (x, y, z). Conversely, once a coordinate base $(\mathcal{O}, I, J, K)$ has been chosen for a space coordinate system, we can show that for each ordered triple of real numbers (x, y, z) there is a unique point P in space whose coordinates are precisely this ordered triple. In fact, there is a unique point X on the x-axis whose $\mathcal{O}, I$-coordinate is x, there is a unique point Y on the y-axis whose $\mathcal{O}, J$-coordinate is y and there is a unique point Z on the z-axis whose $\mathcal{O}, K$-coordinate is z. By Observation 5 there is a unique plane π_1 which contains point X and is parallel to the yz-plane. Similarly there is a unique plane π_2 which contains the point Y and is parallel to the xz-plane, and there is also a unique plane π_3 which contains the point Z and is parallel to the xy-plane. From the fact that the three coordinate planes have exactly one point $\mathcal{O}$ in common, it can be shown that the planes π_1, π_2, and π_3 have exactly one point in common.

<u>Example</u> 1. Consider the plane π, which is parallel to the xy-plane and which contains the point on the z-axis with coordinates $(0, 0, 3)$. Do you agree that every point in this plane will have 3 as its z-coordinate? Why? This suggests to us that a set description of this plane could be $\{P(x, y, z): z = 3\}$. Figure 9.21 depicts π.

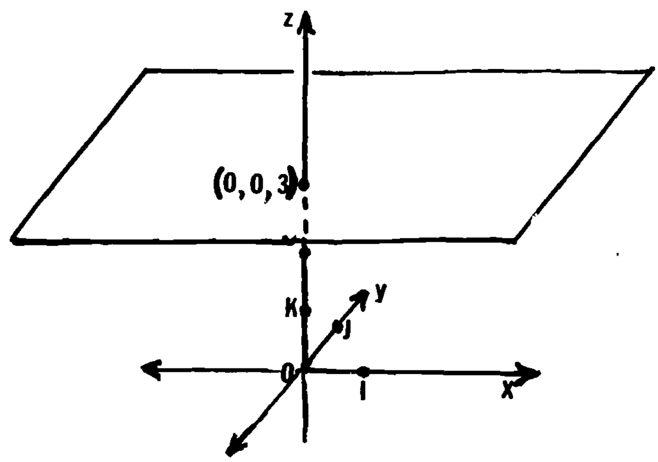


Figure 9.21

From the previously described procedure for assigning coordinates to point P, coordinates are real numbers. To simplify the notation in the discussion which follows we will not include the statement, in our set descriptions, that $x, y, z \in R$.

Example 2. What set is described by $\{P(x, y, z): z > 3\}$? Do you recognize that the set consists of all points in space which are "on one side" of the plane described in Example 1, namely the side which does not contain the origin (point O)? This set of points is an example of an open half-space. Make a diagram selecting a suitable drawing technique to depict this open half-space.

Perhaps you have noted that our coordinatization of affine 3-space, and indeed in our entire discussion in this chapter up to now, we did not consider perpendicularity. In the set of exercises which follows, we will deal with a large variety of situations which can be investigated without introducing perpendicularity.

9.9 Exercises

1. (a) Explain why the y-axis can be described as:

   $\{P(x, y, z): x = 0, z = 0\}$

   (b) Using set notation describe the z-axis.

2. Describe how a z-coordinate is assigned to a point P in space.

3. Describe verbally and sketch the following sets of points:

   (a) $\{P(x, y, z): x = 0, y > 0, z = 0\}$

   (b) $\{P(x, y, z): x < 0, y = 0, z = 0\}$

   (c) $\{P(x, y, z): x = 0, y = 0, z \geq 0\}$

   (d) $\{P(x, y, z): 0 \leq x < y, y = 0, z = 0\}$

   (e) $\{P(x, y, z): y = 0\}$

   (f) $\{P(x, y, z): y > 0\}$

   (g) $\{P(x, y, z): y < 0\}$

   (h) $\{P(x, y, z): y = 5\}$

   *(i) $\{P(x, y, z): 0 < y < 5\}$

4. Using set notation, describe each of the following sets of points:

   (a) All points in the yz-coordinate plane.

   (b) All points on the negative z-axis.

   (c) All points on a plane parallel to the xy-coordinate plane and containing the point Z(0, 0, 5).

   (d) All points of space that are between the xy-coordinate plane and the plane described in (c).

5. Let (2, 0, 4) be the coordinate triple for a point A in space. Using set notation, describe a plane which contains

A and is:

(a) parallel to the xy-coordinate plane.

(b) parallel to the yz-coordinate plane.

(c) parallel to the xz-coordinate plane.

6. Sketch and describe in words the following sets of points:

(a) $\{P(x, y, z): \; x = 4, y = 3\}$

(b) $\{P(x, y, z): \; x = 2, z = 0\}$

*(c) $\{P(x, y, z): \; y = z\}$

7. Use set notation to describe the following sets of points:

(a) A line in the yz-coordinate plane, parallel to the z-axis and containing the point (0, 2, 0).

(b) A line parallel to the z-axis containing the point (2, 3, 0).

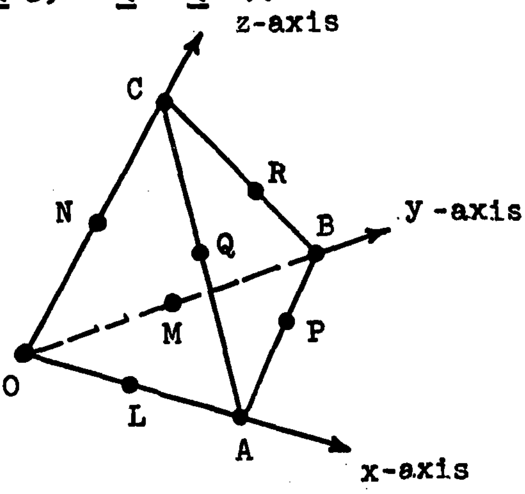
(c) A line in the xy-coordinate plane whose y-coordinate is twice its x-coordinate.

*(d) A plane containing the z-axis and also containing the point P(2, 6, 0). Sketch this plane.

*8. Sketch and describe in words the set of points:

$\{P(x, y, z): \; 0 \le x \le 4, \; 0 \le y \le 3, \; 0 \le z \le 2\}$.

9. One vertex of a tetrahedron (triangular pyramid) is chosen as the origin O, and and the other three vertices A, B, C are chosen as the unit points for a space coordinate system as indicated in the figure.



(a) Write a coordinate triple for each of the vertices O,A,B,C.

(b) If L,M,N,P,Q,R are midpoints of the edges as indicated, express each of these midpoints as a coordinate triple. (Hint: Use two-dimensional coordinates within each coordinate plane, noting that the remaining coordinate is zero in each case.)

9.10 Perpendicularity of Lines and Planes in Space

We have seen that if we choose two lines in space there are three possibilities: they may be parallel, they may intersect, or they may be skew. On the other hand if we choose two planes, or if we choose a line and a plane, then there are only two possibilities: they may either be parallel or they may intersect. The reason for this is that the word "parallel" applied to two lines does not mean the same thing as it does when applied to two planes or to a line and a plane. (In the case of two lines, parallelism includes an extra requirement, namely that the lines be coplanar.)

Similarly, the word "perpendicular" which thus far applies to two lines, must be given a modified meaning when applied to two planes or to a line and a plane. The activities described in this section should help you get a clear picture of the various meanings for the word "perpendicular" in 3-space.

Activity 3. Materials needed: sharp pencils, unlined paper, ruler, and one assistant.

Mark a point P on the paper and draw one line ℓ_1 through P. Place the point of the pencil on P and hold the pencil so that it is perpendicular to the line ℓ_1. Hold the pencil in a different position keeping it perpendicular to the line ℓ_1. What conjecture does this activity suggest concerning the number of lines in space perpendicular to a given line at a point on that line? (See Figure 9.22.)

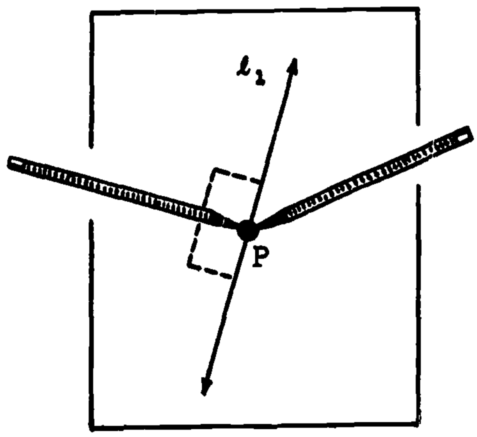


Figure 9.22

Now draw another line ℓ_2 on the paper through point P. (See Figure 9.22.) Repeat the above experiment using line ℓ_2, i.e., hold the pencil in different positions keeping it perpendicular to line ℓ_2. In these new positions, will the pencil be perpendicular to line ℓ_1? Try to find a position for the pencil so that it will be perpendicular to both ℓ_1 and ℓ_2. (See Figure 9.22.)

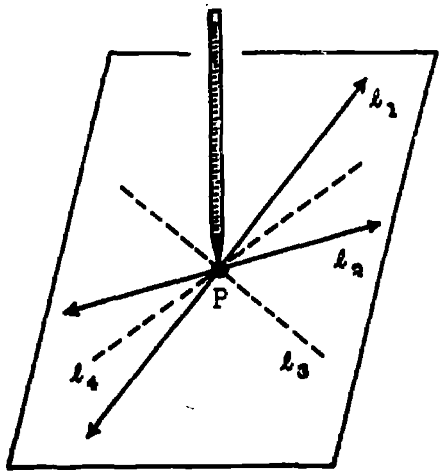


Figure 9.23

When you have found this position keep the pencil fixed and have an assistant draw other lines in the paper through point P (see Figure 9.23). What do you notice about all of these lines?

<u>Question</u> 1. If π is a plane and P a point in π, how many lines are there through P lying in π?

<u>Question</u> 2. If π is a plane, P a point in π, and ℓ_1 a line in π through P, how many lines can be drawn in space through P and perpendicular to line ℓ_1?

<u>Question</u> 3. If π is a plane, P a point in π, ℓ_1 and ℓ_2 two distinct lines in π through P, how many lines can be drawn in space through P and perpendicular to both ℓ_1 and ℓ_2?

<u>Question</u> 4. If a line m in space is perpendicular to each of two intersecting lines ℓ_1 and ℓ_2 at point P, how is line m situated in relation

to all other lines through P in the same plane as ℓ_1 and ℓ_2?

If you have performed the above experiments carefully and answered the questions correctly, you will now appreciate the following definition and observation.

Definition 5. A line m is perpendicular to plane π at P if and only if m is perpendicular to every line in π containing P.

Observation 6. If a line m is perpendicular to each of two intersecting lines in plane π at point P, then m is perpendicular to plane π at P.

Activity 4. Materials needed: a rectangular 3x5 card, unlined paper, sharp pencil, ruler.

Begin by drawing a line ℓ on the 3x5 card perpendicular to one of its longer edges. Place the card so that edge rests on the unlined paper.

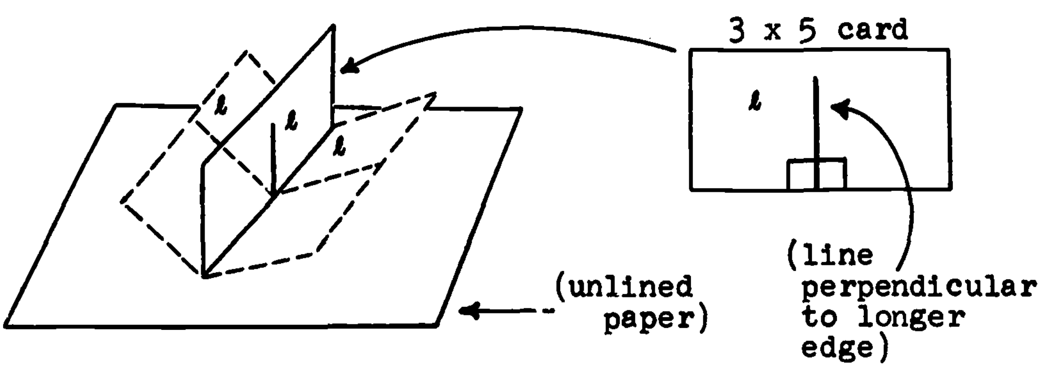


Figure 9.24

Tilt the card so that line ℓ assumes various positions in relation to the plane of the paper (see Figure 9.24). In which of the positions for line ℓ would you be willing to say that the card is perpendicular to the unlined sheet of paper?

Keep the long edge of the card in a fixed position against the paper and rotate the card about this edge until it lies flat on the paper. Trace line ℓ onto the paper, calling this new line ℓ'. Since ℓ was originally perpendicular to the edge of the card, what can you say about ℓ'?

<u>Definition</u> 6. Two intersecting planes π_1 and π_2 are called <u>perpendicular</u> iff there is a line ℓ_1 in π_1, and a line ℓ_2 in π_2, such that each of ℓ_1 and ℓ_2 is perpendicular to the line of intersection of π_1 and π_2, and ℓ_1 is perpendicular to ℓ_2. (See Figure 9.25.)

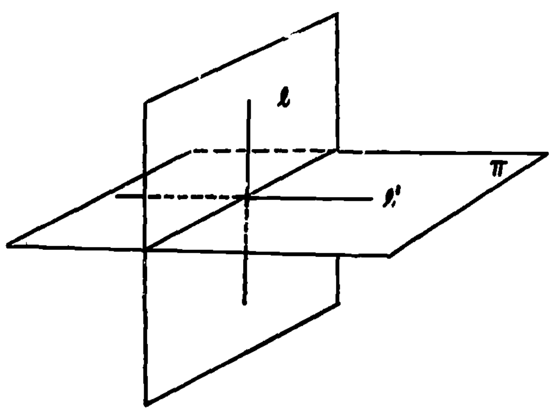


Figure 9.25

Observation 7. If a line ℓ is perpendicular to a plane π, then any plane containing ℓ is perpendicular to the plane π.

Perform an experiment to test this observation, using a pencil and several 3x5 cards.

In a plane, there is one and only one line perpendicular to a given line at a given point on that line. Moreover, there is one and only one line perpendicular to a given line from a given point not on the given line (see Figure 9.26).

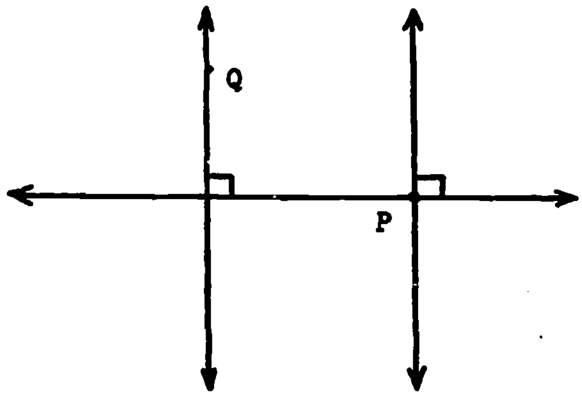


Figure 9.26

Do these properties carry over to perpendicular lines in space? Are there corresponding properties for lines perpendicular to planes? The next activity explores this question.

Activity 5. Materials needed: sharp pencils, unlined paper, ruler.

Mark a point P on the paper and draw one line through P. Place the point of the pencil on P and hold the pencil so that it is perpendicular to the line. Hold the pencil in a different position, keeping it perpendicular to the line. Among the lines perpendicular to the given line at P, how many are perpendicular to the plane of the paper at P?

Next hold one of your pencils perpendicular to the plane of your paper at P. Now hold another pencil as shown in Figure 9.27 (eraser to eraser), and try to make the second pencil perpendicular to the line or to the plane.

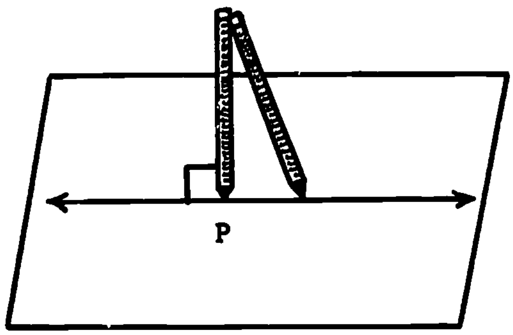


Figure 9.27

Your attempts should suggest that from a point not on a given line there is one and only one perpendicular to the given line, and from a point not on a given plane there is one and only one line perpendicular to that plane.

Lines perpendicular to planes are usually indicated by drawing "vertical" lines as in Figure 9.28.

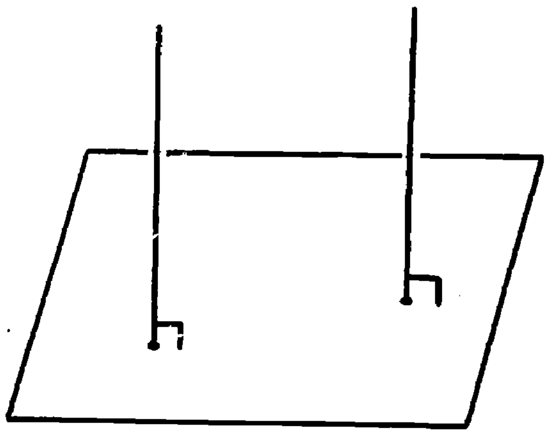


Figure 9.28

9.11 Exercises

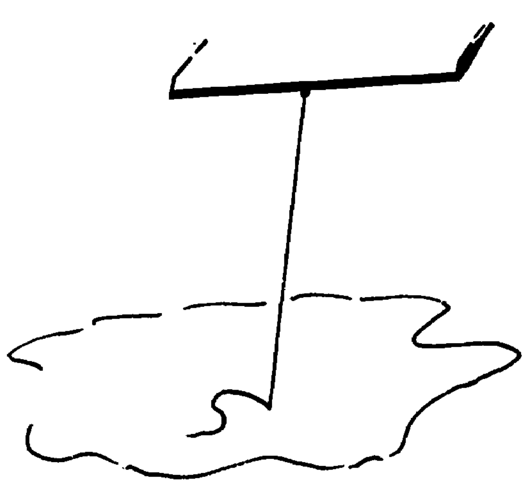
The following exercises involve lines and planes in space, and the relations of perpendicularity and parallelism. In Exercises 1--3 you are asked to perform some experiments and from them draw a conclusion about lines and planes in space.

1. Draw several lines through a point P on a sheet of paper. Can you hold a pencil with point on P so that the pencil is perpendicular to only one line through P? so that it is perpendicular to only two lines through P? so that it is perpendicular to only three lines through P? Conclusion (sample): If P is a point in plane π and m a line through P, m is perpendicular to π at P if m is perpendicular to ______ lines in π through P.
2. With an assistant, hold several pencils perpendicular to the top of a table. What pattern do you see that should hold in general?

9.11 Exercises

The following exercises involve lines and planes in space, and the relations of perpendicularity and parallelism. In Exercises 1 - 3 you are asked to perform some experiments and from them draw a conclusion about lines and planes in space.

1. Draw several lines through a point P on a sheet of paper. Can you hold a pencil with point on P so that the pencil is perpendicular to only one line through P? so that it is perpendicular to only two lines through P? so that it is perpendicular to only three lines through P? Conclusion (sample): If P is a point in plane π and m a line through P, m is perpendicular to π at P if m is perpendicular to ____ lines in π through P.
2. With an assistant, hold several pencils perpendicular to the top of a table. What pattern do you see that should hold in general?
3. Anchor a piece of string to some rigid object such as a table and hold the other end so that it touches the floor.



Now vary the position where the string touches the floor trying to find the position that makes the length of the string shortest. What generalization about distances does this suggest?

In Exercises 4 - 9 determine whether the given statement is true or false. Then describe a physical situation or make a drawing that supports your answer. In these statements "ℓ" denotes line, "π" a plane, and "P" a point. The symbol "$\perp$" denotes "is perpendicular to." (Recall: A true statement must be true without exceptions.)

4. If $\ell_1 \perp \ell_3$ and $\ell_2 \perp \ell_3$, then $\ell_1 \| \ell_2$.
5. If $\ell_1 \perp \pi$ and $\ell_2 \perp \pi$, then $\ell_1 \| \ell_2$.
6. If $\ell_1 \| \ell_2$ and $\ell_1 \perp \pi$, then $\ell_2 \perp \pi$.
7. If $\pi_1 \perp \ell$ and $\pi_2 \perp \ell$, then $\pi_1 \| \pi_2$.
8. If $\ell_1 \perp \ell_2$ and $\ell_2 \| \ell_3$, then $\ell_1 \perp \ell_3$.
9. If $\ell_1 \perp \ell_3$ and $\ell_2 \perp \ell_3$, then ℓ_1 and ℓ_2 are skew.

9.12 Rectangular Coordinate Systems in Space

In Section 9.8 we coordinatized 3-space by introducing three coordinatized lines, called axes. A procedure was then described for designating an ordered triple of real numbers as coordinates for a point in space. In Section 9.10 we explored the various meanings for the word "perpendicular" as it relates to lines and planes in space. In this section we will do some informal work with coordinate systems in which the coordinate axes are mutually prependicular. A system with three mutually perpendicular coordinate axes is called a <u>rectangular coordinate system</u>.

As before, we identify an origin $\mathcal{O}$ and unit points I on the x-axis, J on the y-axis, and K on the z-axis, $\mathcal{O}I = \mathcal{O}J = \mathcal{O}K = 1$.

Figure 9.29 depicts a rectangular coordinate system in 3-space. The positive portions of the coordinate axes are represented by

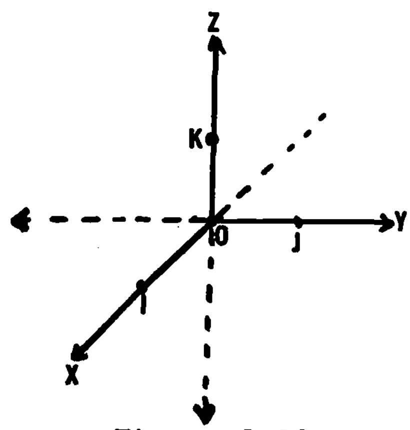


Figure 9.29

solid lines, and the negative portions by broken lines.

Since the x-axis is perpendicular to each of the other two coordinate axes at their point of intersection $\mathcal{O}$, it is perpendicular to the plane determined by these axes. In other words, the x-axis is perpendicular to the yz-coordinate plane. In like manner, the y-axis is perpendicular to the xz-plane, and the z-axis is perpendicular to the xy-plane.

As a simple example of a space coordinate system, consider the miniature three-dimensional space that your classroom constitutes. (See Figure 9.30.)

(1) Let the x-axis be the line where the floor meets the left side wall.

(2) Let the y-axis be the line where the floor meets the front wall.

(3) Let the z-axis be the line where the left side wall meets the front wall.

(4) Let the unit on all three axes be 1 foot.

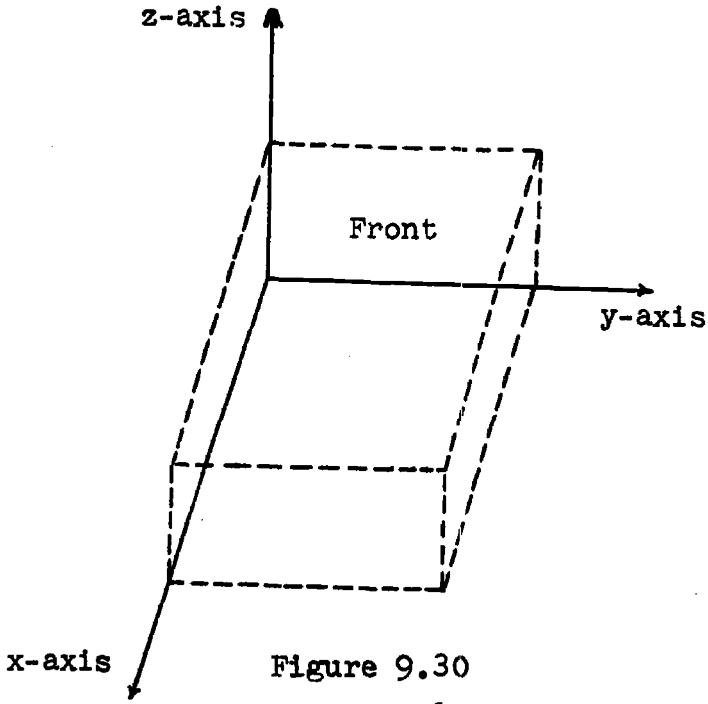


Figure 9.30

If the room is 10 feet high, 26 feet wide, and 36 feet long (front to back), it is not hard to find the coordinates of the following points:

(a)	front, left, lower corner	(A)	(0,0,0)
(b)	middle of the floor	(B)	(18,13,0)
(c)	rear, right, lower corner	(C)	(36,26,0)
(d)	front, right, lower corner	(D)	(0,26,0)
(e)	front, right, upper corner	(E)	(0,26,10)
(f)	middle of the front wall	(F)	(0,13,5)
(g)	middle of the left side wall	(G)	(18,0,5)
(h)	middle of the ceiling	(H)	(18,13,10)
(l)	exact center of the room	(L)	(18,13,5)
(m)	rear, right, upper corner	(M)	(36,26,10)

Several of these points, which are related in an interesting way geometrically, have coordinates that are related also. For instance, points B and C lie on a line with the origin A(0,0,0),

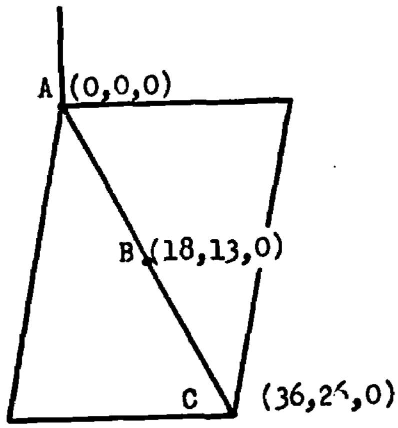


Figure 9.31

and, geometrically (see Figure 9.31), B is the midpoint of segment $\overline{AC}$. We know that in a coordinatized plane the midpoint of a segment (whose endpoints have coordinates (x_1, y_1) and (x_2, y_2)) has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The coordinates of B satisfy this relation if we ignore the 0 in the third position of all three coordinate triples. Is there a general midpoint formula in space?

Consider points L and M and the origin. The exact center of the room should be the midpoint of the diagonal segment from the front, left, lower corner to the rear, right, upper corner. How are the coordinates of these points related?

A(0, 0, 0) L(18, 13, 5) M(36, 26, 10)

The coordinates of the midpoint are the averages of the corresponding coordinates of the endpoints.

$$18 = \frac{0 + 36}{2}, \; 13 = \frac{0 + 26}{2}, \text{ and } 5 = \frac{0 + 10}{2}.$$

If you think this convenient relation was only coincidental, check other triples of points such as A, F, and E, or L, B, and H.

If point M is the midpoint of segment $\overline{KL}$, where K has coordinates (x_1, y_1, z_1) and L has coordinates (x_2, y_2, z_2), then the coordinates of M are given by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

9.13 Exercises

1. Using the coordinate system described in the preceding section, find the coordinates of the following points in the classroom (assuming the dimensions given.)
   (a) Middle of the rear wall.
   (b) Middle of the right side wall.
   (c) Middle of the left side wall.
   (d) Rear, left, upper corner.
   (e) Rear, left, lower corner.
   (f) The midpoint of the segment joining the origin to the exact center of the room.
2. Again using the coordinate system given for your classroom, describe the locations of the points in space with the following coordinates:
   (a) (5, 0, 5)
   (b) (40, 0, 5)
   (c) (-5, 0, 5)

(d) $(0, -5, 5)$

(e) $(0, 0, -5)$

(f) $(-18, 13, 5)$

(g) $(-18, -13, 5)$

3. Use the midpoint formula to calculate the coordinates of the midpoints of the segments with the following endpoints:

   (a) $(5, 10, 15), (7, 6, 3)$

   (b) $(-5, 10, 15), (-7, 6, 3)$

   (c) $(5, -10, 15), (-7, 6, -3)$

   (d) $(5, 10, -15), (7, -6, -3)$

4. Given a coordinate system for space, a translation is a mapping from space to space with rule of assignment of the form $T: (x, y, z) \longrightarrow (x + a, y + b, z + c)$, where a, b, c are real numbers. To locate the image of a point under the translation $T: (x, y, z) \longrightarrow (x + 2, y + 3, z - 2)$ we start at the point, move 2 units parallel to the x-axis, 3 units parallel to the y-axis, and -2 units parallel to the z-axis, and end up at the image of the point. Find the images under this translation of the following points:

   (a) $(0, 0, 0)$

   (b) $(-5, -3, -7)$

   (c) $(11, 7, 2)$

   (d) $(14, -9, 12)$

5. Find the point in space whose image under the translation in Exercise 4 is

   (a) $(0, 0, 0)$

   (b) $(-5, -3, -7)$

(c) (11, 7, 2)

(d) (14, -9, 12)

6. Is every translation in space a one-to-one mapping? Why or why not?

7. Is every translation a mapping of space onto space? Why or why not?

9.14 Distance in Space

In Course II Chapter 6 you found a convenient formula for calculating the distance between two points in a rectangular coordinate system. In particular, if the coordinates of a point T are (x, y) in some rectangular coordinate system for a plane, then the distance from T to the origin is given by $\sqrt{x^2 + y^2}$. Since space coordinate systems are such natural extension of plane coordinate systems, it is reasonable to suspect that in a coordinate system whose axes are mutually perpendicular (a rectangular coordinate system in space) the distance from a point S with coordinates (x, y, z) to the origin is given by $\sqrt{x^2 + y^2 + z^2}$. Or is it? Why not $\sqrt[3]{x^3 + y_3 + z_3}$?

To see that the first given formula is correct, consider the problem of finding the distance from the exact center of the room coordinatized in Section 9.12 to the front, left, lower corner which is the origin.

The point C with coordinates (18, 13, 5) is a vertex of right triangle OBC, (see Figure 9.32), and the distance we are interested in is the hypotenuse $\overline{OC}$ of this triangle. The Pythagorean property of right triangles is suggested as a possible aid in calculating the required length; but only one side of the triangle is known-- BC = 5.

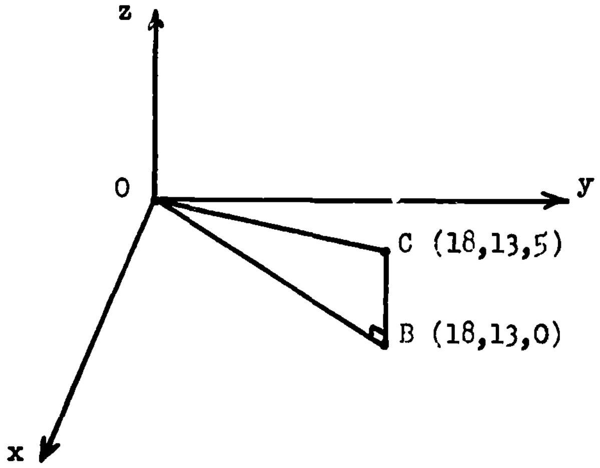


Figure 9.32

Fortunately there is a way we can calculate the length of the other side, $\overline{OB}$.

Since point B lies in the plane coordinatized by the x-axis and the y-axis, and since it has coordinates (18, 13, 0), the distance from B to the origin is $\sqrt{(18)^2 + (13)^2}$. Therefore, the length of $\overline{OB}$ is $\sqrt{(18)^2 + (13)^2}$. Applying the Pythagorean property to triangle OBC, we find that the length of $\overline{OC}$ is

$$\sqrt{\sqrt{(18)^2 + (13)^2}^{\,2} + (5)^2}$$

or

$$\sqrt{(18)^2 + (13)^2 + (5)^2}$$

which is precisely the answer predicted by the formula

$$\sqrt{x^2 + y^2 + z^2}$$

Granting that the formula $\sqrt{x^2 + y^2 + z^2}$ correctly represents the distance from a point S with coordinates (x, y, z) to the origin, can we find the distance from point S to a point other than the origin? Suppose that point T has coordinates (x_1, y_1, z_1). One way to determine the distance ST is as follows. Suppose we map each point of space onto a new point of space by means of a translation. Since every translation is an isometry, the distance ST ought to remain unchanged by the mapping, i.e., if S maps onto S′ and T maps into T′ then ST = S′T′. Now if we choose the translation defined by $(x, y, z) \longrightarrow (x - x_1, y - y_1, z - z_1)$, we observe that under this mapping S maps into $(x - x_1, y - y_1, z - z_1)$, but T maps into $(x_1 - x_1, y_1 - y_1, z_1 - z_1)$, i.e., T maps into (0, 0, 0). Hence the distance from S to T is the same as the distance from $(x - x_1, y - y_1, z - z_1)$ to the origin. This distance is therefore given by the formula

$$\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} .$$

You should recognize this formula as the natural generalization of the corresponding distance formula for two dimensions

$$\sqrt{(x - x_1)^2 + (y - y_1)^2}$$

(See Course II, Chapter 6, Section 20.)

As an example, let us find the distance from the point S with coordinates (3, -1, 5) to the point T with coordinates (2, 1, 3). (See Figure 9.33.)

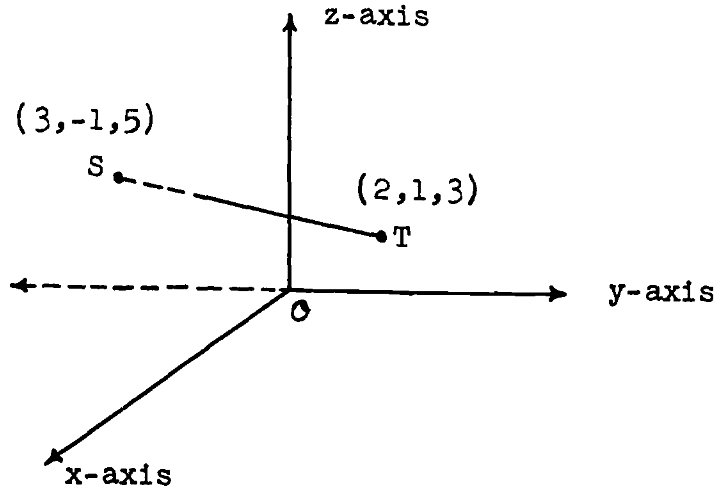


Figure 9.33

Applying the distance formula we obtain:

$$ST = \sqrt{(3 - 2)^2 + (-1 - 1)^2 + (5 - 3)^2}$$

$$= \sqrt{(1)^2 + (-2)^2 + (2)^2}$$

$$= \sqrt{9}$$

$$ST = 3$$

9.15 <u>Exercises</u>

1. Find the distance from the origin (0, 0, 0) of a rectangular coordinate system to the points in space that have the following coordinates in that system:

   (a) (3, 4, 12) (c) (-3, -4, -12)

   (b) (-6, 8, -24) (d) (4, 3, 12)

2. In each of the following, the coordinates of a point P are given in a rectangular coordinate system. First find

the length of segment $\overline{OP}$, then find the coordinates of the midpoint of this segment, and finally find the length of the segment from O to this midpoint.

(a) (6, 8, 24)

(b) (-3, -4, -12)

(c) (3, 4, 5)

(d) (6, 8, 10)

3. Do your findings in Exercise 2 confirm or deny the validity of the midpoint formula adopted in Section 9.12? Explain your answer.

4. Find the distance between each of the following pairs of points:

(a) S(4, 2, 3), T(3, 0, 1)

(b) P(5, -2, 1), Q(1, 2, 3)

(c) $C(1, \frac{2}{3}, 0)$, $D(\frac{1}{3}, 1, -\frac{2}{3})$

(d) M(2, -1, 4), N(-3, 5, 2)

5. Let points A and B have coordinates (7, -3, 6) and (3, -1, 2) respectively, and let M be the midpoint of segment $\overline{AB}$. Calculate each of the following:

(a) The coordinates of M (using the midpoint formula for space).

(b) The distances AM, MB, and AB.

(c) How do your findings in (a) and (b) confirm the validity of the midpoint formula?

6. The vertices of $\triangle ABC$ are A(-3, 2, 0), B(1, -2, 4), C(1, 2, -4). Find the length of the median from vertex C to side $\overline{AB}$.

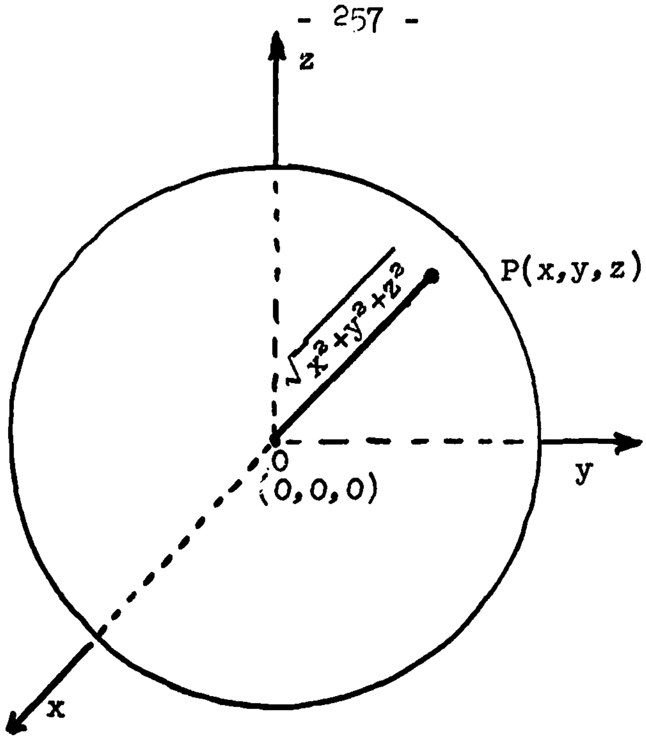
9.16 Surfaces in Space

In the previous sections of this chapter the only subsets of space that were discussed were lines and plances and geometric figures which are made up of intersecting and parallel lines and planes. You are certainly familiar with many other kinds of surfaces and solids in space -- models of spheres, cones, cylinders, and pyramids are common sights in our three-dimensionsal world. In order to treat these figures in the deductive structure of geometry, they must be defined as certain kinds of point sets - subsets of space that satisfy certain conditions.

For example, a sphere S_r, O is a surface with the property that each point on the surface is a fixed distance r from a point O called the center of the sphere.

Definition 7. The sphere S_r,O of radius r and center O is the set of all points Q such that this distance from O to Q is r.

In a rectangular coordinate system for space, this definition implies that if O is taken to be the origin $\mathcal{O}$, having coordinates (0, 0, 0), and if an arbitrary point P on the surface of S_r, O having coordinates (x, y, z) is considered, the distance $\mathcal{O}$P is constant. (See Figure 9.34.)



Using the distance formula derived in Section 9.14, we obtain

$$\text{the radius } r = OP = \sqrt{(x - 0)^2 + (y - 6)^2 + (2 - 0)^2}$$
$$= \sqrt{x^2 + y^2 + z^2}$$

Consequently, $S_{r,O} = \{P(x, y, z) : x^2 + y^2 + z^2 = r^2\}$

The <u>interior</u> of $S_{r,O} = \{P(x, y, z) : x^2 + y^2 + z^2 < r^2\}$

A <u>solid sphere</u> (or ball) is the union of a sphere and its interior.

$$\text{solid sphere } S_{r,O} = \{P(x, y, z) : x^2 + y^2 + z^2 \leq r^2\}$$

<u>Example</u> 1. In set notation, describe the set of points on the surface of a sphere with center at the origin and with radius 5.

Since $r = 5$, $r^2 = 25$, and

$$S_{5,O} = \{P(x, y, z) : x^2 + y^2 + z^2 = 25\}.$$

To most of us, the word "cylinder" brings to mind tin cans, drinking glasses, or pipes. The cylinder is one of the easiest surfaces to visualize. The kind of cylinder which is most common to your experience is the <u>right circular cylinder</u>. If you take a circle in plane π and a line through a point on this circle and perpendicular to π, you generate a right circular cylinder by moving the line parallel to itself, tracing the circle. (See Figure 9.35.)

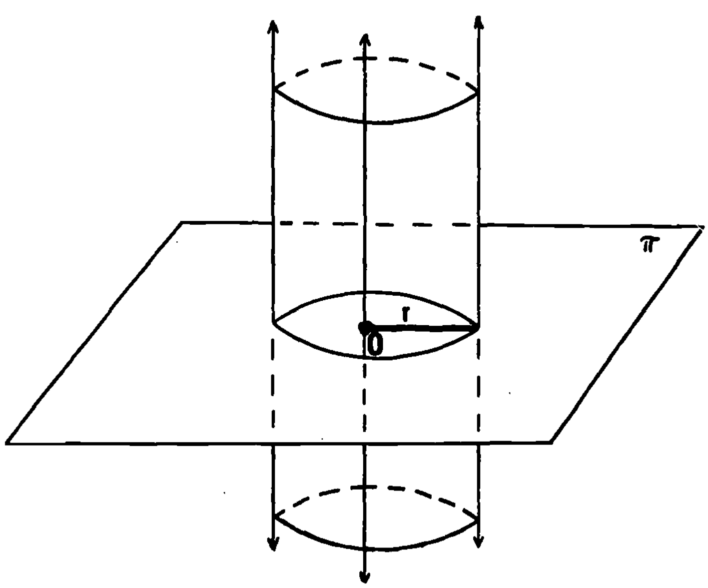


Figure 9.35

The generating line is called a <u>generatrix</u>, and the guiding figure in the plane, a circle in this case, is called a <u>directrix</u>. Instead of considering the generatrix to be a moving line, sweeping out the surface, you could alternatively consider the cylinder to consist of an infinite set of parallel lines. Each of these lines is called an element of the right circular cylinder; the line containing the center of the circle and parallel to the elements is called the axis of the cylinder; and the radius of the directrix circle is called the radius of the cylinder.

Figure 9.36 illustrates a right circular cylinder in coordinate 3-space. In this case, the z-axis is the axis of the cylinder. Let the radius be r.

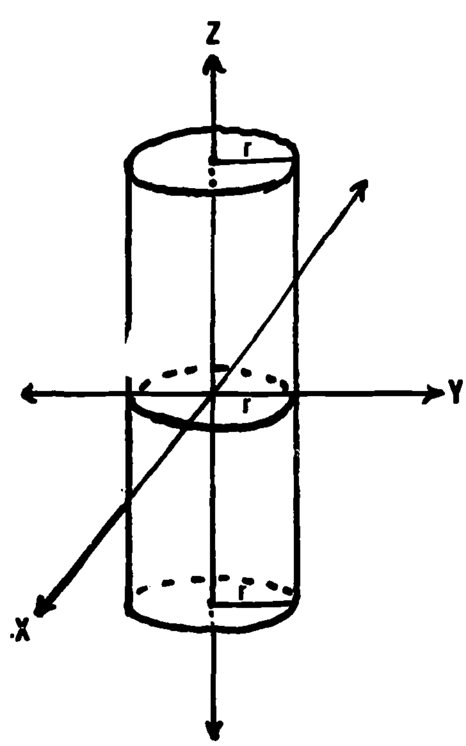


Figure 9.36

We can think of the cylindrical surface as consisting of an infinite stack (set) of circles, all of radius r. This suggests that a good description of the right circular cylinder is $\{P(x, y, z) : x^2 + y^2 = r^2\}$.

Example 2. Describe the set of points on a right circular cylindrical surface which has the y-axis as the axis of the cylinder, and which has a radius of 3 units. The general description of a cylinder of this type is $\{P(x, y, z) : x^2 + z^2 = r^2\}$. Therefore, our cylinder can be described by $\{P(x, y, z) : x^2 + z^2 = 9\}$. Sketch this figure.

Example 3. Describe the intersection of the cylinder of Example 2 with the plane $\{P(x, y, z) : y = 5\}$. The intersection is simply

$\{P(x, y, z) : x^2 + z^2 = 9 \text{ and } y = 5\}$, or

$\{P(x, 5, z) : x^2 + z^2 = 9.\}$

The word "cone" usually brings to mind ice cream cones and Indian wigwams. The cones of our common experience are called right circular cones, or sometimes cones of revolution. We can think of these as being generated by a line (the generatrix) which traces a circle C (the directrix) in a plane π. When generating a right circular cone, the generatrix, in its sweep or tracing of the directrix, passes through a point P in space, not in π. (See Figure 9.37.)

Figure 9.38 shows a portion of a right circular cone in a rectangular coordinate system. For simplicity we chose a cone with vertex at the origin, with axis of the cone the z-axis, and with the property that the plane $z = r$ intersects the cone in a circle of radius $|r|$.

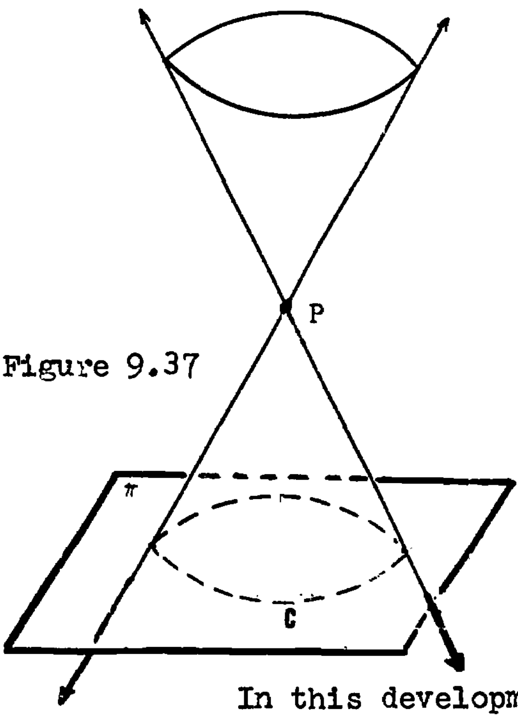


Figure 9.37

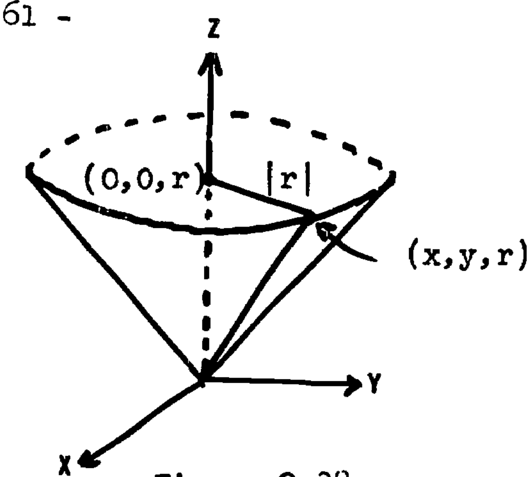


Figure 9.38

In this development it is convenient to consider a right circular cone as shown in Figure 9.38 to consist of an infinite "stack" of circles, all with planes parallel to the xy-coordinate plane, and with varying radii. You will note that for any one of these circles, the radius will be equal to the distance of the plane of that circle from the xy-plane. This suggests that a set description for this right circular is

$$\{P(x, y, z) : x^2 + y^2 = z^2\} \quad .$$

9.17 Exercises

1. Imagine a sphere that is cut (intersected) by a plane. What sort of geometric figure is the set of points in the intersection?

2. Imagine a family of parallel planes intersecting a sphere. Describe the relationship of the figures formed in the intersections.

3. If P and Q are two points on a sphere, imagine the family of planes that contain P and Q. Compare the intersections

4. Sketch the figure formed when a right circular cone is intersected by a plane which does not contain the vertex of the cone but is:
   (a) perpendicular to the axis of the cone.
   (b) parallel to the axis of the cone.
   (c) neither perpendicular nor parallel to the axis of the cone.
5. Sketch the figure formed when a plane intersects a right circular cylinder in case the plane is:
   (a) parallel to an element of the cylinder
   (b) perpendicular to an element of the cylinder
   (c) neither parallel nor perpendicular to any element of the cylinder
6. Describe the cone generated if the directrix is a line.
7. Describe the cylinder generated if the directrix is a line.
8. For each of the following surfaces, describe the figures and make a sketch for each showing the surfaces in coordinate 3-space.
   (a) $\{P(x, y, z) : x^2 + y^2 + z^2 = 1\}$.
   (b) $\{P(x, y, z) : x^2 + y^2 + z^2 = 2\}$.
   (c) $\{P(x, y, z) : x^2 + y^2 = 4\}$.
   (d) $\{P(x, y, z) : y^2 + z^2 = 1\}$.
   (e) $\{P(x, y, z) : x^2 + z^2 = 9\}$.
   (f) $\{P(x, y, z) : x^2 + z^2 = y^2\}$.
   (g) $\{P(x, y, z) : y^2 + z^2 = x^2\}$.
9. Given the sphere described by $\{P(x, y, z) : x^2 + y^2 + z^2 = 4\}$. Describe in detail and within set notation, the intersections

of the given sphere with each of the following. Draw a sketch for each.

(a) $\{P(x, y, z) : x = 1 \text{ and } y = 1\}$.

(b) $\{P(x, y, z) : z = 1\}$.

(c) $\{P(x, y, z) : x^2 + y^2 = 4\}$.

*(d) $\{P(x, y, z) : -1 \leq y \leq 1\}$.

9.18 Summary

The purpose of this chapter was to extend the study of geometry to three-dimensional space. This involved:

(a) The study of planes as subsets of space, and of the relations that exist among planes and lines in space.

(b) The use of deductive logic in obtaining further information about figures in space.

(c) Coordinatization of space.

(d) The study of planes and other common surfaces in space.

(e) Studying the processes for obtaining description of surfaces using set notation.

9.19 Review Exercises

1. Which of the following surfaces suggest a plane?

(a) The surface of a doughnut.

(b) The roof of Grant's Tomb

(c) The surface of the Atlantic Ocean

2. How many planes are there that contain

(a) a given point.

(b) two given points.

(c) two intersecting lines.

3. Determine which of the following statements are true and which are false.

(a) If $\pi_1 || \pi_2$ and $\pi_2 || \pi_3$, then $\pi_1 || \pi_3$.

(b) If $\pi_1 \perp \pi_2$ and $\pi_2 \perp \pi_3$, then $\pi_1 || \pi_3$.

(c) If $\ell \perp m$ and $n \perp m$, then $\ell || n$.

(d) If $\ell || \pi$ and $m || \pi$, then $\ell || m$.

(e) If $\pi_1 || \pi_2$ and $\pi_3 \cap \pi_1 = \ell$ then π_3 intersects π_2.

4. Find the coordinates of the midpoint of $\overline{AB}$ in case

(a) $A = (1, 2, 3)$, $B = (3, 2, 1)$

(b) $A = (17, 4, -3)$, $B = (-45, -32, -12)$

5. Find the distance from $(0, 0, 0)$ to

(a) $(3, 4, 5)$

(b) $(-5, -12, -13)$

6. Find the distance between each of the following pairs of points in space:

(a) $(5, 3, 2)$, $(2, 3, -2)$

(b) $(6, -1, -5)$, $(-6, 4, -5)$

(c) $(4, 3, 0)$, $(-2, 0, 6)$

7. Sketch the following planes in coordinate 3-space.

(a) $\{P(x, y, z) : z = 1\}$.

(b) $\{P(x, y, z) : x = -2\}$.

(c) $\{P(x, y, z) : y = 3\}$.

8. (a) Sketch the sphere described by $\{P(x, y, z) : x^2 + y^2 + z^2 = 9\}$.

(b) If the points on the sphere were translated according to the rule $(x, y, z) \longrightarrow (x - 1, y - 2, z - 3)$, would the resulting surface be a sphere? Why?

(c) Sketch the sphere resulting from translating the sphere in (a) according to the translation rule in (b).

(d) From what you have observed in the first three parts of this exercise, describe the set

$$\{P(x, y, z) : (x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 4\}.$$

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