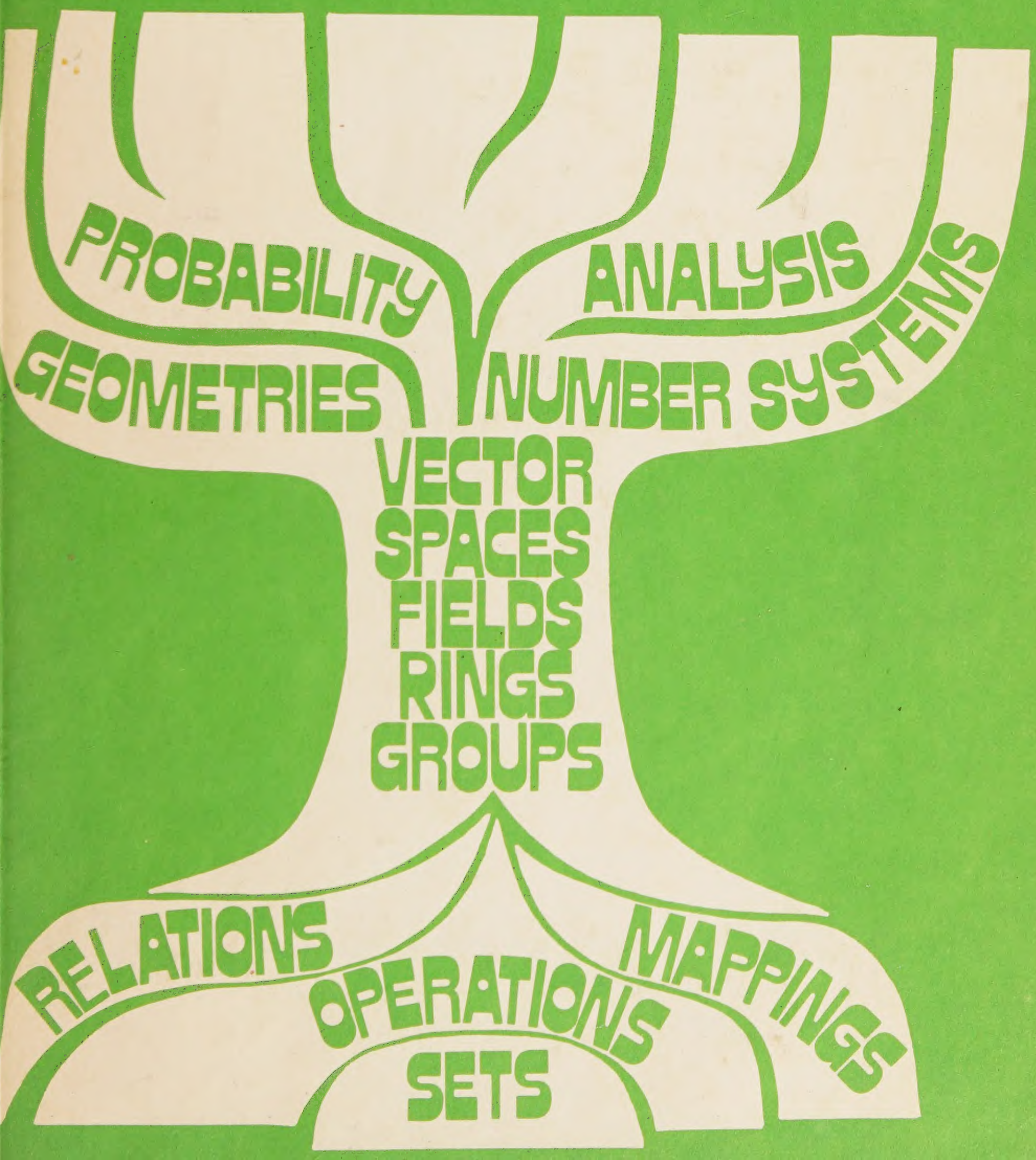
TEACHERS' COMMENTARY WITH ANSWERS AND TESTS

Fehr
Fey
Hill
Camp

Unified Mathematics

Course IV



TEACHERS' COMMENTARY
WITH ANSWERS AND TESTS

UNIFIED MATHEMATICS COURSE IV

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MEMORANDUM TO THE TEACHER

Publishers and authors are interested in learning how to improve their publications. Revision is a continual process, but meaningful feedback from teachers in the classroom is difficult to obtain. Therefore, we would like to ask for your assistance in helping us with advice and guidance in order that we may further refine this program.

As you teach from one of the Courses of <u>Unified Mathematics</u>, and at the end of the course, jot down any information or suggestions about areas that may have presented difficulty or that were especially appealing to the students. Many sound programs have been weakened in revision because some solid feature was innocently removed. Then, tear out this page and send your remarks to us. The following questions may help you focus on the kinds of general matters concerned.

1. How have you used the program?

2. What have been the strongest features?

3. Where has there been difficulty?

4. What would you like to see changed?

5. Any errata: errors such as spelling, mislabeled diagrams, etc.?

6. New aspects that might be introduced?

Thank you for your assistance, and we hope that your experience with this innovative program in <u>Unified Mathematics</u> will be successful.

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CONTENT OUTLINE BY STRANDS

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Fundamental Concepts				
Sets	Definition and Examples, Membership ($\in$), Roster Notation, Subset ($\subset$), Complement, Union ($\cup$), Intersection ($\cap$), Cartesian Product, Empty Set ($\emptyset$), Universe.	Cardinality ($n(A)$) Power Set ($\mathcal{P}(A)$), Application to Voting Coalitions.	Combinatorial Counting of Subsets.	Outcome space; sample space; events; subspaces; power sets ($\mathcal{P}(A)$); Cartesian product.
Binary Relations	Definition and Examples, Arrow Diagrams, Reflexive, Symmetric, Transitive Properties, Equivalence Relations.	Equivalence Relations and Partitions, Partial Orders, Order in Fields.	Applications in Metric Geometry, Trigonometry, Vector Spaces.	Equivalence of free vectors.
Mappings (Functions)	Definition and Examples, Arrow Diagrams, Domain, Co-Domain, Range, Image, Pre-Image, One-to-one, Onto,	Measure Functions, Permutations, Polynomials over Z, Affine Functions, Real Functions - Graphs,	Polynomials over Z, Q, R, degree operations ($+, -, \cdot, \div$) factoring, division algo-	Sequences as functions on Z^+; Trigonometric, sin, cos, tan, as functions of Reals into Reals; Addition formulae, periodicity;

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Mappings (Continued)	Identity, Composition, Inverse, Geometric Transformations.	operations(+, −, ·, ÷), properties of operations.	rithm, graphs, equations, ring structure. Rational Functions --operations (+, −, ·, ÷,). graphs, asymptotes, equations, field structure. Circular Functions.	symmetry, graphs; polynomial, quadratic, graphs; Probability functions, tree diagrams; Exponential, a^x, $a > 0$; e^x, graphs $R \to R^+$; Logarithmic; as inverse of exponential, $R^+ \to R$. Odd/Even. Spiral similarity.
Binary Operations	Definition and Examples, Properties - associative, identity, inverse, commutative, cancellation, distributive. Variety of Examples from W, Z_n, and geometry.	Applications in Number Systems, Composition of Functions, Groups, Fields, Polynomials.	Applications to Matrices, Polynomials, Rational Functions, Vectors.	With complex numbers; on sequences.
Logic	Statement, Variable, Open Sentence, Informal Proofs, Counterexample.	Connectives - and, or, if, iff, negation, Quantifiers - universal, existential. Truth Tables, Substitution,	Application in Algebra, Geometry, Probability.	Algorithms for computing; proof by mathematical induction.

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Logic (Continued)		Inference, Proof-Direct, Indirect.		
Equations	Variable, Domain, Open Sentence, Solution Set. In Z_n, W, Z, Q, the equations: $a + x = b$, $ax = b$, $ax + b = c$. Use of cancellation and inverse properties. Graphs of Solution Sets on Line and in Plane. Equations in two Variables. Absolute Value Conditions.	Equations in Groups and Fields. Linear Systems by Substitution and Application to Geometry with Coordinates, Quadratic Equations and Factoring.	Systems of Linear Equations, Solution by Substitution, Gauss Elimination, Matrix Inversion, Quadratic Equations, Factoring, Completing Square, Quadratic Formula.	Quadratic, real and non-real roots; Discriminant function, formula; Trigonometric, functional equations for sin x and cos x; solving equations by computer techniques; equations of vector lines and planes; parametric representation; Transformation to and from parametric form.
Inequations	Linear Inequations in W, Z, Q, Graphs, Using Absolute Value in Two Variables in Z.	Linear Inequalities in Ordered Fields, Quadratic Inequalities.	Systems of Linear Inequations, Optimization, Linear Programming, Graphical Solution.	Maximum and minimum of a quadratic function.

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Algebraic Structures				
Groups	Definition, Examples - In Number Systems, In Geometric Transformations.	Groups, Subgroups, Complexes, Generators, Permutations, Additive and Multilicative Notation, Powers, Multiples, Elementary Group Theorems, Solution of Equations, Isomorphism.	Applications in Geometric Transformations, Polynomials, Matrices.	Additive Group of complex numbers; free vectors under addition; bound vectors; translations.
Rings	Examples in Number Systems.		Examples - Matrices, Polynomials.	
Fields	Examples - $(Z_p,+,\cdot)$ and $(Q,+,\cdot)$.	Definition, Examples, Elementary Properties, Equations in Fields, Order in Fields, The Real Number Field.	Field of Rational Functions, Field of Scalars in Vector Space.	Field structure of complex numbers $(C,+,\cdot)$.

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Vector Spaces		Examples - Real Functions, Finite Geometries, Co-ordinate Plane.	Vectors in Plane-Free and Bound, Operations on Vectors-Addition and Scalar Multiplication, Application to Physical Problems, Vector Space Structure with Examples, Basis and Linear Dependence, Vector Equations.	Definition and Examples; Structure; Complex Numbers; Scalar Multiplication; Real Vector Spaces; Affine 2 and 3 Space; Euclidean Vector Space; Norm, Distance, Perpendicularity.
<u>Number Systems and Algebra</u>				
Finite Systems $(Z_n,+,\cdot)$	Clock Arithmetic $(Z_n,+),(Z_n,\cdot)$ $(Z_n,+,\cdot)$. Properties: Associative, Commutative, Identity, Inverse, Distributive, Operation Tables, Variable, Equations.	Applied as examples in study of groups and fields. Finite models of co-ordinate geometry.	Applied as examples in matrices and vector spaces.	

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Whole Numbers (W)	Review of arithmetic and properties of operations. Mappings of W to W. Number Theory - Factors, Divisibility, Division Algorithm, Primes, lcm, gcd, Fundamental Theorem of Arithmetic.			Induction, Sequences. Exponents.
Integers (Z)	Negative Integers, The Group (Z,+), Computation in (Z,+) and $(Z,\cdot)$, Ring properties, Equations, Absolute Value, Order in Z, Translation, Dilations, Lattice Points in a Plane.	Reviewed in Groups.		Functions on Z^+; Exponents.

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Rational Numbers (Q)	Fractions, Equivalence Classes of Fractions, Operations on Q, Field Properties, Equation $ax = b$, Decimal Representation, Computation, Ratio, Proportion, Percent, Plane Coordinates, Dilations. Order in Q	Reviewed in Fields. Equations in Fields-Linear Inequalities, Application to Measurement.	Use in Polynomial and Rational Functions.	Exponents.
Real Numbers (R)		Equation $x^n = a$, n^{th}, Roots in Q, Least Upper Bounds, Greatest Lower Bounds, Pythagorean Measurement Problem, Rational Approximation of Real Numbers, Field Properties of $(R,+,\cdot)$, Operations on Radicals, Real Functions.	Application to Line and Plane Coordinate System in Metric Geometry. Solving Polynomial Equations.	Functions of; as Exponents.

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Matrices (M_n)			Definition, Addition, Multiplication, Ring Properties, Transformations, Inverses, Application to Systems of Equations, Vector Space Structure.	Rotation Matrices, Composition of; Isomorphism of 2 x 2 matrices to Complex Numbers; as a Ring, as a Vector Space.
Complex Numbers (C)				Field Structure of Complex Numbers; $(C,+,\cdot)$; Conjugates; Amplitudes; Trigonometric Form; Polar Form; DeMoivre's Theorem; Euler's Formula; $a + bi$; (r,θ); $r(\cos\theta + i\sin\theta)$; Spiral Similarity; Complex Plane.

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Geometry				
Affine (Synthetic)		Incidence and Parallel Axioms, Elementary Theorems, Finite Models, Parallel Projection, Affine Coordinates, Formal Proof.	Lines, Planes in 3-space, Incidence and Parallel Relations.	
Affine (Coordinate)	Lattice Points in a Plane, Linear Conditions, Affine Coordinate Systems.	Line Coordinate System, Plane Coordinate System, Order on a Line, Segments, Rays, Halfplanes, Coordinate Equations for Lines, Ratio of Division on a Line, Slope, Intercept, Parallel Lines, Elementary Affine Theorems.	Plane Vectors, Affine Coordinates in 3-space.	Free Vectors, Bound Vectors, Affine Group of Translations, Affine Transformations.

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Euclidean	Informal Study of Isometries - Reflection in Line and Point, Rotation, Half-turn, Translation, Symmetry, Dilations.	Coordinate Rules for Transformations.	Perpendicularity, Congruence via Transformations-Triangle Congruence, Parallel Lines and Transversals, Angles in a Triangle, Congruence of Polygons and Applications. Similarity via Transformations, Ratio of Similitude, Pythagorean Theorem and Distance Formula, Constructions, Informal Geometry of 3-space, Space Figures - cone cylinder, sphere.	Transformations, Trigonometric Functions, As an inner-product space; Norm, Distance, Perpendicularity.
Vector Spaces			Directed Line Segments, Free Vectors, Equivalence Classes of Vectors, Vector Equation of Line, Applications.	Examples of Vector Spaces; ordered n-tuples: Lines and Planes.

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Measurement	Informal Use of Linear and Angle Measurement With Review	Properties of Measure Function-Unit, Congruence, Additive, Linear, Area, Volume, Angle Measurement. Formulas for Common Figures. Circle and pi. Line Coordinate Systems and Distance.	Distance Function	Coordinatized line, plane, 3-space.
Trigonometry			Sensed Angles, Radian and Degree Measure, Circular Functions--Sine, Cosine, Addition of Sensed Angles, Relations, Law of Cosines, Law of Sines, Applications to Solving Triangles, Vector Decomposition.	

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
<u>Probability and Statistics</u>				
Probability	Frequency Experiments, Stability, Properties of Relative Frequency, Experiment, Trial, Outcome, Event, Counting with Trees, Applications.		Review of Frequency Ideas, Probability Space, Elementary Theorems, Combinatorial Counting, Permutations, Applications.	Review of permutations, Combinations, Factorials; Binomial Theorem. Reduced outcome space; Conditional Probability $(P(A\|B))$; Partitioning of Sample Space, Independent Events.
Descriptive Statistics	Tables of Numerical Data.	Nature of Statistics, Numerical Data, Frequency Tables, Histograms, Frequency Polygons, Mean, Median, Mode, Range, Variance, Standard Deviation, Σ notation.		
<u>Applications</u>				
General	Finite Arithmetic and Coding, Word Problems	Measurement of Segments, Areas, Volumes, Angles,	Coding, Transition, and Transformation Matrices	

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
General (Continued)	and Equations, Ratio, Percent, Coordinate Systems for Location Symmetry, Probability.	Voting Coalition, Analysis Presentation of Statistical Data, Sales Tax, Cost-Revenue-Profit Problems, Latin Squares and Experimental Design.	Systems of Linear Equations, Linear Programming and Optimization, Word Problems and Quadratic Equations, Trigonometry, Probability, Vectors and Force, Velocity, Displacement Problems.	
Models				Properties of Models; Translation into mathematical language, Monte Carlo methods; Circular Functions as Models; simple harmonic motion; A-C electricity; Tides, Radar; Circular Orbits; Organic Growth and Decay; Linear Systems as Models in Economics; Bayes' Theorem; Sampling; Compound Interest; Statistics; Kinematics

SUBJECT	COURSE I	COURSE II	COURSE III	COURSE IV
Computers				Numerical Calculation with Logarithms; Slide Rule; Computer; Algorithmic Analysis; Iteration; Branching; Truncation; Errors and Accuracy; BASIC.
Analysis				
Limits				Of simple Sequences; Intuitive Acceptance of Properties.

TEACHERS' COMMENTARY

UNIFIED MATHEMATICS

COURSE IV

This is the fourth course in a six-year series on secondary school mathematics designed for good college preparatory students. This new series is justified by a number of events that have occurred in the past decade. New subject matter has entered into the secondary school program, new information has become available on how learning is motivated, and a new organization of mathematical knowledge has taken place. Experimentation and research, in the U.S.A. and Europe attest to improvement in mathematical education by virtue of these new developments.

This series of books takes into consideration all the above factors, relying heavily on the work of the Secondary School Mathematics Curriculum Improvement Study to which the authors acknowledge their indebtedness.

REASONS FOR THIS SERIES

During the past twelve years, many countries have been engaged in revising their mathematics curricula. In some cases it has been an updating of the existing traditional program, as in the United States of America. Throughout this period of time we have maintained the division of school mathematics into separate years (or half-years) of the study of arithmetic, algebra, geometry, and trigonometry. We are the only country left in the world that maintains this separation of the branches. Beyond token introduction of new ideas, some rearrangement of the sequence of topics, and (for a few students) acceleration of study so as to permit a year of instruction in calculus, little has been accomplished in the way of bringing more advanced or newer study into the high school. For the latter prupose, it is necessary to make a more efficient organization of the subject matter, one more in harmony with the contemporary conception of the nature of mathematical study.

Reports of the conferences and seminars held recently in Europe and the United States have brought to light ways in which mathematics can be unified for school study, mainly through the use of common concepts at the base of all the traditional branches. Principal among these ideas are sets, relations, mappings, and operations, and the basic structures such as groups, rings, fields, and vector spaces. Extensive experimentation throughout the world has shown the feasibility of teaching a unified program. This series of textbooks is based on the knowledge gained in all these studies and in extensive experimentation during the last ten years.

SUBJECT MATTER CONSIDERATIONS

That formal mathematics at the university and graduate level of study could be organized in terms of its fundamental concepts and basic structures became apparent through the foundational studies at the turn of the twentieth century and the subsequent Bourbaki analysis begun in the 1930's. What was not known, as late as 1960, was how such a unified organization could be used to restructure secondary school mathematics into a form suitable for presentation to students. The reports and seminars referred to above produced guidelines for developing experimental programs. These experiments were carried out in the Nordic countries, England, Germany, and more recently in Belgium, France, and the United States. There has evolved a trend in curriculum development that genuinely reflects the contemporary point of view of mathematical education. It permits the introduction into high-school mathematics of much that has hitherto been considered collegiate instruction.

DESIGN OF THE CURRICULUM

All mathematics can be conceived of as the study of sets, upon which there is imposed a set of axioms and definitions, called the structure. With any set and its structure, many relations, theorems, and applications or activities can be derived. The sets in elementary mathematics are usually those whose elements are called number or points. But the sets could be those of functions, economic phenomena, words of a language, or anything. In our study we shall be mainly concerned with a few structures, their realizations and many useful types of activities that are derived from them.

Since in Course IV, the major part of the program is built on knowledge gained in the previous courses, it may be well to review the structures of those three courses. Course I was primarily an intuitive approach to the fundamental ideas of set, relation, operation, and mapping (function). The number systems--whole numbers, integers, and rationals--were used to illustrate the structures of group, ring, and field, which were further developed in Course II, where the real numbers became a realization of a completely ordered field. Finite number systems (clock arithmetic) were introduced as examples and counter-examples of the structures. In Course III, a brand new structure was introduced--that of matrices. They could be used to exemplify the group, a two-fold operational system of a ring, and--for particular subsets--a field. Here all the activities of composition of functions and solution of equations occur in a new setting. Matrices are immediately put to work in the solution of systems of equations. They shall be in continued use throughout all subsequent study.

In Course I we studied figures in a plane and transformations of the plane. In Course II metric geometry and perpendicularity were extended via transformation to a study of congruence in the plane. Dilations permitted the study of similarity and led to the proof and use of the Pythagorean theorem. In Course II an initial chapter on logic and proof prepared the way for an introduction to axiomatic geometry via the affine plane. After this the study proceeded with a less than axiomatic introduction to both synthetic and coordinate solid geometry of both affine and Euclidean space. An introduction was given to the theory of measure--length, area, and volume. All the foregoing ideas built a foundation of the final chapter of Course III--an introduction to vector spaces. The study of Course III also returned to that of polynomials treated from a modern viewpoint. Again two-fold operational systems, rings, and fields, gave a structure to the study. Here we found the topics of factoring, quadratic equations and graphs, all in a contemporary setting. The study was extended to rational functions and a new 'field' representation resulted.

Probability was introduced in Course I by carrying out experiments and finding relative frequencies of occurrences of events. In Course II the collection of data was used to develop the measures of central tendencies and dispersions as simple descriptive statistics. In Course III, probability was extended into a more formal, set-theoretic presentation and related to combinatorics. This treatment united sets, operations, and algebraic theory in a brand new way of thinking. Finally, the study was extended to a new set of functions--the circular functions--in which all the previously studied geometry, arithmetic, and algebra were put to work. The law of sines and law of cosines were developed and applied to triangle solution and the study ended with the wrapping function which assigns to every real number a sine and a cosine.

In Course I, the language was precise and correct and the learning informal. In Course II, the language remained correct and precise, but the learning took on a more formal aspect and dealt with more abstract notions. In Course III the study became increasingly more mathematical and more structural all of which demanded increasing mental concentration. This is the way all science learning proceeds. In Course IV the study becomes more unified and more applicable.

Course IV is essentially a study of the elementary real functions--polynomials (especially the quadratic), sequences and series, exponential and logarithmic, trigonometric (circular) and probability, and transformations in vector spaces. The making of mathematical models and the applications of these functions is a worthwhile achievement in itself, but these functions also form a strong basis for the study of calculus in the following two courses.

The course begins with the study of BASIC and its use in programming numerical problems for the electronic computer. It is used in subsequent study at every possible opportunity. The aim is not to produce programmers, but to enable the students to understand a computer language and how the computer interprets this language, as well as to appreciate the amount and speed with which otherwise large scale tedious computations can be accomplished in seconds. With the programming comes a deeper insight into the algorithms employed and why they work.

Essentially this is a course on developing and using the function concept. All the previous study on mappings and operations on functions becomes organized into a single concept of a function unifying the separate ideas of domain, range, and a rule of assignment into a single entity. Thus the polynomial function, the exponential function, the trigonometric functions each become a unique concept envisaged by

$$f: R_1 \xrightarrow{m} R_2$$

Sequences are functions on the positive integers, polynomials are functions of real numbers, exponents are functions of real numbers on the positive real numbers, logarithms are functions of R^+ onto R, the sine function is a mapping of all real numbers onto the interval $-1 \leq 0 \leq +1$. Probability is a function of events onto real numbers between 0 and 1, and so on.

While the development of the mathematical theory is foremost, it is also important that its applicability to problems in science, business, technology and so on, become evident. Thus the course makes great use of mathematical modelling. At all times the content is related to all the previous study in Courses I, II and III, giving strength to the unity of the subject.

PSYCHOLOGICAL CONSIDERATIONS

During the last hundred years many psychological theories on learning have been hypothesized and tested. On how the mind comes to gather its information and how it manipulates this information to solve problems, there is very little agreement and much disagreement among the theories. Today there exists no single theory of learning on which to base a complete curriculum and its presentation. Yet in teaching we must adopt some theory on which to present our subject for its acquisition by the mind of the student. It would seem that an eclectic theory, gathering as best we can, from all the theories will give a promising and leeway solution for the classroom teacher.

The students who will enter Course IV at the average chronological age of 15 years, have reached a mature level of intellectual activity and can be challenged with abstract reasoning. Thus the mathematical theory can take a self-contained viewpoint in which the elements are numbers of a specific category, or points of a specified space upon which functions and structures are built. Certainly these elements can be seen in practical situations of a physical, biological, economical, or other nature, but the distinction between pure theory and its application to describe these other phenomena must take on the atmosphere of an approximate isomorphism.

Behaviorism and neobehaviorism, with their ideas of seriated associations, feedbacks, and reinforcements, begin to recede as a learning theory when we enter the higher forms of mental activity required in understanding mathematics. The learning of abstract concepts, the manipulation of ideas, and the ideation of logical conclusions from mental activity, are indeed difficult to explain. The learning of meaning, understanding, and interrelationship of concepts, all done internally within the brain with no recourse to anything but thought, demands newer approaches to knowledge than that which is sensory and intuitive. This mental manipulation and interrelating of ideas, using guided learning (frequently called discovery), self-inquiry, an internal search for generalizations, and recognition of new mental structures, all demand dynamic intellectual activity, without which it is not possible to learn the subject.

In this book, in increasing amount, it is this dynamic type of learning that takes precedence over the usual telling, passive acceptance, memorizing, and reciting--so frequently allied with mathematics study in the past. Of course, description and memory cannot be ignored. Developmental research, discovery, provocative questions, and mental experiments are presented to the student for his consideration. This will enable the student to make precise and clear that which at first is ambiguous and cloudy, and to formulate concepts and properties from which conclusions can be reached and then necessary skills developed for future use.

A student is in a problematic situation when he is seeking an explanation or solution to the situation which is not immediately apparent. He has knowledge (information) and knows many procedures and is in the process of reorganizing all he knows to find the answer. We say he is seeking a gestalt or complete picture, which will clarify the path to the discovery of what is unknown. When this is found, he has added new knowledge to his past accomplishments. It is this sort of approach to the solution of problems that demands dynamic intellectual action, which is indeed learning. Practically all the learning in Course IV demands this type of approach, and the teacher should encourage it at all times.

MOTIVATION

Motivation is always internal. It can be aroused by exterior physical situations or internal questioning, and when it exists there is a genuine interest and desire on the part of the student to inquire into the matter under consideration or investigation. For the college-intending student, a felt or imposed need to prepare himself for college acceptance usually provides sufficient desire to master the subject. However, the natural curiosity of 15-year-olds to understand, to explain, and to acquire new knowledge, especially that which can be used in other disciplines, or to explain the physical events in the world about them, is a strong motivational force. Interest is also provoked by using games, puzzling situations, and competitive challenges that demand the use of logical finesse.

But at this age, the more adult actions of approval, mastery, success, achievement, and individual independence are the main spurs to motivation. Nothing succeeds better than success. So the first approaches are with the simple idea of the subject. As the study continues, the notions become more abstract and more complex and the going becomes harder and more challenging. Here is where the truth must be told--that learning mathematics is hard mental work; it is not easy. In fact all worthwhile intellectual formation requires deep and concentrated mental activity, and the accomplishments are as satisfying and rewarding as those in games of sport, or in musical or artistic performances. In fact, it is the continuing growth of knowledge, in any area, that brings success and satisfaction to the individual. So all chapters contain a number of highly challenging problems the solution to which will give the student a real sense of satisfaction.

PEDAGOGICAL CONSIDERATIONS

On the basis of the foregoing philosophical, curricular, and psychological considerations, certain pedagogical features have been built into the textbook. Some of these are listed here.

The Student's Textbook

The textbook is written for the student--not the teacher. The language and symbolism is precise and correct, but esoteric words and symbols are avoided. The development material is generally more lengthy than that in the usual textbook, but it is designed to aid the student in his study. A few sentences, now and then, require several readings for full comprehension, but this is true of all good mathematical exposition and becomes more essential the higher one proceeds in mathematical study. For the most part, the student will find that the book is readable, one that he can study and interpret. The exercises are routine only when practice for skill is essential. More generally the exercises are thought-provoking and concept-building ones, and they form an integral part of the program of mathematical study. For these reasons, the reading and study of the explanatory sections and the working of some problems is a preclass preparation as well as a postclass verification. This procedure permits, within the class period, a dynamic dialogue between students, and between the class and the teacher, wherein the clarification of uncertain and ambiguous ideas, as well as extensions beyond the textbook development, may take place. Students must learn how to read mathematics.

Learning as a Cognitive Process

As previously remarked, all learning of new mathematical ideas is acquired through problem-solving. By placing the student in a not-too-discomfiting condition of confusion, we activate his thinking--he is no longer a passive listener or dreamer. In an attempt to resolve the confusion, all of us frequently make wrong decisions and unintentional mistakes, which cause us to arrive at blocks to a solution or even false answers. But we learn by our mistakes. In these situations, we soon realize that we must try a new approach to the problem. Of course, to avoid frustration, utter failure, or succumbing to defeat, guiding questions or signposts must be presented to the student from time to time, to place his mental activity on a promising correct path. It is from such activity, with the goal satisfactorily attained, that a concept is formed, a mental construct is built, and the key or solution to new knowledge attained. The final outcome is all the more precise because it is differentiated from that which it is not! Call it an experimental, discovery, gestalt approach to learning--or what you will--it is decidedly superior to mere behavioral associative learning. The type of thinking we urge should be a major goal of all education, not only mathematical, so that for the rest of their lives students can go on learning by their own efforts. It is learning how to learn.

Developing Concepts and Skills

A concept which a student has acquired, is shown by his mental reaction in situations where the concept is present. Important concepts should become permanent knowledge by retention (memory). Their application in specific skills must be perfected through practice. Retention, in Course IV, is provided for in two ways. First, by a spiral approach to learning, that is, returning periodically to each fundamental idea and pursuing it at a deeper level of understanding, necessitating the recall and application of all previously related notions. Secondly, by providing at regular intervals an extensive and intensive set of problems and exercises. These exercises are both re-view and new-view in nature and involve the use of all past and recently studied topics. At the end of each chapter, there is a summary of all the important concepts and specific skills presented in it. Skill is also retained by using the computational and manipulative procedures in the presentation of new topics.

The student may use the summaries to check the extent of his learning by attempting to give satisfactory explanations and/or illustrations of each item as he reads it. Each summary is followed by an evaluative review--a set of problems--by means of which the student can further check on his ability to recall, to apply, and to develop further the content of the chapter. These summaries and reviews serve also as examples of desired outcomes from which the teacher can frame his own test.

Problem-solving, concept formation, developing skills, retention practice and evaluation are prime pedagogical elements built into the program of Course IV.

The Teacher's Role

Every teacher, sooner or later, develops a characteristic manner of conducting the classes under his instruction. At all times the teacher is master of his own class and this is the way it should be. For effective instruction, good teachers adjust the use of the textbook to their own style of teaching, but they also seek new ways of presenting special projects, motivating their students, providing for individual differences in learning ability, and creating genuine interest in learning. The following Teachers' Commentary--chapter by chapter and section by section--has been written for the latter purposes, namely to help those teachers who are searching for new presentations and possibly better ways to carry on their profession. In particular, the commentaries aid the teacher in:

1. Helping students to learn. As students study assigned developmental material, work through assigned experimental tasks, and attempt the solution of problems, they may find the going difficult. The teacher must exercise a great deal of patience so as not to tell the answer or the key to the correct outcome. Students must learn that persistence, mental effort, and patience--all desirable human attributes--are needed to learn mathematics. When students are stumped, another approach, an alteration of the situation, or a simpler problem may help to clarify their thinking. The Commentary supplies some of these approaches.

2. Motivating students. The students who ask "What is the use of this mathematics?" are mostly two kinds, the one showing a genuine interest in applied mathematics, the other showing difficulty in learning the subject. For the first type of student, the teacher should collect a set of uses of each topic in the curriculum. The Teachers' Commentary will supply some uses. For the second type of student we must recognize that trouble and frustration frequently destroy motivation. Here the teacher must be prepared to give the student individual and special help until the pupil has overcome his difficulty. Once the student sees "how" and can "do," he seldom raises the covering-up excuse of "What is the use?" A more concrete illustration, one in which the student has perhaps some experience, may help clarify an otherwise straightforward but hard mathematical development. Both sympathy and empathy help to get a student out of apathy and back to intellectual activity. The Commentary supplies motivation activities other than those in the textbook itself.

3. Meeting individual differences. Even a class of all bright students eventually reflects individual differences in interest and in mathematical ability. There are simple and complex approaches to learning mathematics, as well as sensory (concrete) and abstract, shallow and deep, intuitive, plausible, and formal axiomatic approaches. Depending on the ability of the student, all approaches appear to be hard at some time or other. Learning usually proceeds from a motor-sensory, intuitive, informal, logical manner at adult level. There is a "surface-to-depth" understanding in the development of almost all mathematical ideas; for example, set, number, algebra,

point, geometry, space, operations, function, and so on, are both relatively simple and yet mathematically very deep. There should be examples and problems for the student of lesser ability, who can only wade or float, as well as for the gifted expert who is a deep-sea diver in mathematical thought. The Commentary indicates the type of problems and questions that can be assigned to the average, good, and highly capable students so as to maintain a balance of interest in and time spent on the study.

4. Using the textbook judiciously. Many mathematics teachers fail to give reading and study assignments in the textbook, using it only as a supply of exercises and problems for class and homework. This, in a sense, denies the student the acquisition of one of the major benefits of school education, namely, learning how to get information from the printed page--to learn how to learn. To read exposition, to raise questions about the exposition, to know what he understands and what needs further clarification, and then to carry on reasoned discourse or dialogue with one who knows, are strong means to intellectual growth. The Commentary shows teachers how to make expert use of the textbook.

5. Evaluation. At all times the teacher and the student are concerned as to the amount and quality of learning that is taking place. In school learning, the evaluation of primary importance to the learner is "Do I know it?" or "Can I do it?" He is concerned with his own advancement. For the teacher, "How well is my class meeting the goals and standards I have set?" or "What must be retaught?" All tests that are administered should be looked upon as "opportunities" to discover how well the education program is succeeding; that is--what is good achievement, what is low achievement and how do we improve the achievement? Evaluation of this kind is provided in the problems and exercices, where students can quicly discover what they know and do not know. The summaries, if properly used, form further evidence of the learning that has been acquired. At the end of each chapter there is a "review" which can serve as a chapter test or evaluation for students, before they take a teacher-designed test for the chapter. In Part 3 of the Commentary, sample test items are given as a suggestion to the teacher of what the authors feel is a comprehensive check on the goals to be achieved. These can also serve as a guide to teachers to make their own tests.

6. Goals of mathematical education. Many goals or targets that are to be attained in mathematics instruction have been stated over the past 75 years. They are both general and specific in nature and run from a few to several hundred stated objectives. When teaching a specific topic, it is fairly easy to state the particular kind of mental

operations that one can expect as outcomes of the study. This is done in Teachers' Commentaries. However, at all times the teacher should keep in view the general objectives of the study of secondary-school mathematics. We state only three fundamental objectives that our program attempts to meet. They are: (1) Intellectual formation, (2) Knowledge or information, (3) Application. All of these are culture aims of present-day society.

By intellectual formation is meant the development of mathematical thought. The nature of mathematics, its fundamental concepts and structures, its use of logic in methods of demonstration, its mode of thinking about numerical, spatial, and abstract entities is a unique mental function that all persons should come to know. By information is meant that mathematical knowledge (concepts and skills) developed in mathematical study, which tradition has created and present-day and future society will need to know for everyday professional and cultural activities. By application we mean the way mathematical thought and knowledge are applied in the other disciplines--the use of mathematics for explanation.

In general, the Commentary gives help and clarification on some or all of the following categories of teaching:

Purposes. The goals or targets that students should aim to attain by the study of a topic, section, or a chapter. The possible length of study time required for each section and each chapter.

Materials. A list of media, other than those found in regular classrooms, which can be of value in developing the topic under study. The use of multi-media in teaching, such as overhead projector, physical apparatus, colored crayons, etc.

Getting started. Ways, other than those in the textbook, are suggested for opening the class lesson, so as to gain immediate attention, interest, enthusiasm, and total mental involvement. Other ways of presenting a new topic are described.

Using the textbook. What to assign as reasonable out-of-class study and problem work, ways to use the book in the classroom to teach "how to read" and "how to study" mathematics and the use of the textbook as a reference volume are given. Indeed, in most cases, the textbook tends to become the curriculum and the authority on the mathematics to be learned. How to challenge a textbook is a good learning device.

Other activities. Other games to play or other situations to pose than those given in the textbook for developing mathematical ideas are presented. Other approaches in the development of the main content of each chapter and other applications than those in the exercises in the book also appear.

Possible assignments. Some teachers like to have all students do all the exercises and problems in the book. Of course, it is not necessary to do this, nor is there sufficient time for this in a school year while at the same time being able to cover -- well -- all the material in the textbook. Important exercises, as a basis for the subsequent development of new topics, must be assigned to all students. They are listed in the Teachers' Commentary. Also, reasonable assignments for outside-of-class working time are suggested for average, good, and excellent students.

ENTERING THE PROGRAM AT COURSE IV

All students who have completed Courses I, II and III, or the equivalent, are ready to pursue the work of Course IV. However, students who have followed a regular junior high school program which culminates with a year of algebra would have great difficulty with the program. For these pupils it would be advisable to pursue special topics from the Unified Mathematics Courses II and III, as a one year preparation for the Course IV. These students could enter Course IV in the eleventh school year and then continue into Course V in the twelfth year. This will give adequate preparation to enter a first year university course in analysis.

It is essential in future planning that good students enter the study of Unified Mathematics at the beginning of grade seven. For other students intending to continue university study, the SSMCIS program can begin in grade 8 and continue through Course V in grade 12. Students who transfer from a traditional program in grades 7 and 8 may begin with Course II, with certain topics from Course I included, and then continue through Course V. In either of the two cases, students will be able to enter any university course in analysis without encountering any difficulties.

PROLOGUE

COMPUTER PROGRAMMING IN BASIC

High speed electronic computing machinery is already a well established tool in mathematical problem solving; availability in science, business, and industrial uses can only expand during the productive lifetime of current secondary school students. Thus the authors believe it is essential for mathematically educated students to understand the power, limitations, and "software level" of operation of computers. This alone is sufficient rationale for priority placement of computer programming instruction in this fourth course of Unified Mathematics.

However, computer capabilities influence curriculum and instruction in mathematics in more fundamental ways. When computational drudgery can be taken over by a machine, the range and complexity of problems within the grasp of learners expands greatly. Furthermore, the theoretical tools developed to handle specific types of problems can often be simplified if computational tedium is not a consideration. The algorithmic analysis involved in preparation of a problem for computer solution is a proven aide to learning and understanding of mathematical ideas and processes.

One goal of this Prologue is to teach students to write computer programs in the BASIC language. But this skill is viewed as only a preliminary step toward enlisting the power of computing equipment in mathematical problem solving. The BASIC language was chosen because it is easy to learn and uniquely suited to mathematical problem solving in a time-sharing interactive mode, which we believe is optimal for learning about computers and about mathematics. The illustrative examples throughout the text are chosen from concepts of preceding Unified Mathematics courses, though many will be known by students from more traditional backgrounds. Further computing activities appear in central roles in almost every other chapter of this course.

In most students' first exposure to computers there seem to be two types of barriers: First is learning the technical rules for working with your particular machine or time-sharing system. This includes signing on and off, using the teletype keyboard, and getting a feel for the pace at which things work (Some students get the impression that they must type in responses as quickly as the computer presents its output!). The second stage is performing the careful flow-chart and program analysis to solve a problem. In our experience it is a good idea to have students begin work with the computer by using a "canned" routine such as a game, quiz, or tutorial program with interaction. This stimulates interest by getting immediately in contact with the live machine, and it illustrates the procedural details that will have to accompany later programming efforts.

As you move into the programming activity, keep in mind the fact that this chapter is not a complete manual of BASIC. It would be a good idea to have several copies of such manuals available in the classroom for reference on fine points or unique features of your system's version. Furthermore, it is not essential to cover the entire chapter before moving into the body of Course IV. The first 5 sections provide the essential ideas for subsequent use in this course. Work with function plotting and matrix algebra can be studied when need arises at specific places.

Since debugging is an activity that will be required as soon as students begin writing and running their own programs, section 0.4 can be taken up at any time it seems appropriate. In fact, it can be used as a reference for students when they encounter difficulties--not formally taught at all.

SUGGESTED TIME SCHEDULE

0.0 Computing and Mathematics	1 day
0.1 Problem/Flow Diagram/Program	2 days
0.2 Input/Process/Output	2 days
0.3 Decisions and Loops 0.4 Debugging	4 days
0.5 Computation with Functions	2 days
0.6 Graphs of Functions	1 day
0.7 Dimension, Subscripts, Matrices	2 days
Summary and Review Exercises	1 day
TOTAL	15 days

0.0 COMPUTING AND MATHEMATICS

The purpose of this introductory section is to identify ways in which computers can assist mathematical problem solving and to describe the two tasks of using computers as an aide--problem analysis and programming in a suitable language. This same goal can be accomplished in a variety of ways--including class discussion sharing student knowledge or impressions about computers. It might be a good idea to get students immediately on line with a game, quiz, or tutorial program; let them read the text on their own and come to class the next day ready to begin tackling the task of writing their own programs to solve problems.

0.1 PROBLEM/FLOW DIAGRAM/PROGRAM

Purpose

This section should give an example of several fundamental processes in programming--algorithm analysis, computation and decision making, and the grammar of BASIC. The authors have consciously chosen a non-trivial problem, realizing that students won't have a thorough understanding of all the principles and techniques involved. We hope to whet their appetite with a moderately impressive demonstration of the computer's power to perform repeated applications of an algebraic procedure. Then the balance of the section is to introduce the special, but very mathematical, grammar of BASIC.

Getting Started

Recall with students the Newton method of approximating square roots and then pose the problem to write directions for someone who knew only the four basic arithmetic operations. Let them work on this for a time and pool their ideas for a rudimentary flow chart. Then show them how to convert their flow chart into an acceptable BASIC program explaining briefly each step of the program--without worrying about their memorization of instructions at this time. Then run the program on your computer or using your teletype. Pay particular attention to the somewhat unusual form of LET statements; they are not equations, but indicated substitutions.

Using the Text

Have students read over the BASIC grammar rules and flow chart symbols. Then do some of the exercises from 1 and 2 orally in class.

Other Activities

The modified square root program in Exercise 6 makes the convergence of approximations much clearer by printing out each approximation. Let various students try their own hand at the teletype with their own choice of numbers A and X.

Possible Assignments

Exercises 1-3 and 7-10 are basic and routine. All should be done. Assign the flow chart problems 11-15 to different students or teams of students.

0.2 INPUT/PROCESS/OUTPUT

Purpose

This section explains in detail the BASIC instructions used to read data to the computer and to print the results of computation. After a one day introduction here, students will probably want to refer back to this section and to more elaborate manuals as they refine their techniques for printing out results.

Getting Started

To emphasize the importance of suitable labelling of results and program components, show the students unlabelled printout from the area and perimeter program in the text and ask if anyone can guess what the numbers mean? Then show them the program; it might still be unclear what is being accomplished. Explain the use of REM and PRINT "   " statements. Then demonstrate on the teletype the full program with both INPUT and READ/DATA styles of data read in.

Using the Text

Mention and have students read the remarks about variables in BASIC. This text section is then probably best used as a reference as students try to write programs and find need of techniques for data input and output.

Other Activities

In keeping with the idea of having students spend as much time as possible writing and running their own programs, have them work on Exercises 2-11 as class exercises, running trial programs as soon as they are ready. Be sure to emphasize that there are many correct programs possible in each problem.

You might want to emphasize that a semi-colon placed between parts of a PRINT statement packs the output closer together. Also, if one wishes to abort a program run when INPUT has called for data with a question mark, simply type STOP (check your particular system).

Possible Assignments

Basic assignments should include 1, several from 2-11, and several from 12-20. It is not necessary to have each student do each problem.

0.3 DECISIONS AND LOOPS

Purpose

This section explains and illustrates the fundamental power of stored program computation--looping through a single set of computations many times until some decision criterion has been achieved. There are two basic ways to set up these loops--both are explained in this section.

Getting Started

To give students practice at the hand simulation method of debugging, have them read and work through several trial cases of the decision programs in Examples 1-3. Then run Example 3; the answer is very interesting.

Using the Text

It probably is better to avoid both types of loops in the first day of the section study. Hold the FOR/NEXT statements until the second day at least, selecting exercises (see below) that avoid it at first.

Other Activities

Take the programs of Examples 1-4 and have students write out flow charts diagramming the process embodied in each program. This will point out the looping involved and the routing effect of decisions.

Possible Assignments

On Day 1, assign 3, and several of 9-15. You might want to amplify the latter problems by asking for a flow chart too. Though the FOR/NEXT instruction pair is extremely powerful, it is probably a good idea for students to be able to write toops without it.

On the succeeding days introduce the FOR/NEXT technique and give students plenty of time to design, write, test and refine a variety of programs. You might want to blend instruction of Section 0.4 with this section. This section is the heart of the chapter and worth thorough study.

0.4 DEBUGGING AND ROUNDOFF

Purpose

It is a rare programmer who can write an error free program that correctly solves the problem at hand on the first try. This section suggests and illustrates basic techniques for coping with the puzzling state of affairs when a routine the programmer thought was correct doesn't seem to be performing. It also points out the limitation of machine accuracy by finite decimal representation.

Getting Started

It is entirely possible that students will have encountered the error messages, exponential notation, and need for debugging before arriving at this section. At such times, refer students to read this section. Then sometime during the study of the chapter take some time to pull together the points of the section--particularly the exponential notation and roundoff which will be considered again in the chapter on exponential functions.

Using the Text

As suggested, use this text section as a reference.

Possible Assignments

Exercises 1-20 are good illustrations of algebraic and arithmetic facts.

0.5 COMPUTATION WITH FUNCTIONS

Purpose

To introduce the very useful supply of mathematical functions available in BASIC and the technique for defining functions of one's choosing.

Getting Started

This section is easy going and students can probably read it and begin working on Exercises without much formal instruction.

Other Activities

Work through the programming activities of several exercises--perhaps 1 and 16.

Possible Assignments

The exercises of this section are grouped to accomplish review of mathematical ideas from other courses as well as to show how the computer can be used to explore conjectures. So that students actually do the programming and run their efforts, spend several days. Problems 1-9 on the first day; 10, 12, 13 on the next day; then several from 15-21 and 22-27 on the third day.

0.6 GRAPHS OF FUNCTIONS (OPTIONAL)

Purpose

To explain the rather crude BASIC method of plotting functions, but more important to demonstrate the strategy for planning a graph plot by domain and range estimates. Description and practice in rounding too.

The function plotting program skeleton will be used in succeeding chapters, but can be passed over here without serious prejudice to later efforts.

Getting Started

The key activities in using the graphing program are planning domain and range. Give the class a list of simple functions such as $y = |\frac{1}{3}x|$, $y = \frac{1}{4}x - 2$, $y = (x - 2)^2$, $y = -.1x$ and the direction to determine a sensible domain for graphing the functions and an estimate of the range for the chosen domains. Then pose the problem of deciding the value of each y-axis interval if there can be a maximum of 50 in the plot of the various functions.

When students have suggested reasonable parameters for the graphing activity, call up the skeleton function graphing program that you have stored in advance of the class. Plug in one of the sample functions (line 10 in text) and the DATA (lines 30 and 50 in text); then run the program. Vary the function and DATA parameters for another plot.

Using the Text

It is probably not important for students to digest or remember the total design of the graphing program skeleton. Interested students can read the text where one example is worked through in detail.

Other Activities

Have some students work on a program to print out a table of values for the above functions, using hand plotting to sketch graphs on coordinate paper.

Possible Assignments

The exercises are in two parts. The first 11 give practice in rounding off of decimals. Those in 25-32 give more practice in preparing the function for computer plotting--using estimation of domain and range--an important conceptual skill. Assign several from each part.

0.7 DIMENSION, SUBSCRIPTS, MATRICES

Purpose

To demonstrate the BASIC techniques for handling matrix data and operations and in the process to review linear algebra concepts from Course III.

Getting Started

Before setting students to work on programming with matrices, review some of the uses and language of matrices from Course III. In particular, the a_{ij} notation for the entry in row i and column j of matrix A and the definition of matrix multiplication and its use in translating a system of linear equations to a matrix equation.

Using the Text

Have students read the first example, program a similar example on their own, and read over the list of matrix instructions. Then have them try Exercise 1 in text.

Example 2 uses matrix notation in a slightly different way--double loops for averaging rows and columns--and can be the main topic of the next class meeting.

Other Activities

You will probably want to point out that singly subscripted variables can be created the same as matrices--using DIM A (100) for example to prepare space for a maximum of a_1, a_2, ..., a_{100}, though not all 100 need be used. This indexing of variables is often very useful in problems where numbers must be stored and called on later in the program--particularly in search of primes.

Possible Assignments

Day 1 could include 1-5 and some parts of 7. On Day 2 try 6, 9, 10, 12 with others for extra credit explorations.

CHAPTER 1

SEQUENCES AND SERIES

This chapter introduces sequences as functions (with domain Z^+) and thus builds upon the already extensive work with the function concept in the first three courses of this series. Following as it does the prologue's introduction of BASIC language, the chapter also serves to reinforce and maintain the elementary programming skills developed there, since the iterative nature of sequences lends itself well to programming problems. Note, however, that classes that have not studied the BASIC material in the prologue will not be handicapped in their study of this chapter--or for that matter any subsequent ones.

Sequences and series are important in applications both within and without mathematics itself, and they play a vital role in analysis. Since calculus is to be introduced in the fifth (next) year of this program, it is not too early for students to acquire some of the fundamental language and concepts of sequences. In this connection also the chapter includes an informal treatment of limit (specifically, limit of a sequence). Some more immediate applications are found in such topics as that of compound interest, although a fuller treatment is postponed until Chapter 5 when the logarithmic functions are available. Finally, the concept of proof is extended in the chapter by an introduction to mathematical induction. Induction is then used to prove some of the theorems, and of course it will be used even more frequently in later work.

In terms of behavioral outcomes, students should upon completing the chapter be able to:

1) Define a sequence.
2) Given a "function rule" for a_n, determine any specified term of the sequence.
3) Define a geometric sequence.
4) Use the definition of geometric sequence to derive the formula for the n^{th} term of such a sequence.
5) Determine any specified term of a geometric sequence given:
   a) two terms of the sequence
   b) one term and the common ratio.
6) Use mathematical induction to prove a theorem such as $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
7) Use the appropriate formula to find the sum of any specified number of terms in a geometric sequence.
8) Define an arithmetic sequence.

9) Use the definition of arithmetic sequence to derive the formula for the n^{th} term of such a sequence.

10) Determine any specified term of an arithmetic sequence given:
    a) two terms of the sequence
    b) one term and the common difference.

11) Use the appropriate formula to find the sum of any specified number of terms of an arithmetic sequence.

12) Derive the series associated with any given sequence.

13) Determine whether a geometric sequence--and other simple kinds of sequences--is convergent, and in case it is, determine the limit of the sequence.

SUGGESTED TIME SCHEDULE

1.1 Sequences: Functions that Order	2 days
1.2 Geometric Sequences	2 days
1.3 Mathematical Induction	2 days
1.4 Adding Terms: A Formula for s_n	2 days
1.5 Arithmetic Sequences	2 days
1.6 Series	2 days
1.7 Summary (and Review)	1 day
Testing	1 day
TOTAL	14 days

1.1 SEQUENCES: FUNCTIONS THAT ORDER

Purpose

To develop an understanding of the definition of sequence as a function.

Getting Started

Let the class read and discuss the three "boxed" examples (which are intended as motivation). The discussion can be as brief or as lengthy as class interest and productivity warrant, and here are some of the directions it might take:

Motivational Problem 1. From the preceding chapter students should have no difficulty identifying the output of this program. (Classes that have not done the prologue chapter may omit this problem, or you might explain it briefly.) Ask if the output of the program is a function, a cue for the definition of sequence. (Definition 1.)

Motivational Problem 2. Most students will be intrigued with the Zeno paradox and the fact that you get "closer and closer" to point B without ever getting there. (See related activity under "Other Activities" below.) You might elicit the infinite sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

as a more formal way of saying "there is always some distance left to be covered." This sequence will be examined carefully later in the chapter.

Motivational Problem 3. This example may be related to the formula from physics, $s = \frac{1}{2}gt^2$ (g approximately 32), where s is the total distance fallen after t seconds. Thus,

16, 48, 80, 112, ...

is a sequence in which a_n is the distance fallen during the n^{th} second. Furthermore,

16, 64, 144, 256, ...

is a sequence (the series associated with the sequence above) in which a_n is the total distance fallen at the end of n seconds.

Using the Text

The definition of sequence follows quite naturally from the output of the BASIC program. Use the DISCUSSION QUESTION to see if the class understands the definition. (In this question, the "list" 9, 36, 81, 144, 225 may or may not be a sequence, depending upon the domain. This illustration emphasizes that a sequence is not "just a list," but is a function.)

You may also wish to discuss other examples in the text. In particular, Example 3 deals with the important notion of a real number as an infinite sequence, a notion Course I touched upon when discussing infinite decimals. (As an additional example of this kind, the infinite sequence

1, 1.9, 1.99, 1.999, 1.9999, ...

is associated with the number 2.) Finally, Example 4 makes the point that the range elements of a sequence may be taken from any set at all. Another familiar illustration is that of a class of 25 students, with each student assigned a number from 1 to 25; then the people in the class form a sequence, since each person is the image of some number between 1 and 25, and the "ordering" characteristic of a sequence is apparent.

Other Activities

1. Students are often interested in trying to "guess the rule" for a sequence, a kind of question frequently included in "mental ability" tests of various kinds. Thus, given 2, 5, 10, 17, ..., what is the next number after 17? (One answer is 26, with $a_n = n^2 + 1$.) Some questions of this sort are in the exercises, but you may wish to capitalize on their motivational value at the beginning of the lesson.

2. Try this companion problem with Motivational Problem 2. Draw another segment CD, the same length as segment AB. Suppose one person moves from A toward B in the manner

described before, half the distance, then half the remaining distance, etc. Students should quickly agree that B is a "limit point." Now suppose a second person starts at C. After each move of the first person, the second one flips a coin. If it lands heads, he moves the same distance the first person has just moved; if it lands tails, he stays where he is. Is there a "limit point" in this case? (The answer is "yes" since you can form a sequence, with each term the total distance covered after $\underline{n}$ moves. Clearly the length CD is an upper bound, and so there is a least upper bound.)

Possible Assignments

Exercises 1 through 13, 16 through 18, and 22 constitute a minimal assignment. (Exercise 15 should be discussed with the class, preferably before they work the exercises following it.)

Exercises 21, 23, and 24 deal with Fibonacci sequences, and their assignment is recommended.

Exercises 19, 20, and 22 are the more difficult exercises in this set.

Assign Exercise 25 to students who like puzzle problems. To get the most out of it, you might "perform" beforehand: Thus, if a student starts his sequence with 3, 7, ... then the terms are 3, 7, 10, 17, 27, 44, 71, 115, 186, 301. While he is just starting to add the ten terms you can announce the sum as 781 (11 x 71).

1.2 GEOMETRIC SEQUENCES

Purpose

To develop an understanding of geometric sequences and an intuitive understanding of limit of an infinite sequence.

Getting Started

Discuss the five sequences displayed at the beginning of the text section. Ask students to detect a common property of these sequences, and to list some other sequences having this property. This activity should serve to introduce Definition 2 (geometric sequence) in the text.

Using the Text

Take time to discuss Example 2 with the class, making sure that they understand how interest is compounded and how geometric sequences are involved in the computation. In general, if principal P is invested at r% compounded

$\underline{t}$ times per year, then the total amount at the end of $\underline{n}$ paying periods is $P(1 + \frac{r}{t})^n$. In the example in the text, P is 1000, r is .04, t is 2 (compounded semi-annually), and n is 8. Additional work with compound interest will be done in Chapter 5.

Figures 2.1 and 2.2 and the accompanying discussion deal with the concept of limit of a sequence, but in a very informal way. You may wish to extend the discussion somewhat by issuing "challenges" to the class. Thus in the sequence $(\frac{1}{2^{n-1}})$ how far in the sequence one would have to go so that all remaining terms are less than .0001? .00001? .000001? etc. This sort of activity is related to the second boxed example at the beginning of the chapter, and you may wish to reintroduce it in conjunction with Exercise 25. (The sequence in this instance is <u>not</u> geometric, and students should understand at the outset that it is not only geometric sequences that may converge.)

Possible Assignments

Exercises 1 through 10, 16 through 24, and 27 and 28 constitute a minimal assignment.

Exercises 11 through 15 deal with the geometric mean (as opposed to the arithmetic mean), and assignment is recommended.

Exercises 25 and 26 continue the exploratory treatment of limit, and number 26 should encourage creative output on the part of more capable students.

1.3 MATHEMATICAL INDUCTION

Purpose

To extend the student's concept of proof by adding the method of <u>induction</u> to his proof-making strategies.

Getting Started

It is important that students understand mathematical induction to be a method of <u>deductive</u> proof. (In this sense the label "induction" is unfortunate since it falsely suggests "inductive" as opposed to "deductive" conclusions.) To this end, start by using the two examples at the beginning of the section.

1. "For every $n \in W$, $n^2 - n + 41$ is a prime number." Note that this is a <u>statement</u> and hence is either true or false. Which? One counterexample is enough to show it false, and the text does this. Emphasize however that it "works" <u>forty times</u> before you find the counterexample. Thus, even if one tests a statement--and it

"works"--one million times or more, there is no assurance that it will work the next time.

2. "The sum of the first n whole numbers is $\frac{n(n+1)}{2}$, where n is any positive integer."

   Is this statement true? Is it false? How to decide (without a perhaps futile search for a counterexample) is the subject of this lesson.

Using the Text

Before discussing the proof of

The sum of the first n whole numbers is $\frac{n(n+1)}{2}$

try a few specific cases. Such trials do not of course represent a way to prove the statement, but they help to make the meaning of the statement (and the nature of the problem) clear and to introduce the usual notation. Thus, the text's example:

$$1 + 2 + 3 + 4 + 5 + 6$$

is the sum of the first 6 whole numbers, and this is the same as $\frac{6(6+1)}{2}$. Discuss with the class the fact that the sum above is often written as

$$1 + 2 + 3 + \ldots + 6 = \frac{6(6+1)}{2}.$$

Then ask the class to consider other examples such as

$$1 + 2 + 3 + \ldots + 8 \quad \text{and}$$
$$1 + 2 + 3 + \ldots + 12.$$

But remind them once again that verification of these special instances does not provide a proof--and so back to the search for this proof.

Discuss the two "guarantees" in conjunction with the "infinite ladder" of Figure 1.10. (You may also wish to invoke the analogy of an infinite chain of dominoes set so that when one falls it knocks over the next in line, etc.) Once the nature of the guarantees seems clear, proceed through the proof in the text, discussing each step with the class. Follow this with a careful discussion of the proof of $2 + 4 + 6 + \ldots + 2n = n(n + 1)$.

(Note: Mathematical induction has the reputation of a difficult topic. The method is actually an assumption--one of Peano's postulates characterizing the positive integers--and so all we can do is try to make it seem reasonable. The discussion in the text is designed to help with this pursuit, but in addition it may be advisable to work through a few of the proofs in the exercises before the class starts its assignment.)

Other Activities

1. As an introduction to Exercise 4 you may wish to challenge the class to add the first one hundred positive integers, telling them that you can do it in a matter of seconds--and without pencil and paper. Then show them the "secret":

$$\begin{array}{l}
1 + 2 + 3 + \ldots + 98 + 99 + 100 \leftarrow \text{desired sum } S \\
\underline{100 + 99 + 98 + \ldots + 3 + 2 + 1} \leftarrow \text{desired sum } S \\
\underbrace{101 + 101 + 101 + \ldots + 101 + 101 + 101}_{100 \text{ times}} \leftarrow \text{twice desired sum } S
\end{array}$$

   Thus, $2S = 100(101) = 10100$, and $S = 5050$.
   This method, attributed to Gauss, may be used of course to find any similar sum. Ask students to "invent" other kinds of sums where this technique can be used.

2. A physical model of the Tower of Hanoi will add interest to Exercise 13. These are available commercially, but can be constructed in the classroom.

Possible Assignments

Exercises 1 through 8 and 10 through 11 should be assigned, since there is probably no way to learn induction except by pushing through some proofs. Exercise 12 also should be assigned, because it points up the importance and nature of the <u>two parts</u> of an induction proof. Exercise 9 is optional but profitable. Exercises 13 and 14 also are optional but are highly recommended as a way to enliven a lesson that may seem oppressively difficult to some students.

1.4 ADDING TERMS -- A FORMULA FOR S_n

Purpose

To develop ability to find the sum of a specified number of terms of a geometric sequence.

Getting Started

Reintroduce the example from Section 0.3, reprinted at the beginning of this section. The answer to the question is printed and there should be some curiosity about where the formula came from and why it works. You may also at this time want to broach the two "options" in Exercise 16. There also the answer derives from adding a specified number of terms in a sequence, and this section develops a method for direct computation of such a sum.

Using the Text

The introductory activities discussed above set the stage for Theorem 1.1 in which the desired formula is found. Example 1 precedes the proof simply to make clear the meaning of the theorem.

Next discuss the inductive proof of the theorem with the class, and note that it is one illustration of the power of mathematical induction in proving theorems concerning the positive integers.

Also, develop with the class the fact that

$$a_1\left(\frac{1 - r^n}{1 - r}\right) = a_1\left(\frac{r^n - 1}{r - 1}\right)$$

and so either "form" may be used to find s_n for a geometric sequence. The form on the right is likely to be more convenient when $r > 1$.

Other Activities

Consider a geometric sequence such as

$$1, 2, 4, 8, 16, \ldots$$

and manufacture a new sequence by finding "partial sums" $s_1, s_2, s_3, \ldots$. Thus, for the sequence above, the new sequence would be

$$1, 3, 7, 15, 31, \ldots$$

This kind of exercise not only provides some drill for the present lesson, but also introduces the concept of series, which Section 1.6 discusses in more detail.

Possible Assignments

All eighteen exercises are recommended. Exercise 11 returns to the limit concept. Exercise 17 is somewhat recreational, Exercise 18 deals with a BASIC program, and Exercises 13 and 15 are more difficult than the others.

1.5 ARITHMETIC SEQUENCES

Purpose

To develop ability to determine any specified term and the sum of any specified number of terms of an arithmetic sequence.

Getting Started

Ask the class to study the two BASIC programs printed at the beginning of the section, and to note the difference between them. This should be easy to do and should quickly elicit Definition 1.3 (arithmetic sequence) because of the similarity to that of geometric sequence. Then consider the DISCUSSION QUESTION in the text to be sure that the new definition is clear.

Using the Text

The examples in the text are easy to understand, and students should have no difficulty with them. You may wish to discuss with the class the proof of Theorem 1.2, and to ask for a verbalization: The sum of the first n terms of an arithmetic sequence is $\frac{n}{2}$ times the sum of the first and n^{th} terms. (The role of verbalization in retention and transfer continues to be a major unresolved issue in mathematics education.) Note that Exercise 13 calls for an inductive proof of this same theorem; it may be discussed (or assigned) at the time the text proof is first discussed.

Call attention to the fact that the arithmetic mean (defined for an arithmetic sequence just as the geometric mean is for a geometric sequence) is the common "average" with which most students are familiar.

Other Activities

1. The text notes that the formula

$$1 + 2 + 3 + \ldots + n = \frac{n}{2}(n + 1)$$

(proved in Section 1.3) is but a special case of the sum of the first n terms of an arithmetic sequence, specifically the sequence 1, 2, 3, ..., where $d = 1$. In Section 1.3, Exercise 4 suggests an alternate method for proving this special theorem, and you might ask the class to adapt this method to arithmetic sequences in general. The adaptation should look something like this:

$$\begin{aligned} s_n &= a_1 + (a_1+d) + (a_1+2d) + \ldots + (a_n-d) + a_n \\ s_n &= a_n + (a_n-d) + (a_n-2d) + \ldots + (a_1+d) + a_1 \\ \hline 2(s_n) &= \underbrace{(a_1+a_n) + (a_1+a_n) + (a_1+a_n) + \ldots + (a_n+a_1)+(a_n+a_1)}_{n \text{ addends}} \end{aligned}$$

Sp, $2(s_n) = n(a_1 + a_n)$, and $s_n = \frac{n}{2}(a_1 + a_n)$.

2. While the text speaks only of the arithmetic and the geometric mean, it is quite proper to speak of any number of means between two specified numbers. For example, the three arithmetic means between 2 and 15 are the numbers x, y, x such that $2, x, y, z, 15$ is an arithmetic sequence. (Solution: $4d = 15 - 2 = 13$. So $d = 3\frac{1}{4}$. Therefore, $x = 5\frac{1}{4}$, $y = 8\frac{1}{2}$, $z = 11\frac{3}{4}$.).

   The two geometric means between 1 and 27 are the two numbers x, y such that $1, x, y, 27$ is a geometric sequence. (Solution: $r^3 = 27$; $r = 3$. Therefore, $x = 3$, and $y = 9$.) You may wish to ask the class to consider some exercises of this kind.

Possible Assignments

A minimal assignment should include Exercises 1 through 12 and 19 through 23. Exercise 13 is recommended in order to reinforce the earlier lesson on induction. Note that Exercise 14 returns to one of the three motivating problems posed at the beginning of the chapter.

1.6 SERIES

Purpose

To define series and to provide some experience with determining the limit (or sum) of an infinite series.

Getting Started

Recall the "paradox" (getting from A to B) at the beginning of the chapter, and discuss it in terms of the diagram presented in this section. Actually this diagram (Figure 1.14) is at one an illustration of the physical situation and of the series derived from the given sequence. Thus

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$$ is the series associated with the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ This makes clear the fact that a series is a sequence and clears a path for Definition 1.4 (series).

Using the Text

Take time to discuss Example 2 carefully. This is a kind of infinite series notation that students will see in calculus, and they should understand that it does denote a sequence. An analogous notation for the series in Example 1 would be $3 + 7 + 11 + 15 + 19$.

Since a series is a sequence, "convergence" has the same meaning as in Section 1.2, and here (as there) is treated in an intuitive and informal way. Students should observe again that a geometric sequence (and series) converges if and only if $-1 < r \leq 1$.

Finally the section closes by touching base with an idea first encountered in Course I--rational numbers as infinite repeating decimals. Considering such a decimal as an infinite series provides a legitimate way of expressing it in the form "$\frac{a}{b}$".

Possible Assignment

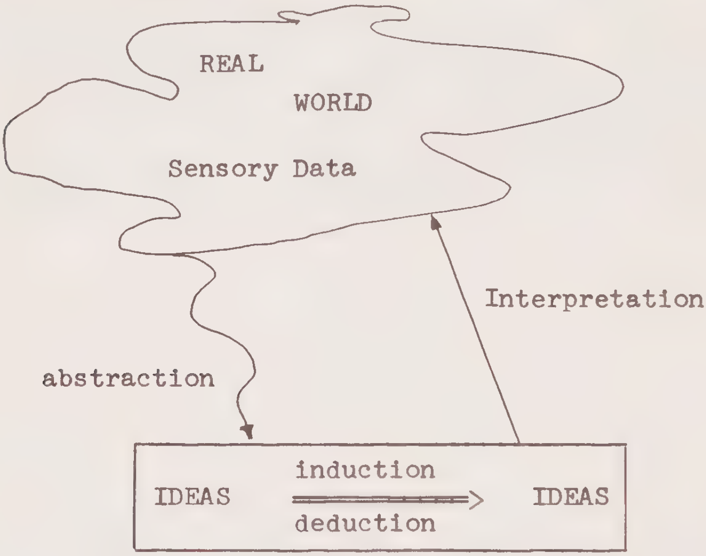
Exercises 1 and 2 might well be handled as discussion questions. Their purpose is that of focusing attention on the "mechanics" of the limit definition, but in such a way as to make clear the nature of the definition.

Exercise 3 through 15 and Exercise 27 should be assigned and at least some of those among Exercises 16 through 26.

CHAPTER 2

MATHEMATICAL MODELS

The fundamental process involved in use of mathematics to solve real world problems is establishing an isomorphism between objects and relations of perception and mathematical ideas which have abstract similarity to those perceptions. In this case the mathematics is called a model of the real world situation.



Mathematical World or Model

The purpose of this chapter is to carefully and consciously examine the process involved in this modelling of real life problem situations--making assumptions, organizing data, translating ideas from one modality into another, interpretation of mathematical results in practical situations, etc. The chapter thus sets the theme for succeeding chapters in this course--mathematics generated from and helping to solve problems in prediction and description of real life phenomena. The authors also believe that the rich variety of problem settings will convince students that their subject is indeed relevant to nearly every science, business, government, and industry. Of course the content given here is only a selection; amplification occurs in succeeding chapters.

In addition to developing mathematical skills such as translating verbal conditions to mathematical language, design of a simulation, or fitting a curve to patterned data, we hope that the problems of the chapter generate vigorous students' debate around such topics as: What are the assumptions you've made to arrive at that decision? Is the exact mathematical 'answer' practical? What non-mathematical boundary conditions must be considered? How does mathematical reasoning contribute to, but not settle, intelligent decision making problems?

As a good general background resource, consult The Man-Made World, produced by the Engineering Concepts Curriculum Project and published by McGraw-Hill.

SUGGESTED TIME SCHEDULE

2.1	What is a Model?	$1\frac{1}{2}$ days
2.2	Language of Science to Language of Mathematics	2 days
2.3	Multivariable Models	1 day
2.4	Modeling Data from Experiments	2 days
2.5	Probability Models and Monte Carlo Methods	2 days
	Summary	$\frac{1}{2}$ day
	TOTAL	9 days

2.1 WHAT IS A MODEL?

Purpose

This section is designed to point out the main features of a mathematical model--it is a collection of mathematical ideas that in some way represent reality. A model is only an approximation and often refinement of the model is necessary to make predictions of sufficient accuracy. Further, many important components of a decision making situation cannot be expressed in suitable mathematical formulation.

Getting Started

Describe the basic conditions of the rocket or typhoid model situations to the class. Then ask the class a sequence of questions about implications of the model, closing with several that indicate limitations of the model. For instance, note that the height function for the rocket takes on negative values for t greater than 3.5. The typhoid model predicts 1,679,616 people sick in 8 days; no provision for limits of the available population.

Using the Text

Have students read the conclusion of section 2.1 in the text and then work in groups on exercises 6, 7, 13, 15.

Other Activities

Contact a school physics teacher to prepare some demonstrations of supposed "physical laws." Illustrate the approximate nature of these formulas when "real" data is gathered. One nice one is Galileo's use of an inclined plane to observe the acceleration due to gravity. His tools available were crude, so the data which led to mathematical abstractions had to be idealized.

Possible Assignments

Exercises 1-5 fit together and involve some computational practice. Then select some others that will generate discussion on modelling assumptions; perhaps 17-20.

2.2 LANGUAGE OF SCIENCE TO LANGUAGE OF MATHEMATICS

Purpose

This section gives practice and strategy hints for setting up the functions and equations that are so often the basis of mathematical models. Further, it explains the use of dimension analysis to check the validity of a model. Since the language of proportional variation is so common in science, it is stressed here, using function concepts.

Getting Started

Pose the verbal statements of Examples 1 and 2 to the class and ask them to construct a mathematical model of the science principles. Guide them by providing input on the meaning of scientific phrasing, pointing out the virtue of using variable names that in some way suggest the quantities represented by each variable. After solving the problems posed by each example, do the dimensional analysis of the remaining examples.

Using the Text

As an optional approach to this section, students might well be able to read the text and begin exercises on their own or working in groups.

Other Activities

Again contact the physics instructor or chemistry instructor to arrange demonstrations of the principles described in examples and exercises.

Possible Assignments

1,2, several from 3-10, 11, 16, 17, and several from 18-24. Discuss 11-15 in class the next day.

2.3 MULTIVARIABLE MODELS

Purpose

This section extends the "word problem" ideas of the preceding section, considering more complex situations. The objective is for students to develop facility in translating conditions into algebraic language. Don't emphasize solving the systems that arise--though some practice will review important skills for linear algebra that were introduced in Course III.

Getting Started

Because the transportation problem in the text is pretty complex, it is a good topic for group problem solving exploration. Before students read the text, put the diagram of figure 2.10 on the board and let students work in groups on producing a solution, presentation to the class, and rational justification of their method. When this activity reaches a successful conclusion or the groups seem to be foundering, pull the ideas together in a full class discussion, emphasizing the problem solving strategy ideas.

Using the Text

Students can probably read Example 1 easily on their own.

Other Activities

For alternative motivational problems, look at the matrix multiplication situations in Chapter 1 of Course III. They generally involve many variables and equations.

Possible Assignments

Exercises 1-5 fit together to solve the problem in the text. This would give students a feeling of closure on that situation. Exercises like 7 and 9 are very good practice for problem solving. You might give students the task of making up story lines of their own to go with problem types.

2.4 MODELLING DATA FROM EXPERIMENTS

Purpose

Application of mathematics to problems from social and biological sciences has given a big boost to the importance of "curve fitting" and the least squares method for determining the best fit. This section is primarily sensitization experience for students--pointing out that not every interesting phenomenon is linear or quadratic or exact in the sense of physical laws. Students should become aware of the possibility of building a model on approximations and of the limitations of inference based on such models.

Getting Started

Students hould be able to readily read this section prior to class discussion. Then in the discussion have students contrast the methods used in the examples with those that have dominated their previous mathematics. Have them discuss limitations of models based on data--for instance, the energy crisis has deeply distorted the usual supply and demand effect on price of gasoline. Note that the models chosen actually lead to some impossible predictions (negative sales, infinite hay production, negative miles per gallon) when applied out of the range of data points.

Using the Text

Have students read prior to class.

Other Activities

Have students work on Exercises 1-4 in teams during class to check the various types of error measurements. Then as a class discuss the virtues and limitations of each.

Possible Assignments

Exercises 6, 7. Then have students choose a situation, 9-10 or 12-14 or 15. Don't spend a long time on this exercise set.

2.5 PROBABILITY MODELS AND MONTE CARLO METHODS

Purpose

This section introduces the concept of simulation--a technique of growing importance in application of mathematics. The focus is on only one type of simulation involving probability experiments. After studying the section students should be able to design simple Monte Carlo methods for estimating probabilities. This activity should also help review basic ideas of probability, particularly the relative frequency basis of probability measures.

Getting Started

Pose several statements to the class in the form "If... is true, then what can you conclude?" Pick contrasting situations such as those in the text to demonstrate the difference in logic between algebra or geometry and probability. Emphasize the fact that a probability model is an attempt to predict the behavior of a system; but unlike the other types of mathematics, the prediction is less definitive.

Using the Text

Present the technique developed in Example 1 and then set the class to working on the succeeding thumbtack activity in pairs or teams. Compare results and try to extend the simulation by using the random digit table in the Appendix. Some subset of the class could go ahead to work on Example 2.

Other Activities

Computer generation of random digits is interesting. You might want to do Exercises 7 and 8 in class. For further ideas, see "A New Way to Teach Probability Statistics" in the April, 1969 Mathematics Teacher.

Possible Assignments

Exercises 1-6 will review important ideas in probability as a background for later work in this course. Several parts of 14 would follow up on the Monte Carlo technique.

SUMMARY

Since the basic purpose of this chapter is to illustrate concepts and strategies that will unify the succeeding work, we recommend that no chapter test be given. Since such examinations frequently have the effect of telling students "All right you've passed this material, now you can forget it!" we prefer to lead them directly into situations where the ideas can be used. It will be important for you as the teacher to keep the strategy ideas alive in the treatment of succeeding material.

CHAPTER 3

QUADRATIC FUNCTIONS, EQUATIONS, AND COMPLEX NUMBERS

To the extent that this chapter has a single objective, it is that of introducing the system of complex numbers as an extension of the real number system. However the pursuit of this objective brings together many existing strands of the unified structure, and in so doing accomplishes other objectives as well.

The chapter opens with a modeling problem using quadratic functions, leading to a reexamination of the relationship between zeros of a function and solutions of an equation, part of which involves a computer search for zeros.

The analysis of quadratic equations results in the quadratic formula and a study of the role of the discriminant in determining whether the equation has any real solutions. The case of no real solutions is the motivation for the development of the complex numbers, a development that is realized by a reexamination of 2 x 2 matrices and a subset of these matrices that is in fact an isomorphic model of the complex numbers. With the complex numbers, the problem of solvability of quadratics is resolved, and the chapter closes with the representation of complex number multiplication as a plane transformation called a spiral similarity. This representation is based on the association of complex numbers with the points of a plane.

In terms of behavioral objectives, the student should, after studying this chapter, be able to:

1) Approximate the zeros of a polynomial function by computing function values at specified intervals.
2) Derive the quadratic formula.
3) Use the quadratic formula to determine all solutions--real and non-real--of a quadratic equation with real coefficients.
4) Identify the discriminant of a quadratic function and describe specifically the relationship between the discriminant and the number of real zeros of the function.
5) Demonstrate the isomorphism between the real numbers and the 2x2 real matrices of type $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$
6) List the basic field properties and demonstrate the possession of these properties by the set of real matrices of type $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$
7) Exhibit the isomorphic correspondence between the complex numbers, using the standard symbols (a+bi),

and the set of real matrices of type $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

8) Interpret complex numbers as points of a plane, and graph those numbers satisfying certain conditions (e.g., $|z| = 1$).
9) Determine the absolute value of any complex number.
10) Add, subtract, multiply, and divide complex numbers, making use of the conjugate in division.
11) Illustrate addition of complex numbers as a plane transformation (parallelogram property).
12) Illustrate multiplication of complex numbers as a spiral similarity.

And in the affective domain, the student should further his appreciation of the role of mathematics in applications (modeling problems), of the uses of computer programming, of the interplay between algebra and geometry, and of the nature of a number system.

SUGGESTED TIME SCHEDULE

3.1	Introduction: A Modeling Problem	1 day
3.2	The Computer Solves Equations	2 days
3.3	Quadratic Equations: Solutions and No Solutions	1 day
3.4	Matrices -- and Some New Numbers	2 days
3.5	The Complex Number System	3 days
3.6	Quadratic Equations: Always a Solution!	1 day
3.7	The Complex Plane	2 days
3.8	Transformation Geometry and Complex Numbers	2 days
	Review and Testing	3 days
	TOTAL	17 days

3.1 INTRODUCTION: A MODELING PROBLEM

Purpose

This section serves as a motivational anchor for the chapter by discussing a modeling problem that makes use of quadratic as well as linear functions. Thus it maintains the mathematical models momentum started in Chapter 2 and also serves to reintroduce the notion of quatratic function.

Getting Started

The section may be read and studied by students individually. However you may wish to set the stage by reviewing with the class the concepts of linear and quadratic functions, and by noting beforehand that they will find in this section a problem showing the use of such functions to answer some significant questions in the field of economics (a new kind of application problem to add to those they saw in Chapter 2).

Using the Text

After students have finished the reading assingment, there should be some class discussion of the four questions included in the body of the text. Answers to these questions are printed below. During the discussion emphasize (if you have not already done so) that such things as linear and quadratic functions are useful in fields of study other than the physical sciences.

Answers to the Discussion Questions:

1) Note that the points (8,0) and (4,4000) were selected (arbitrarily) simply because it looks as though the line through these points is a good "fit" for the seven plotted points. The slope of the line through these points is -1000; so using this slope and the point (8,0), the equation of the line is

$$y = -1000x + 8000, \text{ or}$$
$$p = -1000n + 8000.$$

(Review if necessary the determination of the equation of a line through two given points.)

2) These "break even" points can be located by solving

$$-1000p^2 + 8000\,p = -2000p + 20{,}000,$$

which yields (approximately) \$2.76 and \$7.20. However students need some experience in interpreting graphical data, and suitable approximations can be read from the two intersection points (where revenue equals cost) in Figure 3.2.

3) When $p = 5$, $n = -1000p + 8000$
$= -5000 + 8000$
$= 3000.$

4) This question deviates somewhat from the specific modeling problem at hand, but students should see the relationship--it deals with a method of finding maximum (or minimum) values of a quadratic function, as in the model in this section. It also serves the pedagogical purpose of reviewing some transformation geometry.

$$\text{If } x' = x + \frac{b}{2a} \quad \text{and} \quad y' = y$$

$$\text{then } x = x' - \frac{b}{2a} \quad \text{and} \quad y = y'$$

Thus, $y = ax^2 + bx + c$ becomes

$$y' = a(x' - \frac{b}{2a})^2 + b(x' - \frac{b}{2a}) + c,$$ which simpli-
fies to

$$y' = a(x')^2 + (\frac{b^2}{4a} - \frac{b^2}{2a} + c).$$

But the graph of this function is symmetric to the y-axis, which means that the maximum (or minimum) point of the parabola occurs when $x' = 0$, or when

$$x = -\frac{b}{2a}.$$

3.2 THE COMPUTER SOLVES EQUATIONS

Purpose

Here we review the relationship between zeros of a function and solutions of the associated equation, considering polynomial equations in general before concentrating on quadratics in the next section. Also, following the student's newly acquired skill in BASIC and computer problem solving, this section includes a program format for computer search for zeros.

Getting Started

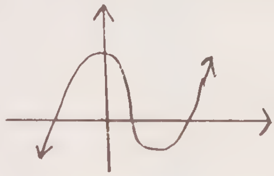


FIG. 1

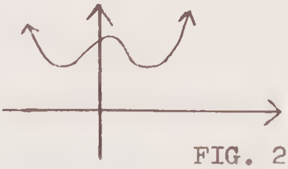


FIG. 2

Sketch a graph such as that in Figure 1, and ask--it should be a review question--how many solutions the equation "$f(x) = 0$" would have in this instance, noting how the answer is shown by the graph. Next, use a graph such as that in Figure 2 to illustrate an equation "$f(x) = 0$" that would have no real solutions. You can then direct attention to the introductory questions at the beginning of the text section, and there should be ready agreement that determining solutions of "$12x^3 + 28x^2 - 7x - 5 = 0$" is tantamount to determining where the graph of "$y = 12x^3 + 28x^2 - 7x - 5$" intersects the x-axis. This section deals with a method for approximating such intersection points.

Using the Text

Students should be able to read the text and study the examples, proceeding at their own rate to the exercises. However there are certain aspects of the topic that probably call for some group discussion, specifically the following:

1) If you want an approximation correct to the nearest <u>integer</u>, you can work with intervals of length 0.5. Thus if the "crossing" is between 2.5 and 3.0, then the best integral approximation is 3.0; there is no need to determine the function values for 2.6, 2.7, 2.8, 2.9. If you want an approximation correct to the nearest <u>tenth</u>, then you can work with intervals of length 0.05. For example, if the x-intercept is between 2.85 and 2.90, then to the nearest tenth it is 2.9. This should be graphically evident to students, and you might help them generalize as follows: For an approximation correct to 10^{-n}, work with intervals of length $5 \times 10^{-(n+1)}$.

2) The class may benefit from a few examples in which they "see" the term of greatest degree "dominate" a polynomial (note the discussion following Example 1 in the text). For instance, for the polynomial $x^3 + 6x^2$, construct the following table:

x	x^3	$\lvert x^3\rvert$	$6x^2$	$x^3 + 6x^2$
0	0	0	0	0
-1	-1	1	6	5
-2	-8	8	24	16
-3	-27	27	54	27
-4	-64	64	96	32
-5	-125	125	150	25
-6	-216	216	216	0
-7	-343	343	294	-49
-10	-1000	1000	600	-400
-100	-1,000,000	1,000,000	60,000	940,000

This table illustrates the principle at hand. As $|x|$ increases, $|x^3|$ is so much greater than the absolute value of all other terms that eventually its sign (positive or negative) dominates and determines the sign (positive or negative) of the entire polynomial. Therefore one knows that as $x \to -\infty$, $x^3 + 6x^2$ eventually becomes and remains negative, thus precluding any more intersections with the x-axis. You may want the class to construct similar tables for other polynomials so that they have a sound intuitive feeling for this phenomenon.

3) In conjunction with the examples in the text, review the point-plotting BASIC program discussed in Section 0.6; it can be helpful in the search for zeros of a function.

4) Review the Factor Theorem from Course III. Many times it too is useful in finding zeros. For instance if the function is cubic and one of the solutions is known, then the Factor Theorem reduces the problem to that of solving a quadratic equation.

Possible Assignment

A minimal assignment would include Exercises 1 and 2, at least some of exercises 4 through 9, exercises 10, 13, 14, and 16 through 20. In addition, exercises 3, 11, and 12 are recommended for classes that have studied the BASIC chapter. Exercises 15 and 21 are optional ones.

3.3 QUADRATIC EQUATIONS--SOLUTIONS AND NO SOLUTIONS

Purpose

In this section the student should come to understand how the quadratic formula is derived, and should be able to apply it to determining the real solutions of any quadratic equation. The section also emphasizes the role of the discriminant in describing the number of real roots.

Getting Started

Ask several students to write quadratic equations on the board. Observe that we know (from Section 3.2) that each of the equations has exactly two, exactly one, or no real solutions. Then announce (by mentally computing the discriminant in each case) how many solutions each of the students' equations has. Then tell the class that determining this number is a simple matter and is the subject of this lesson.

Using the Text

The step-by-step development of the quadratic formula should be discussed in class. The chain of equivalent equations in that development rests primarily on these three principles:

$$r = s \Leftrightarrow r + t = s + t \quad \text{(adding to both sides)}$$

$$r = s \Leftrightarrow rt = st \ (t \neq 0) \quad \text{(multiplying both sides by nonzero number)}$$

$$r^2 = s \Leftrightarrow r = \sqrt{s} \text{ or } r = -\sqrt{s} \ (s > 0)$$

To emphasize the equivalence, ask students to begin with the final formula and "work back" --validating each step-- to $ax^2 + bx + c = 0$. Also note that in the penultimate step in the text development, there are really four possibilities, but only two distinct ones. That is, the sentence $r^2 = s^2/t^2$

is equivalent to

$$r = s/t \quad \text{or} \quad r = -s/-t \quad \text{or} \quad r = -s/t \quad \text{or} \quad r = s/-t.$$

However the first two of these are the same, and the second two are the same. In the text development of the quadratic formula, only the distinct cases--corresponding to the first and third of the above forms--are displayed:

$$x + \frac{b}{2a} = \frac{\sqrt{b^2-4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = \frac{-\sqrt{b^2-4ac}}{2a}$$

Other Activities

1) Ask students to construct a quadratic equation with two given numbers as solutions. Thus, if the numbers are $-2/3$ and $\sqrt{2}$, then $a(x + 2/3)(x - \sqrt{2}) = 0$, where $\underline{a}$ is any nonzero number, is an answer. Then ask them to use the quadratic formula to "recover" the numbers they started with.

2) Ask students to investigate the validity of the quadratic formula in operational systems other than that of the real numbers. (See Exercise 26.)

Possible Assignments

At least half of the exercises numbered 1 through 16 should be assigned. Call attention particularly to exercises 14, 15, 16, so that it is clear that the quadratic formula applies to any quadratic equation having real coefficients, rational or irrational.

Exercise 17 should be assigned; it requires finding the solutions of the equations in the first sixteen exercises (with or without the assistance of a computer). Exercises 20 through 25 should be assigned as a review of the important principle involved in solving quadratics by factoring. Exercises 18, 19, 26, and 27 are optional ones, with the latter two highly recommended for some independent investigation by able students.

3.4 MATRICES AND SOME NEW NUMBERS

Purpose

The immediate objective here is to review the structure of a field and to investigate a subset of the 2 x 2 matrices that is a field. The ultimate purpose for doing so however is that this matrix system will (in Section 3.5) serve as a model for the complex number system.

Getting Started

Because the principal development (that of the complex number system) does not appear until the next section, the present one may seem to be almost misplaced unless you relate it to the themes of the chapter. This can be done by noting the well established fact that not all quadratic equations have solutions. Ask the class to cite examples of equations (such as the two cited in the text) that do not have solutions in W and in Z until appropriate extentions of these sytems are made. Then ask the class to recall another extension they have made (An Answer: "$x^2 = 2$" has no solution in the set Q of rational numbers, but it has two solutions in the real number system R.) Let the class know that the present section is one of considerable drama because it will lay the foundation for still another extension--this time making it possible to solve presently "unsolvable" quadratic equations.

Using the Text

It is recommended here that there be considerable class discussion of the text development so that the purpose of the section is not obscured. It should be clear to students that they have isolated a subset of the 2 x 2 real matrices that has very special properties--it has all the defining properties of a field. This is noteworthy inasmuch as the entire set of 2 x 2 matrices is not a field. It is also significant since, as later sections show, this set of matrices will serve as a model for a structure that may be taken as an extension of the real numbers. (We are soon to see these matrices as complex numbers!)

Expressing the matrices in form aI + bJ is a precursor of the symbolism used in the complex number field. Thus (aI + bJ) + (cI + dJ) will reappear as (a + bi) + (c + di) later. (It is unfortunate, but a fact of life, that "J" plays the role in the matric model that "i" will play for the complex numbers. However the symbol "I" is already so closely associated with the identity matrix that it would be even less satisfying to use it for $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the present context.

The important behavior of this matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ should be emphasized. The fact that $J^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the key that will unlock numbers whose squares are negative. Two of the exercises--those numbered 10 and 16--deal with this product.

Other Activities

Recall with the class the set of rotation matrices of form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

These constitute a subset of the matrices studied in this section--those of type $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Furthermore, every matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ can be written as the product of a real number and a rotation matrix, as follows:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \sqrt{a^2+b^2} \times \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}$$

For example,

$$\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} = \sqrt{13} \times \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}$$

As such, this matrix can be associated with that transformation of the plane which is the composition of a ro tation through θ (where $\tan \theta = 2/3$) and a dilation with scale factor $\sqrt{13}$.

You might ask the class to express other $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ matrices in this product form. Later when these matrices are identified with complex numbers, expressing them in this way will enable each complex number to be associated with a point of the plane. (See "Other Activities" in the Commentary Section 3.7.)

Assignment

It is recommended that all 26 exercises be assigned.

3.5 THE COMPLEX NUMBER SYSTEM

This section enables the student to understand the complex number system as an extension of the reals, using the concept of isomorphism and the special subset of the 2 x 2 matrices which was identified in the preceding section as a field. There is also provision for practice in the arithmetic operations with complex numbers.

Getting Started

The concept of isomorphism is central to this section, and so you may want to review it briefly, perhaps by recalling examples from earlier courses. One such example is that of the two isomorphic systems below.

*	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\odot$	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

The isomorphism is more easily recognized if the second table is rearranged as follows:

$\otimes$	6	8	2	4
6	6	8	2	4
8	8	4	6	2
2	2	6	4	8
4	4	2	8	6

These two systems--which students have seen earlier--may use different labels and may stem from different physical referents, but structurally there is no difference between them. If each label used in one system is replaced by the corresponding label in the other, then the identical structure of the systems is apparent. (In the present section one system refers to matrices and uses the usual matrix notation. The other refers to the real numbers and uses the standard numerals for those numbers.)

You may wish to discuss Exercises 1 through 3 at the time of this introduction; they call for a review of the concept of isomorphism.

Using the Text

Following the review of isomorphism, prove Theorem 3.3 with the class, emphasizing the one-to-one correspondence, and the fact that the theorem justifies identifying a matrix such as $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ with a real number, namely the real number a.

Because of the nature of this section, a step-by-step class development is recommended. In essence the development rests on the following points:

1) We have (in the preceding section) noted the field properties of $(M, +, \cdot)$, where M is the set of 2 x 2 matrices of form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

2) Theorem 3.3 makes it clear that a subset of M--those elements of type $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$--form a subsystem which structurally is simply the real number system.

3) The question then is this: If the real numbers are isomorphic to a subsystem of M, how can we extend the reals to a new system of numbers that will be isomorphic to the entire set M? The structure of M has already been investigated, and so we know in advance about the structure of the new number system--it is going to "behave" precisely as $(M, +, \cdot)$ does.

4) The development starts with the introduction of a correspondent for $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

this new number (called i) is one whose square is -1. And thus we are off and running in the construction of a new number system which will include squares that are negative.

As in all extensions (whole numbers to integers, integers to rationals, etc.) the properties of the original system are preserved (otherwise it would not be an extension) but there are some new properties (e.g., negative squares) since otherwise there would be no need for the extension.

Possible Assignment

All exercises should be assigned or discussed. Be sure to discuss the concept of conjugate and its role in simplifying the quotient (Exercises 25 through 40). Also use the exercises to review the way subtraction and division are defined (in terms of inverses) in a field.

3.6 QUADRATIC EQUATIONS: ALWAYS A SOLUTION

Purpose

This section deals with the principle that in the field of complex numbers every quadratic equation (with real coefficients) has at least one and at most two solutions, and it provides practice in solving such equations.

Using the Text

The present section in a sense provides some closure for the chapter. It all started with a study of quadratic functions and equations and the observation that not all such equations have real number solutions. This led to an exploration of extending the real number system, resulting in the development of the complex numbers. And now we return, this time with the complex numbers in hand, to the question of solvability of quadratics, and find that now all these equations have at least one--and at most two--solutions.

The text section is brief and easy to read, but you may want to work with the class on such simplifications as $\sqrt{-3} = i\sqrt{3}$.

Possible Assignment

It is recommended that the first 15 exercises be assigned. Exercise 16 is optional.

3.7 ROTATING THE NUMBER LINE: THE COMPLEX PLANE

Purpose

This section discusses the representation of the complex numbers by points of a plane, a concept essential to complex analysis. In so doing it makes use of some transformation geometry, and it also discusses the geometric interpretation of complex number addition.

Getting Started

The work here brings together a number of strands from earlier work. To prepare for the principal concern of the section, you might start by recalling the set of matrices isomorphic to the real numbers, leading to the following points:

1) What is the (geometric) effect of multiplying $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes the unit point on the horizontal axis? The effect of course is to map $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ onto the point $\begin{pmatrix} a \\ 0 \end{pmatrix}$. So, since <u>a</u> is any real number, the unit point generates the entire real line.

2) What then is the effect of multiplying the unit point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$? This question leads directly to the text discussion.

Using the Text

Illustrate on the chalkboard (or the overhead projector) the matrix products in the text, and other similar ones. Thus,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

may be interpreted as carrying the point (3,0) through a rotation of 90° to the point (0,3). This is a geometric interpretation based on what we already know about matrices and transformations of the plane. However because of the identification of the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with the number <u>i</u> we may also view the product in terms of numbers:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

You might actually do this "writing over" illustration to emphasize that the product can be associated with (i)(3) or 3i. It should be easy for students to see that all numbers of form ai, where <u>a</u> is real, can be identified in this way with points on the vertical (or "imaginary") axis. It is a short step from there to "filling in" the complex plane.

Be sure to discuss with the class the graphical interpretation of addition of complex numbers ("parallelogram property") which is tantamount to vector addition (2-space) thus emphasizing that a complex number is, considered abstractly, an ordered pair of real numbers (see (2) in Other Activities).

Also note that addition of complex numbers can be interpreted as a translation (this is essentially the "parallelogram property"). Thus, adding $3 + 4i$ is associated with the translation $T_{3,4}$, since it maps (a,b) onto $(a+3,b+4)$; that is $(a+bi)+(3+4i) = (a+3)+(b+4)i$. In the next section a geometric interpretation (in the complex plane) will be given for multiplication of complex numbers.

<u>Other Activities</u>

1) It seems natural enough to associate each complex number with that point in the plane determined by its real component (horizontal axis) and its "imaginary" component (vertical axis). However it also is possible to rationalize this interpretation in other ways. To illustrate, consider the complex number $3+4i$. The product $(3 + 4i)(1)$ may be taken as

$$\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad 5\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus the product--$3+4i$--is represented by the image point of $(1,0)$ under rotation θ, where $\tan\theta = 3/4$, followed by dilation D_5 with center at origin. This point is of course the point $(3,4)$. In general for any complex number $a+bi$ there is an associated transformation represented by

$$\sqrt{a^2+b^2}\begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}$$

This transformation associates any complex number $a+bi$ with the point (a,b), the image of the unit point $(1,0)$.

2) The complex plane suggests that a complex number may be considered as an ordered pair of real numbers--and this is the case. Thus the set of complex numbers is the set of all ordered pairs of reals, with the following definition of addition and multiplication:

$$\begin{aligned} (a,b) \oplus (c,d) &= (a+c,\ b+d) \\ (a,b) \odot (c,d) &= (ac-bd,\ ad+bc) \end{aligned}$$

Students may be interested in investigating analogous definitions of integers as ordered pairs of whole numbers, and of rational numbers as ordered pairs of integers. Ask them to get from a library some books on modern algebra or development of the number system, and to report to the class on these developments of the integers, the rationals, and the complex numbers. (They should also be interested to know that the real numbers cannot be constructed in this way, that is as ordered pairs of rational numbers.)

Possible Assignment

The first thirteen exercises may well be treated as class work; they serve to emphasize that multiplication by *i* can be interpreted geometrically as a quarter-turn.

All other exercises in the set should be assigned.

3.8 TRANSFORMATION GEOMETRY AND COMPLEX NUMBERS

Purpose

This section introduces a transformation interpretation of complex number multiplication. The transformation is called a *spiral similarity* and serves to review the whole concept of similarity.

Getting Started

1) Ask the class to view Figure 3.28 and try to describe a transformation that maps ABC to A'B'C'. (It can be done by a composition of a rotation, translation, and dilation.) This provides the opportunity to review the notion of transformation in general and of similarity in particular.

2) Then point out that this particular kind of similarity can be associated with multiplication of complex numbers, just as addition of complex numbers is associated with a simple translation.

Using the Text

The text has a graphic representation of the geometric interpretation of the product $(2+3i)(3+i)$, where $3+i$ is the complex associate of point B in Figure 3.28. Even so, it may be advisable to go through this step-by-step development on the board or the overhead projector. The product $(2+3i)(3+i)$ is, by distributivity, the same as

$$2(3+i) + 3i(3+i).$$

In the text the steps are done in this order:

$$2(3 + i) + i(3(3 + i))$$

(1) Dilation

(2) Dilation

(3) Quarter-turn

(4) Sum of points

Students should note however that--except for the sum of points, which of necessity must be last--the order of the steps is arbitrary. It may be that you will want to repeat this step-by-step procedure--or ask the class to--for other points and other complex number multipliers.

One question students may ask--and if they don't, you should--is how one would determine that $2+3i$ is the complex number associated with the spiral similarity in Figure 3.28. The question deserves an answer, but not until they have worked on it themselves. (See Exercise 7).

With the introduction of the spiral similarity we now have a geometric model for both addition and multiplication of real numbers, and for both addition and multiplication of complex numbers. You may wish to recap these with the class:

1) Addition of real numbers--Translation on a line
2) Multiplication of real numbers--Dilation on a line
3) Addition of complex numbers--Translation in the plane
4) Multiplication of complex numbers--Spiral similarity.

<u>Possible Assignment</u>

All of the exercises should be assigned.

MAINTAINING SKILLS AND UNDERSTANDING

I and II

There are two sections, I and II, the first following Chapter 3 and the second following Chapter 7. The problems are arranged in the same order as the development in the textbook and are listed under the chapter headings.

These sections do not constitute separate units of study. They are intended for review, practice, enrichment, and adaptation to the individual needs of your students. There are challenging original exercises for the best students, and simple applications and drill exercises for those who need more practice. Not all problems should be required of all students.

At the end of the study of a chapter in a book, the section in these Maintaining Skills and Understanding (MSAU) may be used judiciously as further practice. They can also serve as subsequent study of other chapters as a way of renewing and reinforcing learning. Thus as the study of chapters 4 to 7 is carried on, selected problems from MSAU I can be assigned from time to time. Similarly as chapters 6 and 7 are studied, the teacher can use MSAU II exercises for renewing and keeping up the concepts of chapters 4 and 5.

The teacher should acquaint herself or himself with the solutions of all the problems in both sections before making individual assignments. By working through the solutions the teacher can readily discover the degree of difficulty and thereby determine which exercises best meet the needs of his individual students. It will also give the teacher an idea of the amount of student time the assigned problems will require for solution, enabling the teacher to make reasonable assignments. Roughly, the time required to do <u>all</u> the problems of both sections of MSAU would be three to five weeks of student study, i.e. 15 to 25 clock hours.

The primary use of these exercises is to gain self-reliance--assurance that one has a grasp of what has been developed and studied in the textbook and in class. As such these sections can aid in having the student tackle the problems from time to time, on his own, to see how well he can do and how well he remembers. The answers, and some complete solutions are given in the Teachers Commentary, Part III.

CHAPTER 4

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

This chapter introduces exponential and logarithmic functions and highlights the exponential function $f: x \to e^x$, $x \in R$. The spirit of the development of the two types of functions is consistent with the modelling theme of this textbook--we hope to motivate the study of mathematics through applications.

A recurring emphasis throughout the chapter is on functional properties and the relationships among functions. Students are asked to sketch many variations of exponential and logarithmic functions and to compare results. The "transformational" background from Courses I-III should be reviewed and used wherever possible. (For example, I can sketch the graph of $y = -e^{x+1}$ easily once I know the graph of $y = e^x$; simply translate and reflect.)

Exponential functions, in particular, model growth and decay phenomena (bacteria and cellular growth, radioactive decay). A compound interest problem is used in Chapter 4 to suggest the general mathematical formula for representing growth or decay situations as well as to motivate the existence of the real number e.

SUGGESTED TIME SCHEDULE

4.1	Integral Exponents	3 days
4.2	Rational Number Exponents	2 days
4.3	Real Number Exponents	2 days
4.4	Compound Interest and the Exponential Function	2 days
4.5	Inverses of Exponential Functions--Logarithmic Functions	2 days
4.6	Using Logarithmic Functions	3 days
4.7	Summary	2 days
	TOTAL	16 days

4.1 INTEGRAL EXPONENTS

Purpose

The purposes of section 4.1 are to review the rules for calculating with exponential forms (a^x), where x is a natural number, and to introduce integer exponents. By considering the growth of a rabbit population, the student sees that an exponential function can be used to model a "real" situation. The population function for the rabbits is used to suggest definitions for

$$a^x, \ x \in \text{Integers}$$

and for

$$a^x, \ x \in \text{Rational Numbers (Section 4.2)}$$

Getting Started

Begin by discussing the many ways that growth and decay may occur. For example, one way to accumulate savings is to place five dollars under a mattress each and every month for a period of years. If this were done for n years, the accumulated amount would be given by

$$A = 60n,$$

a linear function. At the end of 2 years, 120 dollars would be saved.

Another procedure would be to place, say 2 dollars under the mattress at the end of the first month, 4 dollars at the end of the second month, 8 at the end of the third month, and so on. Have students guess at the accumulated amount at the end of the 2nd year; this amount is given by

$$A = 2(2^{12t}-1), \text{ where } t = 2$$

an exponential function. (The total would be $8,388,608.) Continue with development in the textbook.

Using the Text

Use the population function $p:t \rightarrow 160 \cdot 2^t$ to motivate a review of the rules for calculating with exponential forms, where the exponent is a natural number. Although the students have studied exponents and operations using exponents in Courses II and III, take time to carefully review the rules for they will be systematically extended in subsequent sections.

Make sure that students understand that Table 2 was constructed <u>without</u> using the function $f:t \rightarrow 160 \cdot 2^t$. The entries in the table can be used to suggest values for 2^t where $t \in \{-1,-2,-3,\ldots\}$.

Assignments

There is enough content in this first section to require that at least two assignments and three days be spent on integer exponents, calculating with exponents (Exercises 1-35), and situations that are modelled by exponential functions (Exercises 36,37,38,41, and 42). In addition, Exercises 39 and 40 introduce important properties of exponential functions and should be assigned. Exercise 45 could be assigned for outside reading with student reports at a later date. The interesting thesis of Vaughan's article is that there are situations that suggest defining $0^0 = 1$.

4.2 RATIONAL NUMBER EXPONENTS

Purpose

To extend the domain of the exponential function to the set of rational numbers and to verify that the rules for calculating with exponents are still valid.

Getting Started

Compare the growth situation suggested for Section 4.1 (put 2,4,8,... dollars under your mattress; how much will you have at the end of two years?) with the situation of the doubling rabbits. By discussing growth (or decay), in general, students will see that growth does not always occur at the end of an "integral interval."

Using the Text

Definition 2 for rational exponents is motivated by referring to the doubling rabbits. The answer to the Discussion Question preceding Definition 2 is:

t	-2	-1.5	-1	-.5	0	.5	1	1.5	2
P(t)	40	56.59	80	113.15	160	226.24	320	452.34	640

If students have studied BASIC, you could make the preceding a "computing" exercise.

In discussing Definition 2, make sure that students understand that negative bases are excluded here only because we want exponential functions to be real functions.

The fact that $x^n = b$ has exactly one positive number solution is crucial to the definition. Although we don't prove it, one way to suggest the result is to have students sketch, say

$$f(x) = x^n - 4$$

for $n = 2,3,4,5,6$. In each case, there will be but one x-intercept corresponding to the unique positive solution of $x^n = 4$ (for $n = 2,3,4,5,6$). In general

$$f(x) = x^n - b$$

can be handled similarly.

The proofs that the rules for calculating with exponents, where the exponents are rational numbers, are quite subtle and should be handled with care. The proof of Lemma 1 is optional, but it does illustrate a nice use of geometric sequences and sums. Together with Lemma 2, the rules can be proved as in the Class Discussion preceding Example 2.

Assignments

Most of the exercises should be assigned with perhaps Exercises 30, 31, and 32 saved for class discussion. Exercise 32 is needed for an exercise in the next section.

4.3 REAL NUMBER EXPONENTS

Purpose

To extend the domain of exponential functions to the set of real numbers; to introduce two new growth situations that are modelled by exponential functions.

Getting Started

Begin by considering $f:x\to 2^x$, $x\in Q$ and ask questions about expressions like $2^{\sqrt{2}}$, 2^{π}, etc. Use the graph of $f:x\to 2^x$, $x\in Q$ to subbest an interpretation for 2^x, x irrational.

Using the Text

Although the graph of $f:x\to 2^x$, $x \in Q$ strongly suggests that "irrational powers" ought to be approximated by rational powers, the definition of an irrational power requires a little work to develop. You will need to recall some vocabulary from Courses II and III; in particular, review the ideas of upper bound, lower bound, least upper bound, and greatest lower bound. These terms were introduced and used to help characterize the real number system in Course II. To review the preceding ideas you could give an exercise like:

Consider the sequence .3, .33, .333, ...

1) In this sequence $a_1 = .3$, what is a_6? a_n?
2) On a number line locate as carefully as possible the first three terms of the sequence.
3) Determine one number that is greater than or equal to any term of the sequence. How many numbers have this property? Answer: One number would be 5. There are an infinite number that would satisfy; they are all upper bounds for the sequence.
4) From the set of all upper bounds, is there a smallest? What is it? Answer: Yes, it is $\frac{1}{3}$, the least upper bound.

By examples, the idea you want to stress is that every increasing sequence of real numbers that is bounded from above, must have a least upper bound (in R).

Go over the Class Discussion Questions carefully. To do Number 1 requires the use of a computer or a relatively powerful electronic calculator. Logarithms can also be used, but they are not introduced until Section 4.6. The answer to Number 2 is subtle. Allow the students to assume that $2^x > 0$ for all rational x (Exercise 13 asks for a proof). Since an irrational power can be approximated by an increasing sequence of positive real numbers which, therefore, has an lub which is positive, the result follows.

Examples 2 and 3 fit together. Mitosis is the process by which the number of cells grows; the growth is exponential. Compound interest is a process by which money grows exponentially.

Assignments

Although Exercise 13 is starred as optional, you should at least discuss the proofs in class. Exercises 16-18 give other growth situations for students to consider with Exercise 18 being particularly relevant in 1974 as the United States enters a period of "zero population growth."

4.4 COMPOUND INTEREST AND THE EXPONENTIAL FUNCTION

Purpose

To discuss compound interest; to introduce the real number e and the exponential function $f: x \to e^x$, $x \in R$; to consider situations where growth or decay is approximately continuous.

Getting Started

Ask students what they know about banks and the way interest is computed. Some banks pay interest more frequently than others; the results given in Example 1 may surprise some students.

Using the Text

Discuss Example 1 thoroughly. The differences among the accumulated totals are surprisingly small and should motivate some discussion about where to bank. (Should I drive the extra miles to get to a bank that compounds more frequently than mine does?) From Example 1, the compound interest situation is used to:

1) define e
2) give a general formula that models continuous growth or decay situations.

As in Section 4.3, least upper bounds play a role in the development.

Compare the result from Example 4 with that of Example 1. Discuss.

Assignments

Exercises 1-14 should all be assigned, perhaps over a number of assignments. Exercises 15-17 set the stage for the next section and should be discussed. Note how Exercises 15 and 16 together prove Exercise 17.

4.5 INVERSES OF EXPONENTIAL FUNCTIONS-LOGARITHMIC FUNCTIONS

Purpose

to introduce logarithmic functions as inverses of exponential functions; to give meaning to the symbol $\log_b$.

Getting Started

Begin by reviewing what one means by an inverse. In general, for a group, combining an element with its inverse produces the identity for the group. Consider $(Z,+)$, $(R^+,\cdot)$, $(Z_3,+)$ and so on. If a group consists of functions, then the identity is a function.

Using the Text

Examples 1-3 give a sequence that leads to a mathematical definition of a logarithmic function. The discussion following the definition is particularly important, for in a systematic way it gives meaning to the symbol $\log_b$. Emphasize Example 5. You should expect students to achieve operational skills in changing expressions from exponential form to logarithmic form and vice versa.

Assignments

Exercise 2 is important for it gives a convenient curve sketching fact about functions and their inverses. A proof of the fact is Exercise 10 which could be assigned before Exercise 2. One proof of Exercise 10(b) is as follows:

Let $A = (a,c)$ and $A' = (c,a)$. Then for any $P = (x,x) \in \{(x,y):y=x\}$,

$$AP = \sqrt{(a-x)^2 + (c-x)^2} = A'P$$

and so all points on $\{(x,y):y=x\}$ are equidistant from A and A'. This makes

$\{(x,y):y=x\}$ the perpendicular bisector of $\overline{AA'}$. Exercises 11-14 anticipate the next section.

4.6 USING LOGARITHMIC FUNCTIONS

Purpose

to introduce the functional properties of $\log_b$, logarithms base 10, and tables of logarithms.

Getting Started

Begin by saying that we know a lot about $f:x \to b^x$, $x \in R$

1) f is a 1-1 mapping of $(R,+)$ onto $(R^+,\cdot)$

and

2) $f(x+y) = f(x)\cdot f(y)$; i.e. f preserves structure (since $(R,+)$ is a group, so is $(R^+,\cdot)$).

What about $\log_b$? Follow the development in the textbook.

Using the Text

Don't expect many students (if any) to recall that if f is an isomorphism, so is f^{-1}. The basic functional property of $\log_b$, that is that

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

comes from the fact that $\log_b$ is an isomorphism because $f:x \to b^x$ is. You may wish to mention this; if not, discuss the proof in the text that is independent of the isomorphism concept. There is quite a lot developed in this section and so three days is probably a reasonable amount of time to spend here.

Today, logarithms are used primarily to help solve exponential equations that usually represent growth or decay phenomena. Hence very little is done in this section on the "usual" computational applications of logarithms. In any case, one still needs to be able to find a logarithm of a number by using a table as shown by Examples 8 and 9.

Assignments

Assign all exercises over a number of assignments. Exercises 10-12 complete the chapter by showing how exponential functions and logarithms can be applied.

CHAPTER 5

CIRCULAR FUNCTIONS: PERIODIC MODELS

Though exponential and logarithmatic functions are growing in importance as models for phenomena in the life sciences, the circular functions remain an indispensible tool for theory and application in the physical sciences. Use of the various right triangle ratios like sine, cosine, secant, and tangent dates back to the astronomy of early Greece, India, and possibly Mesopotamia. While this use of trigonometry to solve triangles remains important, contemporary applied mathematics places even greater emphasis on the periodic extensions of these functions with domain the entire real line--most commonly representing time.

The purpose of this chapter is to develop the definitions and basic properties of the common circular functions and their inverses, using the need to model periodic phenomena as the motivating theme. The chapter begins with a modeling problem designed to quickly review the definitions of sine and cosine and sensed angle measure that were introduced in Course III. Then situations involving tides, circular motion, and alternating current electricity are described to motivate the extension of those functions. The facts that follow are much the same as any basic trigonometry course. Several unusual features should be emphasized. First, in discussing the symmetries of circular functions and the variations such as A sin(CX + D), make use of transformational geometry background the students have from previous courses. Second, several of the fundamental identities are proven using properties of matrices and transformations. Standard proofs can be given in some cases, but the matrix proof is often quite elegant and makes contact with previous ideas in a non-trivial way. Third, the theme of developing mathematical ideas to help describe and predict (i.e. model) physical phenomena is a major thrust of the chapter. Several of the sections on electricity can be skipped without destroying the mathematical continuity, but it is the authors belief that students would miss an important illustration of how mathematics is useful. If the explanations of electrical phenomena seem inadequate, call in a physics or electronics teacher to demonstrate and explain what is going on. The justification for including this material is not an assumption that all students will someday be electronical engineers, but that electrical properties are so fundamental to our modern technology and it is nice to know a little bit about how they operate.

At the end of the chapter students should possess the following basic skills:

1) Ability to determine the period, amplitude, frequency, and graph of A sin (Cx + D) and A cos (Cx + D) along with tangent, cotangent, secant, and cosecant.

2) Fit a circular function to a graph or given boundary conditions of a periodic function.

3) Translate coordinates of points from cartesian to polar or trigonometric complex number form and back.

4) Describe several examples of physical situations modeled by circular functions.

5) State the addition, double angle, and half angle identities for sine and cosine, using these to derive further identities.

SUGGESTED TIME SCHEDULE

5.1 Radar, Sin x, Cos x	$1\frac{1}{2}$ days
5.2 Periodic Functions as Models	$1\frac{1}{2}$ days
5.3 Properties of Cos θ and Sin θ	1 day
5.4 Variations and Applications for Cos θ and Sin θ	3 days
5.5 Rotations, Matrices and f(x+y)	1 day
5.6 Cos 2x, Cos $\frac{1}{2}$x, Sin 2x, Sin $\frac{1}{2}$x	1 day
5.7 Polar Coordinates and Trigonometric Form	1 day
*5.8 DeMoivre Meets $z^n=a$	1 day
*5.9 Electrical Circuits and Circular Functions	2 days
*5.10 Electricity, Complex Numbers, Circular Functions	1 day
5.11 Tangent and Cotangent Functions	$1\frac{1}{2}$ days
5.12 Circular Functions and Equations	2 days
Summary and Review Exercises	$1\frac{1}{2}$ days
TOTAL	19 days

*Optional Sections covering 4 days.

For an excellent general reference of illustrations for the circular functions and their uses consult The Man-Made World published by McGraw-Hill.

5.1 RADAR, SIN x, COS x

Purpose

The purpose of this section is to use a modelling situation to motivate review of the basic definitions of sin, cos, and radian measure. Students should be able to convert degree measure to and from radian measure, and know the ratios of sides in the right isosceles and 30-60-90 triangles, and know the coordinate definitions of sin and cos.

Getting Started

Students probably have seen radar used either live or on films in many settings. Ask them to contribute explanations and particularly probe for the type of information given by a radar scanner. In almost every case the radar locates an object by distance and angle with respect to a reference line. For instance, naval stories involve statements like "Aircraft 25 miles at 270 degrees." In this case the reference ray is due north; point out that this choice is a convention, and due east is equally common.

Using the Text

Recall the definitions of radian and degree measure for sensed angles. Then use 1-10 as class oral exercises, pointing out the special ratios in 45-45-90 and 30-60-90 triangles.

Other Activities

Since radar really deals in polar coordinates, you might want to play the game "Tic-Tac-Toe in Polar Coordinates" described by Joseph Brone in the Mathematics Teacher, February, 1974.

Possible Assignments

A selection from 1-15, all of 16-20, 21, and several from 22-25. Then 31, 32 lead into the next section.

5.2 PERIODIC FUNCTIONS AS MODELS

Purpose

This section motivates the extension of circular functions with several periodic phenomena described. The definition of sine and cosine makes use of an informally described wrapping function.

Getting Started

One way to begin is to have students do exercises 31 and 32 from 5.1 and then read the first three examples of 5.2 prior to class time. Discuss the examples, including emphasis that the tide data will be only roughly approximated by simple functions.

Then explain the wrapping function, the ferris wheel example being perhaps the best example of why curvilinear motion might be converted into vertical and horizontal position components in the way the wrapping function does it.

Using the Text

When the wrapping and extended sine and cosine functions have been defined, go over a selection of exercises from 1-13 as oral class exercises.

Other Activities

Using a circular disk of radius 1 foot (circumference about 6.28 or $6\frac{1}{4}$ feet) wrap a tape measure around the circle, checking x and y coordinates at various points along the way. Particularly important multiples of pi.

Have the science teacher come in and demonstrate electrical periodicity using an oscilloscope or a magnetic generator like that pictured in the text.

Possible Assignements

Most of the exercises in the section should be done. Note particularly how 21 shows the use of unit circle coordinate functions in any similar right triangle.

5.3 PROPERTIES OF COS θ AND SIN θ

Purpose

To illustrate the fundamental periodic and symmetric properties of these functions.

Using the Text

This should be a very easy section for students to get on their own by reading the text.

Possible Assignments

Exercises 1-5, several from 6-10 (perhaps different problems by different class members with results shared before the rest of the class), 11, 13, 14. Exercises 15-23 introduce odd and even functions which you might choose to emphasize. Exercises 26-32 review the law of cosines.

5.4 VARIATIONS AND APPLICATIONS FOR COS θ AND SIN θ

Purpose

Since real life uses of periodic functions seldom have amplitude 1 or period 2π this extremely important section explains how simple variations on sin and cos can be used to model more complex situations. At the end of this section students should be able to completely describe and graph A sin (Cx + D) and A cos (Cx + D).

Getting Started

To demonstrate variations in amplitude and period, make a simple pendulum with string and weight like a fishing sinker. For a fixed length pendulum (string), the period is independent of initial displacement. That is, the angle of deflection from vertical is a periodic function of time with amplitude the initial deflection and the deflection is 0 and max and min at the same time regardless of the initial deflection made. Have students experiment with this, sketching graphs of several variations until they see that the only difference is how high and low the graphs go. Period of such a pendulum is given by

$$p = 2\pi\sqrt{\frac{h}{9.8}}$$

where h is the pendulum length in meters. Thus if you make a pendulum of 1 meter you will get period of about 2 seconds. Next vary the length of the pendulum; say, using a length of .25 meters. This will yield a period of about 1 second.

Now ask the students the more general questions:

1) Give the rule of a function that is just like cos only it reaches a maximum of 2 and a minimum of -2.

2) Give the rule of a function that is just like sine only it has period π.

Using the Text

The composition of functions needed to develop the variations of sine and cosine needed will not be easy for students. They need to study and work plenty of examples with teacher guidance. Perhaps on the first day work through Example 1 and then go to exercises. Return to Example 2 on the next day.

Other Activities

Again, call on the science teacher to demonstrate physical phenomena with different periods, frequency and amplitude.

Possible Assignments

On Day 1, do Exercises 1-6. On Day 2, do Exercises 8-20. Then on Day 3, tackle 21-25.

5.5 ROTATIONS, MATRICES, AND f(x+y)

Purpose

This section derives the basic addition formulas for sine and cosine from which so many other identities can be derived.

Getting Started

Lead the class through discovery of the effect of the special matrices like $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ as transformations of the plane. Have students calculate the matrix products given in Example 1, without looking at the text. Then have them guess the geometric effect.

Using the Text

Have students study the proofs of Theorems 1 and 2 so that they can give justifications for each step. Then work several applications from Exercises 1-5 as class practice work.

Other Activities

Some students might want to check the identities given in Theorems 1 and 2 with a computer program. Remind them that the BASIC circular functions assume radian measure--or really real number domains.

Possible Assignments

Most of the exercises 1-24 should be done to give adequate practice. Then 25-27 make more contact between trigonometric functions and matrices (optional). Exercise 31 is an important class discussion activity emphasizing the unified nature of the course.

5.6 DOUBLE AND HALF ANGLE FORMULAS

Purpose

This section derives the indicated formulas and shows how they can be used to interpolate between known values of cos and sin. It also introduces the idea of an identity and gives some practice in deriving identities. It is the position of the authors, however, that extensive drill on proving identities is not of fundamental importance. Therefore, if your taste is different you might want to supplement the chapter with examples drawn from standard references.

Using the Text

Students can probably quite easily read this short section on their own and begin work with the exercises.

Possible Assignments

Exercises 1-10 and 12-15 are basic practice type.

5.7 POLAR COORDINATES AND TRIGONOMETRIC FORM

Purpose

To develop these two alternatives to cartesian coordinates and develop student facility in working back and forth between forms.

Getting Started

Pose the following problem for the class: How would radar locate the point with cartesian coordinates (15,36)? Guide them to calculation of the distance from origin and the point where the radar beacon intersects the unit circle. They will have to use the table of circular functions to estimate θ.

Using the Text

After running over Definition 6 and Example 2, lead students through the derivation in text of the rule for multiplying complex numbers in polar or trigonometric form. Check to see the commuting dilation and rotation is reasonable to the class.

Other Activities

The matrix proof of complex multiplication can be replaced by the standard argument using addition formulas for practice.

Possible Assignments

1-10, 14, 15, 19 give basic ideas. If you want to review ideas of geometric transformations, do 16-18, 21, 22.

*5.8 DEMOIVRE MEETS $z^n = a$

Purpose

This section is designed to lead students to discovery of DeMoivre's Theorem on powers of complex numbers in trigonometric form and its application to finding nth roots.

Using the Text

The text of this section is written for student discovery on their own or in small groups. It can be used as the outline for a class discovery discussion, too. You will probably want to check at the end of the discovery period that students indeed came up with the correct result!

Possible Assignments

This section is optional, so perhaps 1-5, 8, 11, 17 and 18 are enough. Note how 17 and 18 reestablish contact with groups.

*5.9 ELECTRICAL CIRCUITS AND CIRCULAR FUNCTIONS

Purpose

To demonstrate the way that mathematical formulas model flow of electricity in circuits, explaining the connection between alternating current and terms such as 120 volts, 3 amps, etc.

Getting Started

The effectiveness of this section will be greatly enhanced if the electrical principles enumerated can also be demonstrated, beyond the plausibility argument suggested by water flow in a pipe. Make a visit to the electrical shop teacher or the physics teacher for some wires with alligator clips, some resistors, and a volt- or ammeter. Better yet, poll the class to find out if you have some hi-fi or electronics buffs and let them plan a demonstration.

Using the Text

Have students read the text after any demonstration. Then work through several circuits in Exercises 1-10.

Possible Assignments

Exercises 1-10, 16-23. Exercise 11 gives a hint of integration by approximation. It might be done as a class exercise.

5.10 ELECTRICITY, COMPLEX NUMBERS, CIRCULAR FUNCTIONS

Purpose

If you can handle the electrical facts of this section very informally--simply acknowledging that "this is the way it happens"--then the mathematical development will be a beautiful example of unification.

Getting Started

On separate overhead transparencies sketch graphs of two functions--perhaps two periods of sin x and cos x. Then superimpose the two graphs and let the class guide you to a sketch of the sum of the two functions. If you take sin and cos you will notice that the result is periodic with max and min in different places than either of the components. Then casually say to the class that in some kinds of electrical circuits the impedance to flow of current is often the sum of two such out of phase separate impedances. An analogy might be that water coming through one valve surges like sine, but a second type of impedance in the pipe sends water around a loop before it goes on, delaying the surge from its usual peak time.

Using the Text

Now work through the proof of Theorem 6 and have students read the paragraph on complex numbers and electricity that follows it.

Other Activities

If you feel very uncomfortable with the electronics, skip the section!

Possible Assignments

Some of Exercises 1-15 will be routine practice with circular functions and the distance function. Exercises 16-18 illustrate an important relation between graphs of functions and their translates.

5.11 TANGENT AND COTANGENT FUNCTIONS

Purpose

To define and develop basic properties and graphs of these two functions. Secant and Cosecant are mentioned at the end of the section.

Getting Started

Present students with a copy of Figure 5.75 with coordinates of point P' omitted. Have them figure out what those coordinates ought to be. Then give the definition of tan x and pose the next problem of determining the graph and period of tangent.

Using the Text

By this stage of the chapter students should be able to generate the basic ideas of the section on their own, reading the text for confirmation. After checking the definitions of cot, sec, csc, work on several of Exercises 1-20 as oral practice in class.

Other Activities

A more general challenge to the class at the outset might be to say simply: O.K. you've got these two basic functions sine and cosine. What operations should be perform on them to generate new functions? Sum, difference, and constant multiple have been done already. Product and quotient and reciprocal remain to be investigated.

Possible Assignments

Exercises 1-20 are simple, but useful practice with the definitions. Then 27-30 and some identities from 36 would be a good thought type. In class the next day tackle the discussion question 35 and 37.

5.12 CIRCULAR FUNCTIONS AND EQUATIONS

Purpose

This section demonstrates trigonometric equations, a few techniques for solution, and an informal introduction to the inverse circular functions.

Getting Started

Draw the graph of a function--not necessarily one with a particular algebraic rule--on the overhead or chalk board. Then ask students to demonstrate graphically how they would solve various equations of the type "f(x) = a." Then take a periodic example like cos x and ask to solve the equation when a = .5. Point out the multiplicity of solutions.

Using the Text

Then have students read Examples 1-3 and begin work on some exercises from 1-20. Perhaps splitting the problems among the class members would see that the maximum number were worked and illustrated for all to see.

Possible Assignments

The practice in Exercises 1-20 and applications in 26-28 is basic. Inverse functions will not really be needed until well into calculus.

CHAPTER 6

CONDITIONAL PROBABILITY

This chapter reviews previous study of combinatorics and probability theory and extends the study to conditional probability and independent events. The new major probability concepts that occur are

Conditional probability and its symbol $P(A|B)$
Partitioning of the probability space
Total probability
Independent events

Each of these basic ideas are introduced by an illustrative example upon which the mathematical theory is developed. This is followed by a number of illustrative examples of the theory and a set of exercises in which the theory and application are extended. Much use is made of tree diagrams as an analysis of the probability space. A diagrammatic representation of partitioned space is pre-requisite and essential for the understanding of what appears to be a very complex algebraic formula. In fact, it is recommended that most teaching of the chapter be done in the sequence--investigating a situation, making a diagram, solving several simple examples informally, developing the probabilistic theory (theorems and formulas), more involved examples, and finally the exercises. The ultimate goal is to have the students solve problems by recognizing the probabilistic situation, making a suitable diagram, and applying the theory--not just substituting, in a formula.

SUGGESTED TIME SCHEDULE

6.0 Introduction	1 hr.
6.1 Counting Outcomes and Assignment of Probabilities	3 hrs.
6.2 Conditional Probability	4 hrs.
6.3 More on Conditional Probability	4 hrs.
6.4 Independent Events	2 hrs.
Summary, Review, Text	2 hrs.
TOTAL	16 hrs.

Two references for further problems are (a) Probability by G.E. Bates and Fifty Challenging Problems in Probability by F. Mosteller, both published by Addison-Wesley Company. For class and individual use it will be most useful to have coins, spinners, cubical dice, tetrahedral dice, and statistical data on categories of events, car accidents, births, medical treatment, and the like. The use of prepared visuals on probability spaces with an overhead projector is a real time saver in class instruction.

6.0 INTRODUCTION

Before beginning this chapter, the class should have been assigned to read and come to class prepared to discuss the gambler's problem. In the discussion, raise the question why it is necessary to consider the playing of nine games in order to arrive at a meaningful answer. However, do not solve the problem at this time. As an added example propose another conditioned probability situation similar to the following. The probability that a whole number less than or equal to 1000, selected at random, is divisible by 5 is of course $\frac{1}{5}$. Now what is the probability that the number is divisible by 5 given that the number selected is an even number? Why is the answer again $\frac{1}{5}$? Try the same problem using divisible by 10 and divisible by 4. Why are the two probabilities different? They are $\frac{1}{10}$ and $\frac{1}{5}$. Again, allow the answer to be undecided until the end of Section 6.2.

A plausible assignment is to study Section 6.1 as far as Example 1.

6.1 COUNTING OUTCOMES AND PROBABILITY ASSIGNMENTS

Purpose

The main purpose of this section is to review the previous study of permutations, combinations, and elementary probability. All of these concepts are a basis for the study of the rest of the chapter. In particular, the aims are:

1) To develop the concept and ability to create a probability space;
2) To develop sophisticated counting and the use of permutations, $(n)_r$;
3) To develop the concept of combinations and their use in counting $\binom{n}{r}$;
4) To review the use of factorial notation, $n!$;
5) To combine aims 1) to 4) in real probabilistic situations.

Open the first class discussion with the problem on the throwing of two dice. Review the concept of "equally likely events." Present a case of throwing one die and spinning a dial around five equally spaced sectors (of a spinner), numbered 1 to 5, and considering the representation of several outcome spaces, and the manner of assigning probabilities to each outcome. This should lead to the fundamental counting theorem by using the die face first, the spinner dial second, and vice versa. Then develop the fundamental theory of permutations and the accompanying symbolization and use of factorials.

Assign a selected set of exercises at the end of this section as outside of class work.

The second class hour can start with the development of the theory of combinations and developing the combination formula. Examples 3 and 4 should be reviewed in class and the instructor should supply several new problems calling for the use of both permutations and combinations (see Chapter 6 of Course III). The outside of class assignment should be selected problems from Exercises 1-30 at the end of the section. A few of these can be done as a class exercise with all students participating in the solutions.

A third class session should be given over to reviewing the fundamental concepts and theorems of a probability space. Here the teacher can use pages 243, 244, Course III stating the properties as theorems to be proved. Then Exercises 35 to 37 can be assigned and in all cases Venn diagrams should be used to guide the study. Uncompleted problems should be assigned as out of class work.

6.2 CONDITIONAL PROBABILITY

The essential idea of conditional probability lies in the concept of a reduced outcome set and its symbolic representation as a probability space by $P(A|B)$. There are two key goals in the introduction (1) to show that $(S, P(A|B))$ is a probability space and hence we can apply all previously proved probability theorems to this space and (2) the conditional probability can be obtained directly from the probability assignments in the original outcome space, S.

Start with an illustration, not in the textbooks, e.g. the probability that in tossing a die, a two is face up, given it is an even number, or that a card drawing from a deck is a heart given that it is a nine; or that a random counting number < 101 selected at random is divisible by 7, given that the number is odd, etc. In each case draw the sample space, show the events and the subspace and the two ways of finding $P(A|B)$. Close your discussion with the development and use of the formula

$$P(A|B) = \frac{P(A\cap B)}{P(B)}\ .$$

Assign as homework, a study of the Section 6.2 through example number 4.

In the second class hour review the assignment to be sure it is understood. Go over examples that show the relations

$$P(A|B) \lesseqgtr P(A)$$

The equality relation will be important in Section 6.4. For the remainder of the class period teach the use of trees in depicting a sample space. Take example 5, 6, and 7 and similar examples from other sources. Assign the study of the textbook through Example 7 and Exercises 1,2 and 3.

The third class hour can be given over to clarifying any of the previous study, as well as taking up selected exercises at the end of the section, e.g. Nos. 6, 7, and 18. Assign Exercises 2,4,5,8,9,10,12,14. Use remaining exercises for additional work in class or at home.

6.3 MORE ON CONDITIONAL PROBABILITY

The essential aim of this section is to relate conditional probability to the sample space. This will enable us to work two ways, (1) given several $P(A|B_1)$, $P(A|B_2)$ etc., we find $P(A)$ or (2) we can find $P(B_i|A)$ from the $P(A|B_i)$ for any number of partitions B_i.

Begin in the first hour by using the textbook example opening this section. Analyze the probability by ordinary selection, then refer to partitioning. Develop the total probability (for partitioning into 2 parts) and apply the formula. Follow this with Example 1 and the proof of the total probability theorem 2. Assign the study of Example 2 and make a tree diagram for it. Also Exercises 1,2 and 3 at the end of the section.

During the second class hour, review the home assignment and then take up the illustration following Example 2. Do not stress only the memorization of the formula but rather the construction of tree diagrams and their use via the formula. Be sure all branches are labelled correctly as $P(B)$, $P(A_i|B_j)$ etc. Study Example 3 and assign Exercises 4,6,7,8 and *5 for extra credit.

In the third class hour review the home assignment, redevelop the total probability formula and do Exercise 9 in class as an example. Assign Exercises 10,11,12, and give a review and summary of the section.

6.4 INDEPENDENT EVENTS

The chief goal of this section is to define independence. Proceed in the order of proving

if $P(A|B) = P(A)$ then $P(B|A) = P(B)$;
then consider examples where intuitively events A and B appear to be independent of each other and

$$P(A) \cdot P(B) = P(A \cap B);$$

finally postulate the formal definition. Note that at some time we can tell that the events are independent, but that at other times we make the test. Note also that the dependence may be so slight that we can use independence to find an approximate answer. Assign the study of the section as far as Example 3 and Exercises 1,2,3,10 and 11.

The second class hour can begin with a study of Example 3 and the proof of Theorem 4. Examine examples 4 and 5 and stress the use of 2 x 2 category tables. Define independence for three events and take up at once Exercises 14 and 15 at the end of the chapter. Assign a selected set of exercises from 5 to 14 and do the remaining unassigned exercises in class.

REVIEW AND SUMMARY

Use the first class hour to review and summarize all major concepts and formulas of the chapter. For each idea written on the board, the student should be able to give a definition, or description or use that indicates understanding. Begin the review exercises as class work, with the total class as an operating team, and assign study of the chapter and working of remaining exercises as outside work.

CHAPTER 7

VECTORS, VECTOR SPACES, AND GEOMETRY

The purpose of this chapter is to extend students' understanding of geometry and of some fundamental concepts of linear algebra.

The precise meaning of a vector cannot be given until a vector space has been defined. Therefore, we begin by considering two models of vectors and vector spaces--arrows that represent displacements and n-tuples of real numbers that represent coordinates in space. The underlying structure of these two models suggests a definition for a vector space and proceeds by concentrating on R^2 and R^3, coordinatized 2-space and 3-space, respectively.

As you teach this chapter be aware that all theorems and applications are true in R^n although for the most part they are illustrated for R^2 and R^3. This fact should be pointed out to students, but at this stage it is not necessary to go beyond 3-space. The geometry (lines and planes) and the algebra (vector spaces and subspaces) reinforce each other nicely in 2- and 3-space and hence should be emphasized.

The following are some recommendations for teaching Chapter 7:

1) Be careful not to assign too many exercises for one night's assignment. If you have taught vector geometry, than you know that often there is not a unique vector representation for a figure in, say, 3-space. Because there are many possible answers for exercises in this chapter, checking homework becomes an important part of learning.
2) In line with 1), consider, spirally, homework exercises on a particular topic rather than giving them in one concentrated dose.
3) If your students have access to a computer system, apply the BASIC skills to help solve some of the systems of linear equations of Section 7.6.

SUGGESTED TIME SCHEDULE

7.1 Introduction	1 day
7.2 Arrows and Mathematics	2 days
7.3 Lines in R^2 and R^3	3-4 days
7.4 Linear Combinations and Planes	3 days
7.5 Non-Parametric Representations of Lines and Planes	3 days

7.6	Vector Spaces, Geometry, and Systems of Equations	2 days
7.7	Measurements in Space	2 days
7.8	Applications of the Inner Product	2 days
	Summary	1 day
	TOTAL	19-20 days

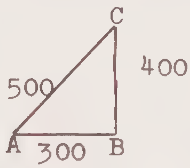
7.1 INTRODUCTION

Purpose

To motivate the use of "arrows" to represent phenomena that have magnitude and direction; to use arrows to help solve problems from architecture, the automotive industry, and aviation; to use applications of arrows to motivate their subsequent study (mathematically).

Getting Started

One way to begin is to recall that the students were introduced to arrows and vector spaces in Course III, Chapter 9: Vector Spaces. The major emphasis there was on arrows and their applications in 2-space. Ask students to consider a plane that flies from A to B, 300 miles east of A and then to C, 400 miles north of B. An equivalent displacement is to fly directly to C as given below:



Recall that $\overrightarrow{AC}$ is called the resultant, the arrow sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$; i.e.

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}.$$

Note that the arrow sum of the two vectors may be found by completing a triangle, if the arrows are located as in the preceding figure. Proceed with Example 1 in the textbook.

Using the Text

In the first two sections, we are careful to represent displacements with arrows and to call them arrows, not vectors. Although it is true that we may also call certain types of arrows vectors, we do not want to introduce the association too soon. Technically, a vector is an element of a vector space and so vectors may be arrows, ordered pairs, functions, ordered triples, and so on. We do not want students to think that vector means arrow and that's all it means. Of course, historically the term vector first arose in connection with displacements and so we do chose to begin this chapter with some applications. By the end of the next section, arrows will be called vectors because they are--they are elements of a vector space.

Discuss Examples 1 and 2, highlighting the application of mathematics to help solve the two problems. You'll probably need to review some trigonometry, especially the laws of sines and cosines.

Assignments

Assign all of the exercises.

7.2 ARROWS AND MATHEMATICS

Purpose

To review some of the ideas from Course III relating arrows, arrow sums, coordinates, and translations; to review the concept of vector space; to recall that bound arrows (in the plane, space, etc.) with arrow addition form a vector space over the real numbers.

Getting Started

Return to the mobile hanging problem from the first section to introduce the concept of a bound arrow. Proceed with development in the textbook.

Using the Text

This section reviews much content that was introduced in Course III. The major ideas are that in the plane (or space) there is a 1-1 correspondence between bound arrows, ordered pairs (or triples) of real numbers, and translations. The idea that an arrow can represent a translation is extremely important and will be a major ingredient in our development of the geometry in this chapter; the association, arrow $\longleftrightarrow$ translation, is natural.

Assignments

Exercises 1-8 are relatively straight forward, but should be assigned. Exercises 9-13 show that a vector can take various forms, and Exercises 14 and 15 ask students to apply the mathematics as they did in Section 8.1.

7.3 LINES IN R^2 AND R^3

Purpose

There are many ideas that are developed in Section 7.3 and you should proceed slowly. The procedures that are used to find equations for lines carry over to equations for planes and other geometric figures. Section 7.3 sets a base for much of what follows; the purposes are:

1) to identify $(R^n,+)$ over R as representing an important family of vector spaces with $(R^2,+)$ and $(R^3,+)$ as distinguished members;
2) to develop a procedure for finding vector and parametric equations of lines in R^2 and R^3.

Getting Started

You could begin by skipping the introduction to the section and by considering the problem of determining equations of a line. Begin in 2-space and recall how the equation of a line can be found using points and slopes. In particular for the line through (0,0) and (-3,6), its equation is

$$y = -2x$$

That is, the students should recognize that in R^2, finding equations of lines is no big thing. Give a few other examples to reinforce this idea.

Now consider the line through (0,0,0) and (-3,6,2) and ask for suggestions. Unfortunately, the simple procedure from R^2 does not extend easily and so let's return to R^2 to see if we can handle equations of lines in a way that will generalize to any space.

Using the Text

Go over Theorem 1 paying particular attention to R^2 and R^3. Note how the operations in $(R^3,+)$ over R are similar to those of R^2 (Example 1). Proceed with the development in the text. The important observation is that for a line through the origin, its vector and parametric equations are extremely easy to determine (in fact, easier than using slopes and points).

Examples 3 and 4 will require some attention. The major geometric idea is that the image of a line ℓ under a translation is a line ℓ' that is parallel to ℓ. An example that you might give is to start with

$$\ell = \{r(-4,2) : r \in R\}.$$

Have students sketch ℓ and the images of ℓ under the following translations:

1) $(x,y) \to (x+1,\ y+1) = (x,y) + (1,1)$
2) $(x,y) \to (x-4,\ y+3) = (x,y) + (-4,3)$
3) $(x,y) \to (x+2,\ y-5) = (x,y) + (2,-5)$

Ask students to relate each image to ℓ (For example, in 1) the image is a line parallel to ℓ and containing (1,1)).

We hope that students will understand that not all lines in 2- or 3-space are vector lines. Vector lines are subspaces (having the vector space properties) while other lines containing vectors need not be. Vector lines must pass through the origin; any line in 2- or 3-space is either a vector line or a translate of a vector line.

Assignments

In this and in subsequent sets of exercises, students answers may vary considerably depending on how they solved the problems "vectorially." Exercise 10 is necessary for Exercises 11-16; Exercise 18 and 19 show that there is more than one way to give a vector equation for a line. Exercises 21-24 are important and should be assigned.

7.4 LINEAR COMBINATIONS AND PLANES

Purpose

To develop procedures for finding vector and parametric representations for planes.

Getting Started

As background, a vector plane is a set of all linear combinations of two non-zero vectors that are not in the same vector line. Any plane is a vector plane or a translate of a vector plane. This is an analogous situation to that in Section 7.3 where any line was related to a vector line.

Using the Text

Use $R^2 = \{(x,y) : x,y \in R\}$

$= \{x(1,0) + y(0,1) : x,y \in R\}$

to motivate the general form for a plane.

Following Example 1, the answers to the Discussion Questions are:

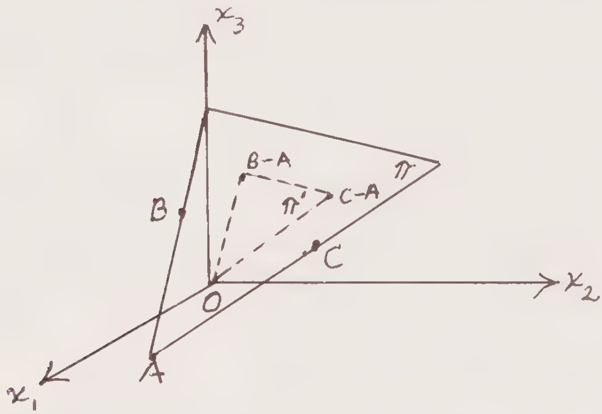
1) If $Q \in \{r(-4,5,1)\}$, then
$\{r(-4,5,1) + s(1,1,-3) : r,s \in R\}$ represents a line.
2) $(-3,6,-2) = (-4,5,1) + (1,1,-3)$ and
$(-2,7,-5) = (-4,5,1) + 2(1,1,-3)$.

3) No

4) Yes

5) An infinite number

You should be careful to emphasize the informal geometry that can help students visualize the process for finding equations of planes. In particular, when discussing Example 3, a diagram like the one below might help:



To find an equation for π, find an equation for π' and then use the translation that maps O onto A or B or C to find an equation for π.

Assignments

Exercises 1-5 and 8-10 are developmental and should be assigned. Exercises 6 and 7 show why the phrase "vector plane" is used to describe certain types of planes. Exercises 12-18 should not all be assigned for one homework session. Be prepared for a number of different answers, each correct, for each of Exercises 12-18. As Exercise 19 shows, although vector representations may look different, they can represent the same plane.

7.5 NON-PARAMETRIC REPRESENTATIONS OF LINES AND PLANES

Purpose

To relate parametric and non-parametric representations; to develop procedures for going from one representation to another.

Getting Started

It might be helpful to begin by recalling that students have studied non-parametric equations before. For example, in R^2, $x + 2y = 6$

$$y = x^2 + 2x - 3$$

$$y = \sin x$$

are non-parametric (or standard) equations for a line, a parabola, and a sine curve, respectively. In R^2, the reason the equations are called non-parametric is that they are "pure." That is, the equations are given entirely in terms of x and y (or x_1 and x_2, depending on how the coordinate axes are labelled).

Using the Text

Elimination of parameters and introduction of parameters are the two procedures that link parametric and non-parametric forms. For your information, it is not always possible to eliminate parameters as in

$$(x_1, x_2) = r(1,1) + s(0,1)$$

In this text, however, we do not include examples or exercises in which eliminations are not possible.

Theorems 3 and 4 are important results. Students should understand that we have "constructively" generated lines and planes, we have represented them parametrically, and now via elimination of parameters we can conclude that lines and planes can also be represented non-parametrically by linear equations.

Introducing parameters is a little subtle. To introduce parameters requires that the system of non-parametric equations be in reduced form; that is, where some of the variables or components are each expressed in terms of the others. In the next section, students will see that the Gauss-Jordan method that they learned in Course III will transform a system of equations into reduced form. For example, the system

$$x_1 + x_2 + x_3 = 6$$

$$x_1 - x_2 + x_3 = 8$$

cannot be put into reduced form directly. It is not possible to express, say x_1 and x_2 each in terms of x_3. Try it.

Assignments

Exercises 8-12 present a more intuitive way to introduce parameters than does the reduced form process. The method is less general however and only works when you are able to identify beforehand the geometric nature of the system. Exercise 13 should be assigned.

7.6 VECTOR SPACES, GEOMETRY, AND SYSTEMS OF EQUATIONS

Purpose

To continue work with parametric and non-parametric forms; to relate the algebra and geometry of lines and planes.

Getting Started

Begin with Example 1 and discuss the geometric possibilities of the solution set.

Using the Text

The content of this section is an extension of work first introduced in Course III. The solution set of a system of linear equations may be empty, contain a single element, or an infinite number of elements. The content of Sections 7.1-7.5 will allow students to describe and to represent the solution set in various forms.

If you feel students need to review the Gauss-Jordan procedure, go back to Course III and work through some systems that have unique solutions. Point out that if a system of 3 linear equations in 3 variables has a unique solution, then geometrically we have 3 planes that intersect in one point.

Assignments

Exercises 1-4 are systems of equations to solve. The solution sets are either empty or have an infinite number of elements. You may want to supplement these 4 with additional systems with unique solutions.

7.7 MEASUREMENTS IN SPACE

Purpose

To introduce the inner product of vectors; to relate the inner product to measurements in space.

Getting Started

Begin by discussing angles in space--angles between two planes, between two lines, between a plane and a line. In the case of two lines, the lines may be parallel, skew, or intersecting. Even in the case where two lines are skew, there is an angle between them; it is formed by projecting one of the skew lines so that it intersects the other. Floor, wall, and ceiling lines can be used to illustrate the angles between skew lines.

Using the Text

In this section, we only discuss angles between lines although other angles in space can be handled in a similar fashion. Spend considerable time discussing the angles of triangular face ABC of tetrahedron ABCD.

Assignments

Most of the exercises should be assigned. Exercise 15 is particularly interesting for it shows that the inner product is not an associative operation.

7.8 APPLICATIONS OF THE INNER PRODUCT

Purpose

To introduce the concept of work and to relate work to inner products; to develop a procedure for finding equations of planes that are given in terms of perpendicularity.

Getting Started

Begin by contrasting the difference between work and force. A layman's interpretation of work is often not the scientist's interpretation and the difference should be made clear. One exerts force to hold a 50 lb. weight over one's head, but no work is accomplished even though the holder might exclaim, "That was hard work!" For the scientist, work has a special meaning; it implies that force is exerted and an object is moved. If no motion occurs, no work is done.

Using the Text

The material of Section 7.8 should probably be developed over three days with the emphasis on work completed before beginning the material on lines, planes, and vectors normal to lines and planes. If you have time, you might want to discuss the unit normal form of a plane to help simplify finding distances to planes.

If π is represented by $ax + by + cz = 0$, then we know that

1) (a,b,c) is normal to π
2) $|(a,b,c)| = \sqrt{a^2 + b^2 + c^2}$
3) $\frac{1}{\sqrt{a^2+b^2+c^2}}(a,b,c) = \left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2bc^2}}, \frac{c}{\sqrt{a^2+b^2bc^2}}\right)$

   is a unit vector normal to π.

Because of 3), if we rewrite $ax + by + cz = 0$ as

$$\frac{ax + by + cz}{\sqrt{a^2 + b^2 + c^2}} = 0,$$

we call the new equation the <u>unit</u> <u>normal</u> form for π. That is, we can determine a unit normal by observation.

<u>Example</u>. From $3x - 2y + z = 0$, we get $\frac{3x-2y+z}{\sqrt{14}} = 0$ and a unit normal would be $\left(\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$

The unit normal form is convenient for finding the distance that a plane is from the origin or from other points. Refer to any standard analytic geometry textbook for examples.

<u>Assignments</u>

Exercises 1-7 could be assigned for one homework session. Exercises 8-20 all relate to lines and planes; to supplement Exercise 14, you could review the relationship between perpendicular lines in a plane and their slopes.

EPILOGUE

UNIFIED MATHEMATICS

The major purpose of this chapter is to exhibit part of the unified nature of mathematics via Euler's Equation

$$re^{ix} = r(\cos x + i\sin x)$$

It is true that to develop the formula with mathematical rigor requires a good deal of analysis; however, it would be a shame to pass up an opportunity to relate exponents, circular functions, complex numbers, and vectors (all topics from this course) in the name of mathematical purity. This is not to say that we should develop the formula without recognizing the mathematical questions that are left unanswered and so, when appropriate, students are told that although the results are correct, the development is informal. We hope that this Epilogue is a fitting conclusion to our Unified Mathematics: Course IV.

SUGGESTED TIME SCHEDULE

E.1 e Revisited	2 days
E.2 e^{ix}	2-3 days
E.3 Some Applications	2-3 days
E.4 Summary	1 day
TOTAL	7-9 days

E.1 e REVISITED

Purpose

To introduce the infinite series representation for e^x.

Getting Started and Using the Text

Begin by recalling that in Chapter 4, we accepted as a definition for e,

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n .$$

That is, for n large,

$$e \approx \left(1 + \frac{1}{n}\right)^n$$

and hence

$$e^x \approx ((1 + \frac{1}{n})^n)^x$$

$$\approx (1 + \frac{1}{n})^{nx}$$

In limit form,

$$e^x = \lim_{n\to\infty} (1 + \frac{1}{n})^{nx}$$

The preceding should be intuitively clear although we have not demonstrated that

$$\lim_{n\to\infty} (1 + \frac{1}{n})^{nx}$$

does exist for any real number x. If your students have access to a computer system, you might have them investigate $(1 + \frac{1}{n})^{nx}$ for various values of x (say $x = \frac{1}{2}, 1, \frac{3}{2}, 2$) as n increases. The results should suggest that for <u>each</u> x, $(1 + \frac{1}{n})^{nx}$ does converge.

The proof that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ is usually considered when students study the Maclaurin series of a function in calculus. Without tools from calculus, our approach is to consider the expansion of $(1 + \frac{x}{m})^m$ and to note what happens as m gets large (using the fact that $\frac{1}{m} \to 0$ as $m \to \infty$). Although the development is informal, the results are correct. As an additional exercise, you could have students compare, with the aid of a computer, the sum of the first few terms of $(1 + \frac{x}{m})^m$ for a given x and large m with the sum of the first few terms of $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$. The difference should be small.

<u>Assignments</u>

Assign at least Exercises 1-3. Exercises 4-6 are optional but do review some relationships among circular and exponential functions; the results are interesting and hence the exercises should probably be assigned.

E.2 e^{ix}

Purpose

To suggest that $e^{ix} = \cos x + i \sin x$ and

$$re^{ix} = r(\cos x + i \sin x)$$

for $r, x \in R$.

Getting Started

There is no a priori reason to expect that symbols like e^{2i} or $e^{\pi i}$ will have any meaning at all. Instead, appeal to your students' mathematical curiosities (we hope they have some) and consider extending the domain of $f: x \to e^x$ to include the pure imaginary numbers (those of the form bi, $b \neq 0$). A logical place to begin is with

$$e^x = \lim_{m \to \infty} \left(1 + \frac{x}{m}\right)^m$$

and to try substituting ix for x to get

$$e^{ix} = \lim_{m \to \infty} \left(1 + \frac{ix}{m}\right)^m$$

Some authors simply take the preceding as a definition and get away cheaply. Our students, with their background of vectors, complex numbers, and so on, should be ready for the argument that is presented in the textbook.

Using the Text

Follow the sequence of steps in the text that suggest as m gets large, $\left(1 + \frac{ix}{m}\right)^m$ approaches the complex number of magnitude 1 and direction angle x. The subtle part of the argument occurs in the last stages where we chose to write 0 in two forms (because $\frac{x}{m} \to 0$ and $\alpha_m \to 0$, $\alpha_m \to \frac{x}{m}$ and $m\alpha_m \to x$).

Euler's Equation, $re^{ix} = r(\cos x + i \sin x)$, has been called the most aesthetically beautiful formula in all of elementary mathematics. Certainly, the result that

$$e^{i\pi} + 1 = 0$$

unites the historically important numbers ($e, i, \pi, 0$ and 1) into one amazing result. Ask your students to think about it.

Assignments

Assign all of the exercises. In Exercise 10, students are asked to extend the development in the textbook to consider numbers like $e^{2+\pi i}$. Most students will probably rewrite $e^{2+\pi i}$ as $e^2 \cdot e^{\pi i}$, look up e^2 in the table and write

$$\begin{aligned} e^{2+\pi i} &= e^2 \cdot e^{\pi i} \\ &\approx (7.39)(-1) \\ &\approx -7.39 \end{aligned}$$

The process is correct and comes from the definition

$$e^{a+bi} = e^a(\cos b + i \sin b).$$

Don't overdo the exercise, however, by introducing the preceding definition.

E.3 SOME APPLICATIONS

Purpose

To present some mathematical and non-mathematical applications of e^{ix}.

Getting Started and Using the Text

In the spirit of unification and interrelationships among mathematical topics, this section reviews some previously introduced topics from a new point of view. The subsection on Electrical Circuits is optional for those students who studied about electrical circuits as part of Chapter 5.

Assignments

Not all of Exercises 1-10 need be assigned. Exercises 11-16 are nice and introduce new results. Exercise 18 asks students to verify that the laws for operating with exponents are still valid when the exponent is a pure imaginary number.

SOLUTIONS AND ANSWERS TO EXERCISES

CHAPTER 0

Section 0.1 (Pages 6-9)

1. (a) 441
   (b) 147
   (c) 2
   (d) -27
   (e) AB^C
   (f) $\frac{1}{B^2}$
   (g) $\frac{A+B}{CD}$
   (h) 44
   (i) A^6
   (j) A^8
   (k) $4x^2 + 3x - 8$
   (l) rx - 5 = 243

2. (a) A = 3.14*R↑2
   (b) C = 3.14*D
   (c) P = 2*L + 2*W
   (d) A = S↑2
   (e) (2*x+3)/(x↑2-1)
   (f) (A-B)/(C-(D/E))
   (g) 3*x↑2 + 2*x+5

3. (a) 2.5, not 2½ and 3.25, not 3¼
   (b) 7*X + 3*Y
   (c) 3250*76
   (d) 15/5
   (e) parens missing somewhere
   (f) nothing after ↑
   (g) 32∠ = 64
   (h) r*X

4. (a) $\sqrt{240} \approx 15.4919$
   (b) $\sqrt{8} \approx 2.82843$
   (c) $\sqrt{5.46} \approx 2.33666$
   (d) $\sqrt{935} \approx 30.5778$

5. In the first case, the computer would go into a never-ending loop with no criterion for when to print an answer.
   In the second case, same problem although the computer would be getting better and better approximations at each turn through the loop.

6. This program prints each successive approximation, not only the last.

7. Errors - Line 20 LET Y = 3*A, RUN does not have line number

Output - 18

8. Errors - No value assigned to A; probable correction
   4 READ X,A
   Output - 41.6

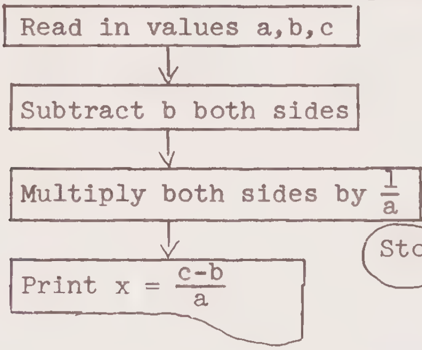
9. Errors - RUN does not get a line number, no PRINT
   Output - If Z PRINT Y, output is 78.5

10. Errors - 12 GØ TØ occurs before PRINT, so the computer will simply report ØUT ØF DATA IN 10, having read each value of M in turn, but given no instruction on what to do with it.
    Output - If change line 15 to 11,

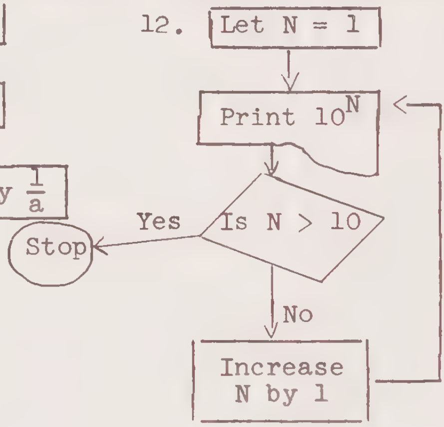
```
17
26
41
23
ØUT ØF DATA IN 10
```

In 11-15 various answers are possible. Samples are given:

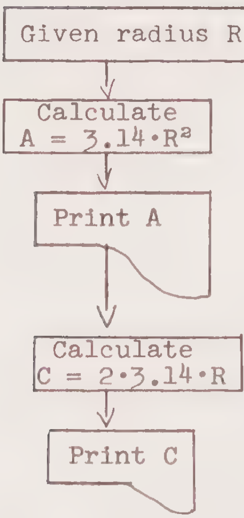
11.



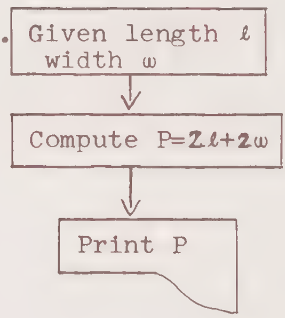
12.

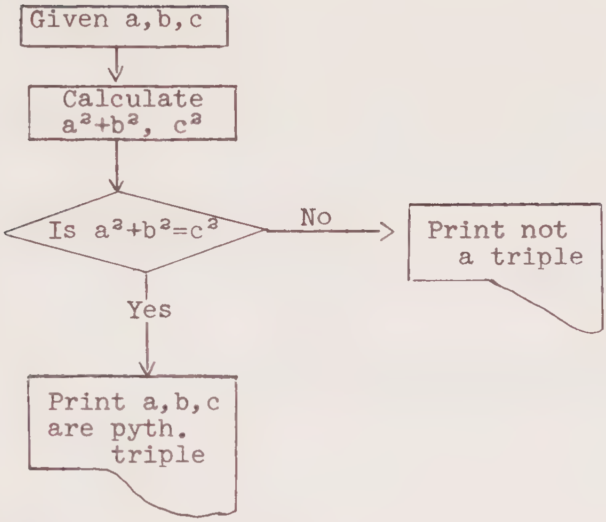


13.



14.



15. 

Section 0.2 (pp.14-17)

1. Printout will be:

```
SCALE CØNVERSIØN
FARENHEIT TØ CENTIGRADE

PICK A NUMBER
? 212                  (as example)

FARENHEIT TEMP 212, EQUALS CENTIGRADE TEMP 100
```

2.
```
10 INPUT A,B,C
20 LET X = (C-B)/A
30 PRINT X
40 END
```

3.
```
10 READ B,H
20 LET A = ½*B*H
30 PRINT A
40 END
```

4.
```
10 READ X
20 LET Y = X↑2 + 5*x + 6
30 PRINT Y            [Note: More elegant methods follow
40 GØ TØ 10            in 1.3, 1.4]
50 DATA -10,-9,-8,-7,-6,-5,-4,-3,-2,-1
60 DATA 0,1,2,3,4,5,6,7,8,9,10
70 END
```

5.
```
10 INPUT A,B
20 PRINT "X↑2 +  "; A+B; "*X + "; A*B
30 END
```

6.
```
10 PRINT "N"; "1/N"; "1/N↑2"
20 READ N
30 PRINT N; 1/N; 1/N↑2
40 GØ TØ 20
50 DATA 1,2,3,4,5,6,7,8,9,10
60 END
```

7.
```
10 INPUT A=B=C
20 PRINT "X = "; (A*C)/B
30 END
```

8.
```
10 INPUT A=B
20 LET Y = 100*(A/B)
30 PRINT A; "IS"; Y; "PERCENT ØF"; B
40 END
```

9.
```
10 INPUT C
20 LET F = 1.8*C+32
30 PRINT "C="; C, "F="; F
40 END
```

10.
```
10 INPUT Y
20 LET M = Y*(36/39.37)
30 PRINT "YARDS"; Y, "METERS"; M
40 END
```

11.
```
10 INPUT P
20 LET Ø = 16*P
30 PRINT Ø
40 END
```

In 12-20 many different REM and PRINT statements are possible. We give only selected samples here.

12. This program averages 5 numbers. It might be elaborated as follows:

```
1  REM AVERAGE 5 NUMBERS
2  PRINT "ENTER 5 NUMBERS"
4  INPUT A,B,C,D,E
6  LET Y = (A+B+C+D+E)/5
8  PRINT "AVERAGE OF"; A; "," ;B; ",";C;",";C;","
         ;E; "IS"; Y
10 END
```

13. Calculates slope of $\overleftrightarrow{XY}$, given coordinates of X,Y

14. Prints squares of the given numbers

15. Converts a given number of feet to inches, yards, and meters

16. Sums the squares of the first 10 odd whole numbers

17. Calculates circumference and area of circle with given diameter

18. Prints a matrix, given the entries, and 3 times the matrix

19. Calculates mean and variance for a set of 5 numbers. Add statements:
```
10 REM MEAN
15 PRINT "ENTER 5 NUMBERS"
40 PRINT "MEAN ØF"; A;B;C;D;E; "IS"; M
45 REM VARIANCE
60 PRINT "VARIANCE IS"; V/5
```

20. Prints X^3 for given numbers

21. This procedure is simply a formalization of solving linear equations by elimination.
   1 Multiply first equation by $\frac{1}{a}$, then subtract c times it from the second equation
   2 Multiply second equation by $\frac{a}{(ad-bd)}$, then subtract $\frac{b}{a}$ times it from first equation.

22.
```
10 READ A,B,C,D,E,F
20 LET M = A*D-B*C
30 LET X = (D*E-B*F)/M
35 LET Y = (A*F-C*E)/M
40 PRINT "X = "; X,"Y = "; Y
```

23. (a) x = 11, y = -6 (b) x = 42.074, y = 139.376
(c) x = -1.47826, y = 6.78261 (d) x = 94, y = 34

24.
```
10 PRINT "N"; "N!"
20 LET F = 1
30 READ N
40 LET F = N*F
50 PRINT N; F
60 GØ TØ 30
70 DATA 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15
80 END
```

0.3 Section 0.3 (PP. 23-26)

1. (a) No [NEXT X] instruction between 200 and 250 to close loop
   (b) This is all right. A better print form would result from including [50 PRINT "X";"Y"] and [300 PRINT X; Y]

2. (a) All right as is. Note 60^3 and $(-60)^3$ dominate the expression, so this is a reasonable range to explore.
   (b) FØR/NEXT loop done incorrectly.
   [10 FØR X=-60 TØ 60]
   [25 GØ TØ 35]
   [35 NEXT X]

3. (a) To avoid repeat print of headings, make
   [05 PRINT "FARENHEIT", "CENTIGRADE"]
   [40 LET C = C+5]
   (b) Interchange instructions now in 25,30 or will print for C=105
   [15 LET F=(9/5)*C+32]

4. (a) [30 LET X=S+I↑2]
   Better printout if
   [05 PRINT "N", "SUM I↑2 TØ N"]
   (b) Loops crossed by 40 and 50. After each sum to N, must reset S at 0
   [47 LET S = 0]

5. (a) All right, except more descriptive
   [34 PRINT "ABSØLUTE VALUE ØF";X;"IS";Y]
   [40 PRINT "ABSØLUTE VALUE ØF"; X; "IS"; X]
   (b) Faulty decision in 40
   [40 IF Y < X THEN 45]
   [42 PRINT Y]
   [44 GØ TØ 50]

6. For the program in Exercise 1(a)

```
100 LET X = -10
150 LET Y = X↑2+5*X+6
200 PRINT Y
250 LET X = X+1
300 IF X > 10 THEN 400
350 GØ TØ 150
400 END
```

For the program in Exercise 4(a)

```
 10 LET N = 1
 15 LET S = 0
 20 LET I = 1
 30 LET S = S+I↑2
 40 LET I = I+1
 50 IF I > N THEN 65
 60 GØ TØ 30
 65 PRINT "N IS"; N, "SOU IS"; S
 70 LET N = N+1
 80 IF N > 20 THEN 100
 90 GØ TØ 15
100 END
```

7. Loops overlap in (b). This won't run.

8. (a)

```
┌──>5
│┌─>9
││  11
│└──13
└───15
    17
```

(b) loops cross; this won't run correctly.

```
 ┌─>5
┌┼─>9
││  11
│└──13
└───15
    17
```

In 9-27, many programs are acceptable. Encourage variety!

9.

```
10 PRINT "SØLUTIØNS ARE"
20 FØR X = -20 TØ 20
30 IF (X-3)< = 5 THEN 50
40 PRINT X
50 NEXT X
60 END
```

10.

```
10 PRINT "SØLUTIØNS ARE"
20 FØR X = -10 TØ 10
30 LET Y = 3*X↑2 + 5*X -4
40 LET Z = X↑2 - 4
50 IF Z = 0 THEN 80
60 IF Y/Z <>2 THEN 80
70 PRINT X
80 NEXT X
90 END
```

11. Generalize by replacing 2 with m

```
10 LET X = 10
20 IF 2↑X > 100000 THEN 50
30 LET X = X+1
40 GØ TØ 20
50 PRINT "2↑"; X; "="; 2↑X
60 END
```

12.
```
10 FØR N=1 TØ 79 STEP 2
20 PRINT N↑2,
30 NEXT N
40 END
```

NOTE: The comma after N↑2, in 20 will lead computer to make succeeding prints on the same line until full. Then it shifts to the next line.

13.
```
10 FØR N=1 TØ 1000
20 LET X=S+1/N
30 NEXT N
40 PRINT S
50 END
```

14.
```
10 PRINT "MILES";"KILOMETERS"
20 FØR M=0 TØ 100 STEP 5
30 PRINT M, (5/3)*M
40 NEXT M
50 END
```

15.
```
10 PRINT "METERS"; "YARDS"
20 FØR M , 100 TØ 1500 STEP 100
30 PRINT M; (39.37/36)*M
40 NEXT M
50 PRINT 3000; (39.37/36)*3000
60 PRINT 5000; (39.37/36)*5000
70 PRINT 10000;(39.37/36)*10000
80 END
```

16. This can be done with the following program:
```
 10 LET N = 1
 20 LET S = N*1000 + S
 30 IF S > = 75000 THEN 60
 40 LET N = N+1
 50 GØ TØ 20
 60 PRINT N; "WINS PAYS"; S
 70 FØR M , 1 TØ 20
 80 LET T = T+M*1000
 90 NEXT M
100 PRINT "20 WINS PAYS"; T
120 PRINT "DIFFERENCE IS"; T-75000
130 END
```

Option (b) is better if he wins 12 or more games ($78,000). For 20 wins he will earn $210,000.

17. $$\sum_{J=1}^{N} (2*J+1) = N^2$$

A program would be:
```
10 PRINT "N"; "SUM ØF FIRST N ØDDS"
20 FØR N = 1 TØ 10
30 LET S = 0
40 FØR J = 1 TØ N
50 LET S = S+(2*J+1)
60 NEXT J
70 PRINT N; S
80 NEXT N
90 END
```

18.
```
10 FØR N = 1 TØ 50
20 LET S = S + ((-1)↑(N+1))*(1/(2*N-1))
30 NEXT N
40 PRINT "P1 IS ABOUT"; 4*S
50 END
```

```
19. 05 PRINT "SQUARE FEET"; "SQUARE INCHES"
    10 FØR F = 1 TØ 10
    20 PRINT F; 144*F
    30 NEXT F
    40 END

20. 05 PRINT "SQUARE FEET"; "SQUARE YARDS"
    10 FØR F = 10 TØ 500 STEP 20
    20 PRINT F; F/9
    30 NEXT F
    40 END

21. 10 PRINT "GIVE PRICE IN SQUARE FEET, DOLLARS"
    20 INPUT F,D
    30 LET Y = F/9
    40 PRINT "CØST IS"; D/Y; "DØLLARS PER SQUARE YARD"
    50 END

22. 10 PRINT "SØLUTION PAIRS"
    20 PRINT "X"; "Y"
    30 FØR X = 0 TØ 20
    40 FØR Y = 0 TØ 20
    50 IF 3*X + 2*Y > 5 THEN 70
    60 PRINT X; Y
    70 NEXT Y
    80 NEXT X
    90 END

23. 10 PRINT "SØLUTIONS ØF ABS(X-5)<3"
    20 FØR X = -20 TØ 20
    30 IF ABS(X-5)>=3 THEN 50
    40 PRINT X
    50 NEXT X
    60 END

24. 10 READ Y
    20 FØR N = 1 TØ 9
    30 READ X
    40 IF Y >= X THEN 60
    50 LET Y = X
    60 NEXT N
    70 DATA -20,32,-14,-8,12,43,95,-18,0,22
    80 PRINT Y; "IS LARGEST"
    90 END

25. 10 PRINT "PYTHAGØREAN TRIPLES"
    20 PRINT "X"; "Y"; "Z"
    30 FØR X = 1 TØ 25
    40 FØR Y = 1 TØ 25
    50 LET Z = 1
    60 IF X↑2+Y↑2<Z↑2 THEN 90
    70 PRINT X; Y; Z
    80 GØ TØ 120
    90 IF X↑2+Y↑2<Z↑2 THEN 120
   100 LET Z = Z+1
   110 GØ TØ 60
   120 NEXT Y
   130 NEXT X
   140 END
```

NOTE: An easier version using INT(X) and SQR(X) can be done later in the chapter. This is also much more efficient:

```
10 FØR X = 1 TØ 25
20 FØR Y = 1 TØ 25
30 IF INT(SQR(X↑2+Y↑2))<>SQR(X↑2+Y↑2) THEN 50
40 PRINT X; Y; Z
50 NEXT Y
60 NEXT X
70 END
```

26.
```
10 INPUT D
20 PRINT "RATE"; "INTEREST"
30 FØR R = .03 TØ .08 STEP .005
40 LET I = D*R*1
50 PRINT R; I
60 NEXT R
70 END
```

27.
```
10 LET P = 100
20 FØR N = 1 TØ 20
30 LET P = P + .0125*P
40 NEXT N
50 PRINT P
60 END
```

28. (1) $\begin{pmatrix} 1 & -1 \\ -4 & 5 \end{pmatrix}$ (2) $\begin{pmatrix} -2 & -5 \\ -5 & -12 \end{pmatrix}$ (3) $\begin{pmatrix} .4 & -1 \\ -.5 & 1.5 \end{pmatrix}$

(4) No inverse, ad-bc=0 (5) $\begin{pmatrix} .230769 & -.153846 \\ .153846 & .230769 \end{pmatrix}$

29. Must include ad-bc≠0!

30.
```
10 PRINT "N"; "N!"
15 LET S = 1
20 FØR N = 1 TØ 10
30 LET S = N*S
40 PRINT N,S
50 NEXT N
60 END
```

Section 0.4 (PP. 32-33)

1. 2.34578E-6 2. 453987E+4 3. 47890.6 4. -907846E+5

5. .567839 6. -234.568 7. 1.33333 8. 109638E+3

9. 107374E+4 10. 832120E+4 11. 5200 12. 6432100

13. .000004567 14. 4536210000000 15. .00000657

16. 92000 17. 6800 18. 400 19. XYE(N+M)

20. (X/Y)E(N-M) 21. [70 PRINT "X="; X1]
[40 LET B = Y1 - M*X1]or[Y2-M*X2]

22. (a) Y = 1.5*X + 0
(b) Y = -.6*X + 2.6
(c) Y = 1.92308E-2*X+-3.19769

23. LIST

```
 10 INPUT N,R
 15 LET A = 1
 20 FØR X = N TØ N-R+1 STEP -1
 30 LET A = A*X
 40 NEXT X
 45 LET B = 1
 50 FØR Y = R TØ 1 STEP -1
 60 LET B = B*Y
 70 NEXT Y
 80 LET C = A/B
 90 PRINT N; "BINØMIAL"; R; "="; C
100 END
```

(a) 120 (b) 28 (c) 11 (d) Since R = 0, both loops are skipped and output is 5 BINØMIAL 0 = 1.

24. By hand $\binom{20}{15} = 15504$. Answers by computer will depend on machine capacity.

<u>Section 0.5 (pp. 35-38)</u>

1.
```
10 INPUT A B
20 FØR X = -20 TØ 20
30 IF ABS(X-A)<>B THEN 50
40 PRINT X,
50 NEXT X
60 END
```

2. Same as 1. with new [30 IF ABS(X-A)>=B THEN 50]

3. ① $\{x: |x-5=8\} = \{x: x-5=8 \text{ or } x-5=-8\}$
$= \{x: x=13 \text{ or } x=-3\}$
$= \{13,-3\}$

② $\{x: |x-5|<8\} = \{x: x-5<8 \text{ and } x-5>-8\}$
$= \{x: -3<x<13\}$

4.
```
10 INPUT A,B
20 PRINT "X= "; B+A; "ØR" ; B-A
30 END
```

5. 

6. 

{x: distance x to -4 is more than 1}

7. One helpful program would be:
```
05 PRINT "ABS(A+B)"; "ABS(A)+ABS(B)"
10 FØR A = -10 TØ 10
20 FØR B = -10 TØ 10
30 PRINT ABS(A+B); "ABS(A)+ABS(B)
40 NEXT B
50 NEXT A
60 END
```

True relationship is $|a+b| \leq |a|+|b|$. Equality holds if a,b have same sign or if one is 0.

8. As in 7 with revised lines 05 and 30.
   $\forall a,b \ |a-b| \geq |a| - |b|$

9.
```
10 INPUT X1, Y1, X2, Y2
20 PRINT ABS(X2-X1) + ABS(Y2-Y1)
30 END
```

10.
```
10 INPUT X1, Y1, X2, Y2
20 PRINT SQR((X2-X1)↑2 + (Y2-Y1)↑2)
30 END
```

11.
```
10 INPUT X,Y,Z
20 PRINT SQR(X↑2 + Y↑2 + Z↑2)
30 END
```

12.
```
 10 READ P1, P2, S1, S2, T1, T2
 20 LET D1 = SQR((S1-P1)↑2 + (S2-P2)↑2)
 30 LET D2 = SQR((T1-S1)↑2 + (T2-S2)↑2)
 40 LET D3 = SQR((T1-P1)↑2 + (T2-P2)↑2)
 50 PRINT "IF P=(";P1;",";P2;"), S=(";S1;",";S2;"),T=(";
 60 PRINT T1; ","; T2; ") THEN"
 70 PRINT
 80 IF D1 + D2 >= D3 THEN 130
 90 PRINT "TRIANGLE INEQUALITY FAILS"
100 PRINT
110 PRINT "    PS + ST < PT"
120 PRINT "    "; D1; "+"; D2; "<"; D3
125 GØ TØ 170
130 PRINT "TRIANGLE INEQUALITY HOLDS"
140 PRINT
150 PRINT "     PS + ST >= PT"
160 PRINT "     "; D1; "+"; D2; ">="; D3
165 PRINT
170 GØ TØ 10
180 DATA 2,3,5,8,7,-2,-1,3,4,0,4,-3
190 END
```

13. Using the program of 12 with ABS in place of SQR and no other changes, students will discover that the taxi metric does satisfy the triangle inequality. It does not have the usual euclidean property that equality holds iff the points are collinear with S between P and T. Have students explore this for some related generalization.

```
20 LET D1 = ABS(S1-P1) + ABS(S2-P2)
30 LET D2 = ABS(T1-S1) + ABS(T2-S2)
40 LET D3 = ABS(T1-P1) + ABS(T2-P2)
```

14.
```
10 DEF FNF(X) = X↑3 - X
20 FØR X = -5 TØ 5
30 PRINT X, F(X), SGN(F(X))
15 PRINT "X", "F(X)", "SIGN F(X)"
17 PRINT
35 NEXT X
40 END
```

15.
```
10 INPUT N
20 IF INT(N/3) = N/3 THEN 50
30 PRINT "3 DØES NØT DIVIDE"; N
40 GØ TØ 60
50 PRINT N; "=3* "; N/3
60 END
```

16.
```
10 INPUT N,M
20 IF INT(N/M)=N/M THEN 50
30 PRINT M; "DØES NØT DIVIDE"; M
40 GØ TØ 60
50 PRINT N; "/"; M; "="; N/M
60 END
```

17.
```
15 PRINT "FACTØRS ØF"; N; "ARE"
10 INPUT N
20 FØR X = 1 TØ N/2
30 IF INT(N/X)<>N/X THEN 50
40 PRINT X,
50 NEXT X
60 END
```

18.
```
10 INPUT N
20 IF INT(N/2)=N/2 THEN
30 FØR X = 3 TØ SQR(N) STEP 1
40 IF INT(N/X)<>N/X THEN 60
50 GØ TØ 90
60 NEXT X
70 PRINT N; "IS PRIME'
80 GØ TØ 100
90 PRINT N; "IS CØMPØSITE"; N; "="; X; "*"; N/X
100 END
```

19.
```
10 PRINT 2,
20 FØR N = 3 TØ 500 STEP 2
30 FØR X = 2 TØ SQR(N)
40 IF INT(N/X)=N/X THEN 70
50 NEXT X
60 PRINT N,
70 NEXT N
80 END
```

20.
```
10 INPUT N,M
20 FØR X = N TO N*M STEP N
30 FOR Y = M TO M*N STEP M
40 IF X = Y THEN 70
50 NEXT Y
60 NEXT X
70 PRINT X; "="; X/N; "*"; N; "="; Y/M; "*"; M
```

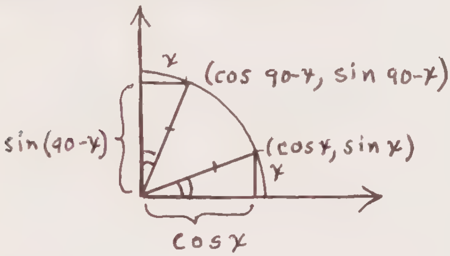
21.
```
10 INPUT N,M
30 FØR J = 1 TØ N
40 LET X = N/J
50 IF X<>INT(X) THEN 70
60 IF M/X = INT(M/X) THEN 80
70 NEXT J
80 PRINT "GCD IS" ; X
```

```
22. 10 PRINT "DEGREES"; "SINE"; "COSINE
    20 DEF FNR(D) = 3.1416*(D/180)
    30 FØR D = 0 TØ 180 STEP 10
    40 LET X = FNR(D)
    50 PRINT D, SIN(X), COS(X)
    60 NEXT D
    70 END
```

23. For a suitable program change 22 as follows

```
    10 PRINT "DEGREES"; "COSINE(X)"; SINE(90-X)"
    30 FØR D = 0 TØ 90 STEP 9
    50 PRINT D, COS(X), SIN(3.1416/2-X)
```

For all X, Cos X = Sin 90-X




```
24. 10 INPUT A,B,C
    20 LET R = 3.1416*(C/180)
    30 LET C1 = SQR(A↑2 + B↑2 - 2*A*B*COS(R))
    40 PRINT C1
    50 END
```

25. (a) 5 (b) ≈ 5.7 (c) ≈ 6.46529 (d) ≈ 10.86

```
26. 10 INPUT A,B,C,A1
    20 LET A = 3.1416*(A/180)
    30 LET B = 3.1416*(B/180)
    40 LET C = 3.1416*(C/180)
    50 PRINT "B1="; A1*SIN(B)/SIN(A)
    60 PRINT "C1="; A1*SIN(C)/SIN(A)
    70 END
```

27. (a) B1 ≈ 8.66 C1 = 10   (b) B1 ≈ 1.11 C1 ≈ 1.11   (c) B1 ≈ 1.02 C1 ≈ .44

28. (a)

```
10 DEF FNC(I)=2.54*I
20 FØR I=1 TØ 40 STEP 2
30 PRINT I, FNC(I)
40 NEXT I
50 END
```

(b)

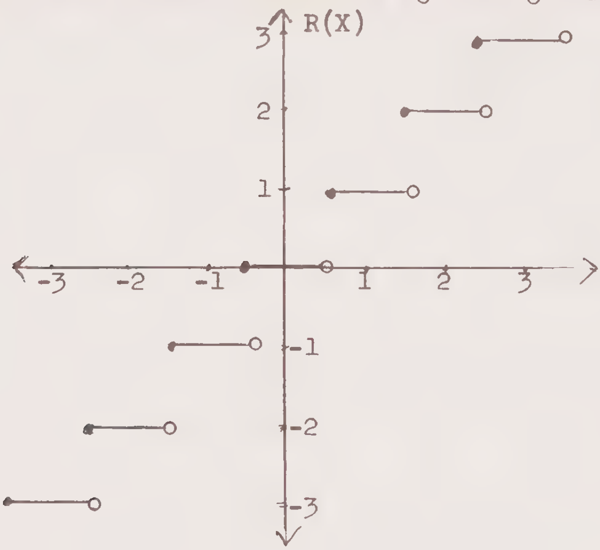
```
10 LET X=2.54↑3
20 DEF FNC(I)=X*I
30 FØR I=1 TØ 100 STEP 10
40 PRINT I, FNC(I)
50 NEXT I
60 END
```

29. Rounds of any given number according to usual rule.

Section 0.6 (pp.42-43)

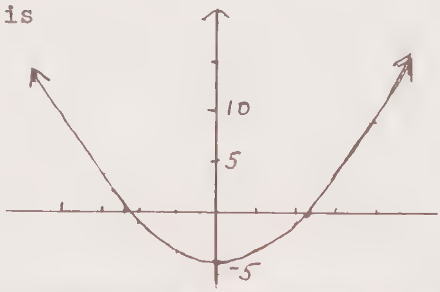
1. 23, 23   2. 0, 0   3. 1, 0   4. -12, -13   5. -13, -13
6. 234560   7. 234560   8. 0   9. -245670   10. 0

11. FNR(X) is the usual round-off procedure--round to the nearest whole number. FNR(X) = X_0 IFF $X_0 - .5 \leq X < X_0 + .5$.



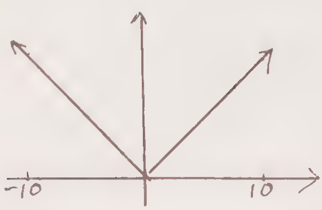
12. [-5, 20]   13. .5   14. 50   15. 50   16. 32
17. 32   18. 18   19. 18   20. 18   21. 18
22. 2   23. 2   24. 0

Normal graph is

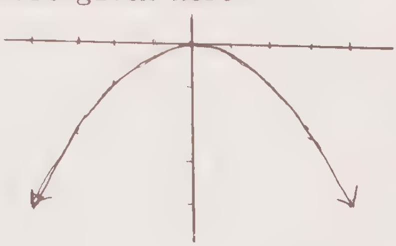


In 13-25, the continuous graphs are given here:

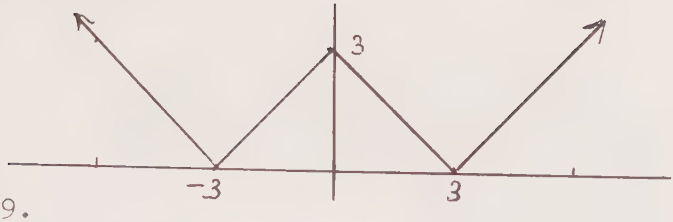
25.



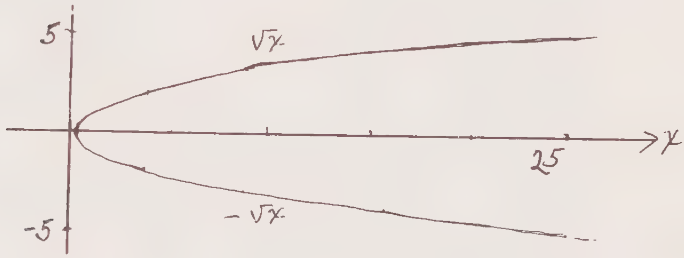
26.



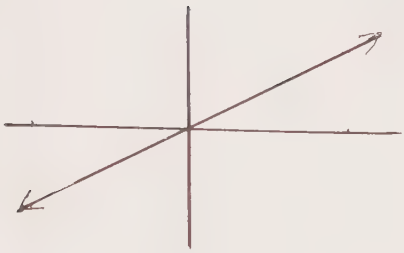
27.



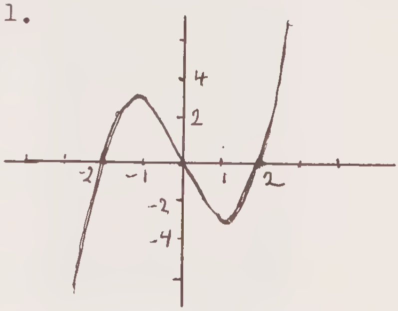
28., 29.



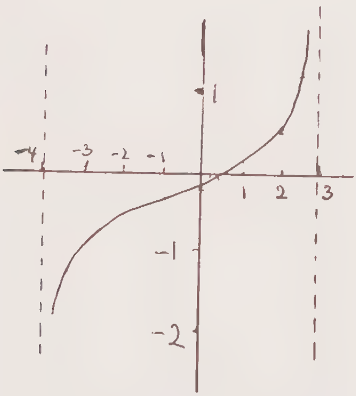
30.



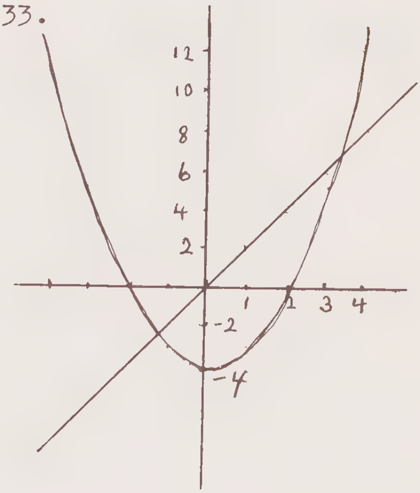
31.



32.



33.



34. Intersection is (1,2)

Section 0.7 (pp.48-52)

1. Program:
```
 05 DIM A(4,4), B(4,4)
 10 MAT READ A,B
 20 DATA 3,4,-2,-1,2,0,7,18,-1,2,0,6
 30 DATA -3,4,1.1,.4,-3,4,2.1,-.5,6
 40 DATA 12,-3,4.1,0,0,1,0,-3,-6,-5,82
 50 MAT C = A+B
 60 MAT D = (2)*A
 70 MAT E = A-B
 80 MAT F = A*B
 90 MAT G = INV(A)
100 MAT H = INV(B)
110 MAT J = TRN(A)
115 MAT K = TRN(B)
120 MAT PRINT C; D; E; F;
130 MAT PRINT G; H; J; K;
140 END
```

2. Program:
```
 10 DIM A(3,3), B(3,3), M(3,3)
 20 MAT READ A,B
 30 DATA 1,2,3,4,5,6,7,8,9
 40 DATA 9,8,7,6,5,4,3,2,1
 50 MAT C = A+B
 60 MAT D = A-B
 70 MAT E = A*B
 80 MAT F = INV(A)
 90 MAT G = INV(B)
100 MAT H = (-1)*A
110 MAT M = ZER
120 MAT I = A+M
130 MAT PRINT C; D; E; F;
140 MAT PRINT G; H; I;
150 END
```

3. Program:
```
 10 DIM C(3,4), D(4,3)
 20 MAT READ C,D
 30 DATA 1,2,-5,2,2,3,8,1
 40 DATA 1,-2,-3,5,1,-2,3
 50 DATA 0,-4,3,2,7,1,6,9,8
 60 MAT E = C+D
 70 MAT F = C*D
 80 MAT G = INV(C)
 90 MAT H = (3)*C
100 MAT I = TRN(C)
110 MAT J = (-1)*C
120 MAT PRINT E; F; G; H;
130 MAT PRINT I; J;
140 END
```

4. (a) $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -1.33333 \\ 1 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} -7 & 3 \\ -2 & 1 \end{pmatrix}$

5. (a) $x = 4,\ y = 2,\ z = -3$ (b) $x = 2,\ y = -3$

(c) $x = -1,\ y = 4,\ z = 3$

6. $\left.\begin{aligned} 44A + 36B &= 780 \\ A + B &= 20 \end{aligned}\right\} \therefore A = 7.5 \quad B = 12.5$

7. (a) T (b) F (c) T (d) F (e) F (f) T

(g) T (h) T

8.
```
10 DIN A(  ), B(  ), C(  )
20 MAT READ A,B,C
30 MAT D = C-B
40 MAT E = INV(A)
50 MAT F = E*D
60 MAT PRINT F
70 DATA
80 END
```

9.
$$\begin{array}{ll} & \text{Liquid} \quad \text{Vapor} \\ \text{Day 1} & [51 \quad 19] = [60 \quad 10]\begin{bmatrix} 5/6 & 1/6 \\ 1/10 & 9/10 \end{bmatrix} \\ \text{Day 2} & [44.4 \quad 25.6] = [51 \quad 19]\begin{bmatrix} 5/6 & 1/6 \\ 1/10 & 9/10 \end{bmatrix} \\ \text{Day 3} & [39.56 \quad 30.44] = [44.4 \quad 25.6]\begin{bmatrix} 5/6 & 1/6 \\ 1/10 & 9/10 \end{bmatrix} \\ \text{Day 4} & [36 \quad 34] = [39.56 \quad 30.44]\begin{bmatrix} 5/6 & 1/6 \\ 1/10 & 9/10 \end{bmatrix} \\ \text{Day 5} & [33.4 \quad 36.6] = [36 \quad 34]\begin{bmatrix} 5/6 & 1/6 \\ 1/10 & 9/10 \end{bmatrix} \end{array}$$

10. Rotation 90° $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Reflection $y=x$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(a) 180° rotation $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (b) 270° rotation $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ reflection x-axis (d) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ rotation 90°

11. Message Matrix $\begin{bmatrix} 12 & 0 & 19 \\ 7 & 26 & 19 \\ 8 & 12 & 4 \end{bmatrix}$

Coded Message $\begin{bmatrix} 83 & 86 & 96 \\ 34 & 64 & 61 \\ 34 & 100 & 35 \end{bmatrix}$

Decoded Matrix $\begin{bmatrix} 0 & 26 & 3 \\ 0 & 24 & 26 \\ 14 & 5 & 5 \end{bmatrix}$ ——> Message: A DAY OFF

Decoding Matrix $\frac{1}{33}\begin{bmatrix} 6 & 6 & -9 \\ -5 & 17 & 2 \\ 4 & -7 & 5 \end{bmatrix}$ if you did it by hand.

12. Averages:

DAY		LOCATION	
1 -	.833333	1	2.21428
2 -	1.6	2	2.64286
3 -	1.96666	3	.985714
4 -	2.5		
5 -	2.96666		
6 -	2.53333		
7 -	1.23333		

Program:

```
 10 DIM A(7,3)
 20 MAT READ A
 30 FØR I = 1 TØ 7
 35 LET S = 0
 40 FØR J = 1 TØ 3
 50 LET S = S+A(I,J)
 60 NEXT J
 70 PRINT "DAY"; I; "AVERAGE"; S/3
 80 NEXT I
 90 FØR K = 1 TØ 3
100 LET T = 0
110 FØR M = 1 TØ 7
120 LET T = T+A(M,K)
130 NEXT M
140 PRINT "LOCATION" M; "AVERAGE" T/7
150 NEXT K
160 END
```

*13. Program:

```
 10 DIM A(5,3), B(5,3), C(5,3), D(3,1)
 20 MAT READ A,B,C,D,
 30 DATA 20,33,17,7,12,8,16,10,7,10,9,12
 40 DATA 8,6,4,12,21,24,11,14,23,21,5,8,11
 50 DATA 11,17,6,4,3,37,48,8,6,12,11,17,13
 60 DATA 14,19,23,20,10,9,11,7500,10000,14500
 65 REM SALES VØLUME
 70 MAT X = A*D
 80 MAT Y = B*D
 90 MAT Z = C*D
100 PRINT "SALES BY ØFFICE AND MØNTH"
110 PRINT
112 PRINT "JANUARY"
113 MAT PRINT X
114 PRINT
115 PRINT "FEBRUARY"
116 MAT PRINT Y
117 PRINT
118 PRINT "MARCH"
119 MAT PRINT Z
125 REM AVERAGES ØF UNIT SALES
130 PRINT
133 MAT M = A+B
135 MAT M = M+C
140 MAT V = (1/3)*M
145 PRINT "AVERAGE SALES BY MØDEL AND ØFFICE"
147 PRINT
150 MAT PRINT V
160 MAT S = V*D
```

```
170 PRINT
180 PRINT "AVERAGE DØLLAR SALES"
190 MAT PRINT S
200 PRINT
201 FØR I = 1 TØ 5
210 LET T = T+S(I,1)
220 NEXT I
230 PRINT "TØTAL QUARTER SALES"
240 PRINT 3*T
250 END
```

Print Out:

```
SALES BY ØFFICE AND MØNTH

JANUARY

726500
288500
321500
339000
178000

FEBRUARY

648000
556000
323500
439000
128500

MARCH

873500
324500
460500
662500
324500

AVERAGE SALES BY MØDEL AND ØFFICE

23.0        34.0         16.3333
 8.0        12.6667      14.0
18.0         9.33333      9.66667
13.3333     14.3333      16.3333
 8.0         6.33333      6.0

AVERAGE DØLLAR SALES

749333.
389667.
368500.
480167.
210333.

TØTAL QUARTER SALES

6.594E+6
```

14.
```
10 DIM A(M,N)
20 FØR I = 1 TØ M
30 FØR J = 1 TØ N
40 READ A(I,J)
50 NEXT J
60 NEXT I
70 DATA
```

Review Exercises (pp. 53-54)

1. 3*X+Z   2. 2*Y, not 2Y   3. No comma used between IF THEN

4. No LET for matrix commands

5. A10, A11 not allowable variables

6. 7<=X, no GØ TØ with IF THEN

7. 3*(X↑2) - 5*X+2   8. (5*X+7)/X↑3   9. (2/X + 3)↑5

10. ABS(ABS(A) - ABS(B))

11.
```
 05 REM PRIME SEARCH
 10 DIM P(500)
 20 LET P(1) = 2
 30 LET P(2) = 3
 35 REM J CØUNTS PRIMES
 40 LET J = 2
 50 FØR N = 5 TØ 10 000 STEP 2
 60 FØR I = 1 TØ J
 70 IF INT(N/P(I)) = N/P(I) THEN 110
 80 IF P(I) > SQR(N) THEN 90
 85 NEXT I
 90 LET J = J+1
100 LET P(J) = N
110 NEXT N
115 REM PRINT PRIMES
120 FØR I = 1 TØ J
130 PRINT P(I),
140 NEXT I
150 END
```

12.
```
10 REM HEX PERIM, AREA
20 PRINT "SIDE   PERIMETER   AREA "
30 INPUT S
40 LET P = 6*S
50 LET A = 6*(S↑2)*(1.732/4)
60 PRINT S, P, A
70 END
```

13.
```
10 INPUT M
20 PRINT "METERS   FEET-INCHES"
30 LET I = 39.37*M
40 LET F = INT(I/12)
50 LET I = I-12*F
60 PRINT M, F; I
70 END
```

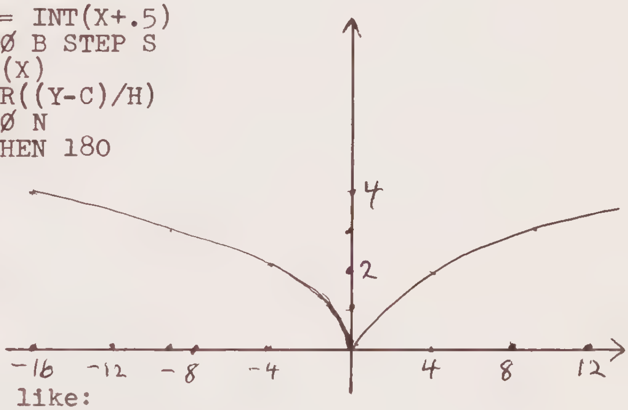
14. Introduce the following steps:
```
25 FØR M = 5 TØ 6 STEP .05
65 NEXT M
Delete 10
```

15.
```
10 FØR X = -10 TØ 5
20 IF 3*X+5 > 8 THEN 30
25 PRINT X,
30 NEXT X
40 END
```

16.
```
05 PRINT "X"; "Y"
10 FØR X = 1 TØ 11
20 FØR Y = 1 TØ 11
30 IF ABS(X-6)+ABS(Y-6)>5 THEN 50
40 PRINT X; Y
50 NEXT Y
60 NEXT X
70 END
```

17.
```
 10 DEF FNF(X) = SQR(ABS(X))
 20 READ A,B,S,C,D,N
 30 DATA -16,16,.5,0,4,40
 40 LET H = (D-C)/N
 50 PRINT "Y-AXIS FRØM" C; "TØ" D; "IN STEPS ØF" H
 60 PRINT
 70 FØR I = 0 TØ N
 80 PRINT "-";
 90 NEXT I
100 DEF FNR(X) = INT(X+.5)
110 FØR X = A TØ B STEP S
120 LET Y = FNF(X)
130 LET Y1 = FNR((Y-C)/H)
140 FØR I = 0 TØ N
150 IF Y1 = I THEN 180
160 PRINT "  ";
170 GØ TØ 190
180 PRINT "*";
190 NEXT I
200 PRINT "  ";
210 PRINT X
220 NEXT X
230 END
```

Actual graph looks like:



18. x = 18.1667   y = 7.5   z = .75

19. $||a|-|b|| \leq |a-b|$

20.
```
 10 DIM A(10,10)
 20 FØR I = 1 TØ 10
 30 FØR J = 1 TØ 10
 40 IF J < I THEN 70
 50 LET A(I,J) = J-1
 60 GØ TØ 80
 70 LET A(I, J) = I-1
 80 NEXT J
 90 NEXT I
100 MAT PRINT A;
110 END
```

21.
```
 10 INPUT A,B
 20 LET S = ((A+B)/26)-INT((A+B)/26)
 30 LET S = 26*S
 40 LET S = INT(S+.5)
 50 PRINT A; "+"; B; "="; S; "MOD 26"
 60 LET P = ((A*B)/26)-INT((A*B)/26)
 70 LET P = 26*P
 80 LET P = INT(P+.5)
 90 PRINT A; "*"; B; "="; P; "MOD 26"
100 END
```

Note: Lines 40 and 80 are roundoff instructions in case the roundoff in earlier calculations has left non-integer values for S and P.

22. (a) sum $\begin{pmatrix} -1 & 3.1 \\ 6.2 & 11.3 \end{pmatrix}$ difference $\begin{pmatrix} 5 & 2.9 \\ -8.2 & -1.3 \end{pmatrix}$

product $\begin{pmatrix} 15.6 & 19.1 \\ 39 & 31.4 \end{pmatrix}$

(b) sum $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ difference $\begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$

product $\begin{pmatrix} -30 & -36 & -42 \\ -66 & -81 & -96 \\ -102 & -126 & -150 \end{pmatrix}$

CHAPTER 1

Section 1.1 (pp.59-60)

1. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

2. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25},$

3. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}$

4. -6, -2, 2, 6, 10

5. $1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}$

6. 1, 2, 3, 4, 5

7. 2, 2, 2, 2, 2 (This is a <u>constant</u> sequence; it is a special case of a constant function.)

8. 2, 4, 8, 16, 32

9. -2, 4, -8, 16, -32

10. 1, 3, 6, 10, 15

11. yes; $a_n = |n-5|$

12. no; The domain is $\{2,4,6,8,10,12,14,16,18,20\}$, not the proper kind for a sequence.

13. yes; $a_n = 3n^2 + 5n - 6$

14.
```
10 LET N = 1
20 LET A = 3*N+20
30 IF A >= 500 THEN 70
40 PRINT N,A
50 LET N = N+1
60 GØ TØ 20
70 END
```

15. For $a_n = 2n-1$, the first four terms are 1,3,5,7
For $a_n = (n-1)(n-2)(n-3)+2n-1$, the first four terms are 1,3,5,13.
The lesson of this exercise is that listing some terms of a sequence does not determine the sequence.

16. $a_n = 2n+1$ 17. $a_n = (-1)^{n-1}$ 18. $a_n = 2^{n-1}$

19. $a_n = n(n+1)$ 20. $a_1 = 1$, $a_n = n(a_{n-1})$ when $n > 1$

21. $a_n = a_{n-2} + a_{n-1}$ $(n > 2)$; $a_1 = 1$, $a_2 = 1$

22. $a_n = 4(-\frac{1}{2})^{n-1}$ 23. 21, 34, 55, 89, 144

24.
```
 10 LET A1 = 2
 20 LET A2 = 3
 30 PRINT 1, A1
 40 PRINT 2, A2
 50 LET N = 3
 60 LET A3 = A2 + A1
 70 LET A1 = A2
 80 LET A2 = A3
 90 PRINT N, A2
100 LET N = N+1
110 IF N < 21 THEN 60
120 END
```

25. Let a be the first term, and b the second. Then the first ten terms are:
a, b, a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, 8a+13b, 13a+21b, 21a+34b.
The sum of these terms is 55a + 88b. Note that the seventh term is 5a+8b, and 11(5a+8b) = 55a + 88b. Therefore you can find the sum of the first ten terms by taking eleven times the seventh term.

Section 1.2 (pp.65-68)

1. -2, -6, -18, -54, -162; $(-2)3^{n-1}$

2. 1, -3, 9, -27, 81; $(-3)^{n-1}$

3. 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$; $4(\frac{1}{2})^{n-1}$ 4. 4, 6, 9, $13\frac{1}{2}$; $4(\frac{3}{2})^{n-1}$

5. 6000, 60000; $6(10)^{n-1}$ 6. $\frac{16}{3}$, $\frac{16}{9}$; $144(\frac{1}{3})^{n-1}$

7. 486, 729; $144(\frac{3}{2})^{n-1}$ 8. -5, 5; $(-1)^{n-1}5$

9. $8(2^6) = 512$ 10. $8(\frac{1}{2})^6 = \frac{1}{8}$

11. We want 2, x, 72, where $\frac{x}{2} = \frac{72}{x}$. Thus $x^2 = 144$, and $x = 12$. -12 is also a possible solution since 2, -12, 72 is a geometric sequence with $r = -6$. However the geometric mean of two positive numbers a and b is usually defined as the positive number x such that a, x, b is a geometric sequence. Thus, in general, $x = \sqrt{ab}$.

12. $\sqrt{48}$, or $4\sqrt{3}$ 13. $\sqrt{ab}$

14. $\frac{x}{a} = \frac{a}{y}$. So $a = \sqrt{xy}$, $x = \frac{a^2}{y}$, $y = \frac{a^2}{x}$

15. $(AD)^2 + (CD)^2 = 16$ and $(5-AD)^2 + (CD)^2 = 9$
$\therefore 10(AD) = 32$, $AD = 3.2$, $BD = 1.8$.
Also $(CD)^2 = (3.2)(1.8) = 5.76$. $DC = 2.4$

16. The sequence is 50000, 50000(1.1), $50000(1.1)^2$, ...
The fifth term is $50000(1.1)^4 \approx 73205$

17. $6000(.85)^4 = \$\ 4335$

18. Geometric; $r = \frac{1}{2}$ Converges to 0

19. Geometric; $r = -2$ Does not converge

20. Geometric; $r = .9$ Converges to 0

21. Geometric; $r = -1$ Does not converge

22. Geometric; $r = -\frac{1}{2}$ Converges to 0

23. Not geometric. Converges to 0

24. Not geometric. Converges to 5

25. (a) $n = 11$ (b) $n = 1001$ (c) $n = 1,000,001$

26. Answers vary

27. Yes. The sequence a_1c, a_1cr, $a_1cr^2, \ldots$, $a_1cr^{n-1}, \ldots$ is geometric. The common ratio is r.

8. No. The sequence a_1+c, a_1r+c, a_1r^2+c, ..., $a_1r^{n-1}+c, \ldots$ is not geometric since $\frac{a_1r+c}{a_1+c} \neq \frac{a_1r^2+c}{a_1r+c}$

29.
```
10 FØR N = 1 TØ 50
20 LET A1 = 2↑(-N)
30 LET A2 = 1 + 2↑(-N)
40 PRINT N, A1, A2
50 NEXT N
60 END
```
(Note: To most computers, $1 + 2^{-50} = 1$. This is called round off error; $1 + 2^{-50}$, rounded to the seven or ten digits your computer may hold, is exactly one.)

30. The first term is π. The second is $2(\frac{1}{2}\pi) = \pi$.
The third is $4(\frac{1}{4}\pi) = \pi$. In general,
$a_n = 2^{n-1}(\frac{1}{2^{n-1}}\pi) = \pi$.
Thus the sequence is the constant sequence

$$\pi, \pi, \pi, \ldots,$$

which converges to π.

31. The sequence here is $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{8}$, ..., where $a_n = \frac{\pi}{2^n}$.
This sequence converges to 0.

Section 1.3 pp.72-74)

1.
```
10 LET S = 0
20 FØR X = 1 TØ 50
30 LET S = S+X
40 NEXT X
50 PRINT S
60 END
```

Formula: $\frac{n(n+1)}{2} = \frac{50(51)}{2} = 1275$

2. $\frac{100(101)}{2} = 5050$ 3. $\frac{1000(1001)}{2} = 500,500$

4. $2S = \underbrace{(n+1)+(n+1)+\ldots+(n+1)}_{n \text{ addends}} = n(n+1)$

So $S = \frac{n(n+1)}{2}$

5. The sum of the first n even integers is the product of n and n+1.
$n = 5 : 2 + 4 + 6 + \ldots + 10 = 30 = 5(6)$
$n = 8 : 2 + 4 + 6 + \ldots + 16 = 72 = 8(9)$
$n = 10 : 2 + 4 + 6 + \ldots + 20 = 110 = 10(11)$

Proof: For $n=1$, $2=1(2)$
Assume: $2+4+6+\ldots+2k = k(k+1)$
Then: $2+4+6+\ldots+2k+2(k+1) = k(k+1)+2(k+1)$
$= (k+1)(k+2)$

6. The sum of the first n multiples of 3 is $\frac{3}{2}$ times the product of n and n+1.
$n = 5: 3 + 6 + 9 + 12 + 15 = 45 = \frac{3}{2}(5)(6).$
$n = 8: 3 + 6 + 9 + \ldots + 24 = 108 = \frac{3}{2}(8)(9).$
$n = 10: 3 + 6 + 9 + \ldots + 30 = 165 = \frac{3}{2}(10)(11).$

Proof: For $n = 1$, $3(1) = \frac{3}{2}(1)(2)$
Assume: $3+6+9+\ldots+3k = \frac{3k(k+1)}{2}$
Then: $3+6+9+\ldots+3k+3(k+1) = \frac{3k(k+1)}{2} + 3(k+1)$
$= 3(k+1)(\frac{k}{2}+1)$
$= \frac{3}{2}(k+1)(k+2)$

7. The sum of the first n multiples of 4 is twice the product of n and n+1
$n = 5: 4 + 8 + 12 + 16 + 20 = 60 = 2(5)(6)$
$n = 8: 4 + 8 + 12 + \ldots + 32 = 144 = 2(8)(9)$
$n = 10: 4 + 8 + 12 + \ldots + 40 = 220 = 2(10)(11)$

Proof: For $n = 1$, $4(1) = 2(1)(2)$
Assume: $4+8+12+\ldots+4k = 2k(k+1)$
Then: $4+8+12+\ldots+4k+4(k+1) = 2k(k+1)+4(k+1)$
$= 2(k+1)(k+2)$

8. The sum of the first n multiples of 5 is $\frac{5}{2}$ times the product of n and n+1
$n = 5: 5 + 10 + 15 + 20 + 25 = 75 = \frac{5}{2}(5)(6)$
$n = 8: 5 + 10 + 15 + \ldots + 40 = 180 = \frac{5}{2}(8)(9)$
$n = 10: 5 + 10 + 15 + \ldots + 100 = 275 = \frac{5}{2}(10(11)$

Proof: For $n = 1$, $5(1) = 5$

Assume: $5 + 10 + 15 + \ldots + 5k = \frac{5k(k+1)}{2}$

Then: $5+10+15+\ldots+5k + 5(k+1) = \frac{5k(k+1)}{2} + 5(k+1)$

$= 5(K+1)(\frac{k}{2} + 1)$

$= \frac{5}{2}(k+1)(k+2)$

9. The generalization is this: If p is any number, then the sum of the first n multiples of p is $\frac{p}{2}n(n+1)$; i.e.,

$$p + 2p + 3p + \ldots + np = \frac{p}{2}n(n+1)$$

Proof: For $n = 1$, $p = \frac{p}{2}(1)(2)$

Assume: $p+2p+3p+\ldots+kp = \frac{p}{2}k(k+1)$

Then: $p+2p+3p+\ldots+kp+(k+1)p = \frac{p}{2}k(k+1)+(k+1)p$

$= p(k+1)(\frac{k}{2} +1)$

$= \frac{p}{2}(k+1)(k+2)$

Alternate Proof:

$1+2+3+\ldots+n = \frac{n}{2}(n+1)$ (previously proved)

multiply both sides by p,

$p + 2p + 3p + \ldots + np = \frac{p}{2}n(n+1)$

10. The sum of the first n odd numbers is n^2.

$n = 5$: $1 + 3 + 5 + 7 + 9 = 25 = 5^2$

$n = 8$: $1 + 3 + 5 + \ldots + (2\cdot8-1) =$

$1 + 3 + 5 + \ldots + 15 = 64 = 8^2$

$n =10$: $1 + 3 + 5 + \ldots + (2\cdot10-1) =$

$1 + 3 + 5 + \ldots + 19 = 100 = 10^2$

Proof: For $n = 1$, $1 = 1^2$

Assume $1+3+5+\ldots+(2k-1) = k^2$

Then $1+3+5+\ldots+(2k-1)+[2(k+1)-1]=k^2+[2(k+1)-1]$

$=k^2+2k+1$

$=(k+1)^2$

11. The sum of the squares of the first n whole numbers is $\frac{n(n+1)(2n+1)}{6}$

$n = 5$: $1^2+2^2+3^2+4^2+5^2 = 55 = \frac{5(6)(11)}{6}$

$n = 8$: $1^2+2^2+3^2+\ldots+8^2 = 204 = \frac{8(9)(17)}{6}$

$n =10$: $1^2+2^2+3^2+\ldots+10^2=385 = \frac{10(11)(21)}{6}$

Proof: For $n = 1$, $1^2 = \frac{1(2)(3)}{6}$

Assume: $1^2+2^2+3^2+\ldots+k^2 = \frac{k(k+1)(2k+1)}{6}$

Then: $1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+2)[2(k+1)+1]}{6}$$

12. A) The statement asserts that the sum of the first $\underline{n}$ even integers is $n(n+1)+10$. This is <u>not</u> true for $n = 1$, since $2 \neq 1(2)+10$. Note, however, that the second part of the induction proof can be carried through (although of course it is useless without the first part) as follows:

Assume $2+4+6+\ldots+2k = k(k+1)+10$

Then $2+4+6+\ldots+2k+2(k+1) = k(k+1)+10+2(k+1)$
$\qquad = (k+1)(k+2)+10$

In summary, this shows that if you ever get a foot on the first rung of the ladder, you are guaranteed to be able to get to the next rung. The trouble is that you can't ever get on a rung!

B) This statement asserts that the sum of the first $\underline{n}$ even whole numbers is n^2+2n-1. It <u>is</u> true for $n=1$, since $2 = 1+2-1$. Here you are able to get on the first rung all right, but your progress to successive rungs is not guaranteed, since the second part of the inductive argument <u>fails</u>:

Assume: $2+4+\ldots+2k = k^2 + 2k - 1$

Then: $2+4+\ldots+2k+2(k+1) = k^2+2k-1 + 2(k+1)$
$\qquad = k^2+4k+1$

For the argument to succeed, this result must be the same as

$$(k+1)^2 + 2(k+1) - 1 = k^2 + 4k + 2$$

Obviously, since $2 \neq 1$, it isn't!

13. If $n = 1$ (one disc), 1 move is required, and $1 = 2^1-1$

Next, assume that for $\underline{k}$ discs the number of required moves is 2^k-1. Then consider $k+1$ discs and for convenience call the bottom (largest) disc L. Leaving L stationary, it will take 2^k-1 moves to get the top $\underline{k}$ discs on another peg. Then use one move to get L on a vacant peg. Finally it will take 2^k-1 moves to get the other $\underline{k}$ discs in position above L. Therefore, for $k+1$ discs the number of moves is

$$(2^k-1)+1+(2^k-1) = 2\cdot 2^k-1$$
$$= 2^{k+1}-1.$$

14. If this proof were valid, you could prove all sorts of fascinating things: All horses are the same color, all people the same age, etc. Of course it isn't valid. The fallacy stems from the fact that induction is based on the whole numbers which are ordered so that each has a <u>successor</u> and no two occupy the same

position "in line." The argument in this exercise tries to apply the inductive assumptions to a proof about <u>sets</u>, which are not ordered in the same way as the whole numbers. For example, of two distinct sets, each having ten elements, which comes <u>first</u>?

<u>Section 1.4 (pp.77-78)</u>

1. $1+2+4+8 = 15$; $(1)\frac{2^4-1}{2-1} = 15$

2. $\frac{1}{2} + \frac{3}{2} + \frac{9}{2} + \frac{27}{2} = 20$; $(\frac{1}{2})\frac{3^4-1}{3-1} = \frac{1}{2}(40) = 20$

3. $-4+2+(-1)+\frac{1}{2} = -\frac{5}{2}$; $(-4)\frac{1-(-\frac{1}{2})^4}{1-(-\frac{1}{2})} = (-4)\frac{\frac{15}{16}}{\frac{3}{2}} = -\frac{5}{2}$

4. $-4 + 4 - 4 + 4 = 0$; $(-4)\frac{(-1)^4-1}{-1-1} = 0$

5. $$S_5 = 2\left[\frac{1-(\frac{1}{10})^5}{1-\frac{1}{10}}\right] = 2\left[\frac{\frac{99999}{100000}}{\frac{9}{10}}\right] = \frac{22222}{10000} = 2.2222$$

6. $$S_{10} = 1\left[\frac{(-2)^{10}-1}{2-1}\right] = 1023$$

7. $$S_{10} = 1\left[\frac{(-2)^{10}-1}{-2-1}\right] = \frac{1023}{-3} = -341$$

8. $$S_5 = \sqrt{2}\left[\frac{(-\sqrt{2})^5-1}{-\sqrt{2}-1}\right] = \sqrt{2}\left[\frac{-4\sqrt{2}-1}{-\sqrt{2}-1}\right] = \frac{8+\sqrt{2}}{1+\sqrt{2}}.$$

9. $$S_6 = 32\left[\frac{(\frac{3}{2})^6-1}{\frac{3}{2}-1}\right] = 32\left[\frac{\frac{729}{64}-1}{\frac{1}{2}}\right] = 32\left[\frac{665}{64}\cdot\frac{2}{1}\right] = 665$$

10. $$S_{20} = 32\left[\frac{1-(\frac{3}{4})^{20}}{1-\frac{3}{4}}\right] = 128\left(1-(\tfrac{3}{4})^{20}\right)$$

11. $S_1 = 1$, $S_2 = \frac{3}{2}$, $S_3 = \frac{7}{4}$, $S_4 = \frac{15}{8}$, $S_5 = \frac{31}{16}$, $S_{10} = \frac{1023}{512}$, $S_{20} = \frac{1048576}{524288}$. The sum will never exceed 2.

   Since $S_n = \dfrac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 2[1-(\frac{1}{2})^n]$, the sum will always be the product of 2 and some positive number less than one. Note however that the sum can be made arbitrarily close to 2 by taking <u>n</u> large enough. Thus, S_{20} differs from 2 by only $\frac{1}{524288}$.

12. If $r = 1$, the denominator in the formula is 0, and thus the formula is meaningless. However in the case of $r = 1$, the sequence is constant, and so $S_n = na_1$.

13. $-10 = 2\left(\frac{r^4-1}{r-1}\right)$. So $r^3 + r^2 + r + 6 = 0$ $r = -2$ is a solution. Therefore the first four terms are $2,-4,8,-16$.

14. $r^5 = \frac{128}{4} = 32$. So $r = 2$, and $a_1 = 1$. $S_5 = \frac{2^5-1}{2-1} = 31$.

15. Since $a_2 = (a_1)^2$, $a_1 = r$, and the sequence is of form $r, r^2, \_\_\_, \_\_\_, \frac{1}{3}(r^2)^2$.

    But since $a_5 = r^2 \cdot r^3 = r^5$, $r^5 = \frac{1}{3}r^4$, $r = \frac{1}{3}$.

    So, the first five terms are $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}$.

16. Choose B. At the end of 30 days, Option A would amount to \$30,000. Option B would total

$$S_{30} = 1\,\frac{2^{30}-1}{2-1} = 2^{30}-1 = \$232,881,023.$$

17. $a_1 = \frac{3}{4}$, $r = \frac{3}{4}$, $a_5 = \frac{3}{4}\left(\frac{3}{4}\right)^4 = \frac{243}{1024}$

    (Note that this problem differs from Example 2. Here it is not the <u>sum</u> we are trying to find.)

18.
```
10 REM PRØGRAM BY LØØP PRØCEDURE
20 INPUT A1, R, N
30 LET S = 0
40 FØR M = 1 TØ N
50 LET A = A1*R↑(M-1)
60 LET S = S+A
70 NEXT M
80 PRINT "THE SUM ØF THE SEQUENCE IS" S
90 END
```

```
10 REM PRØGRAM USING THEØREM 1
20 INPUT A1, R, N
30 LET S = A1*(1-R↑N)/(1-R)
40 PRINT "THE SUM ØF THE SEQUENCE IS" S
50 END
```

Section 1.5 (pp.83-84)

1. $7,15,22,29,36,43$ 2. $7,4,1,-2,-5,-8$

3. $\sqrt{2},\sqrt{2}+10,\sqrt{2}+20,\sqrt{2}+30,\sqrt{2}+40,\sqrt{2}+50$

4. $9,9+y,9+2y,9+3y,9+4y,9+5y$

5. $-6,1,8,15,22,29$ 6. $\frac{1}{5}, \frac{3}{5}, 1, \frac{7}{5}, \frac{9}{5}, \frac{11}{5}$

7. $x,y,2y-x, 3y-2x, 4y-3x, 5y-4x$ 8. 465   9. 370

10. -175 11. 27   12. 15,250

13. For $n = 1$, $a_1 = \frac{1}{2}(a_1+a_1)$

Assume: $a_1+(a_1+d)+\ldots+[a_1+(k-1)d] = \frac{k}{2}(a_1+a_k)$

$= \frac{k}{2}[a_1+(a_1+(k-1)d)]$

Then: $a_1+(a_1+d)+\ldots+[a_1+(k-1)d]+(a_1+kd) =$

$\frac{k}{2}[2a_1+(k-1)d]+(a_1+kd) =$

$\frac{k+1}{2}[a_1+(a_1+kd)] =$

$\frac{k+1}{2}(a_1+a_{k+1})$.

14. The sequence is 16, 48, 80, ..., with $a_1=16$, $d=32$.
$S_8 = 4[16+(16+7(32))] = 4(16+224) = 960$.

15. The sequence is 20, 30, 40, ... $S_7 = \frac{7}{2}(20+80) = 350$

16. The sequence is 15000, 14500, 14000, ...
$a_{19} = 15000 + 18(-500) = 6000$

17. $7506.49

18.
```
10 REM PRØGRAM WITHØUT FØRMULA
20 INPUT A1, D
30 LET S = 0
40 FØR N = 1 TØ 20
50 LET S = S+A1
60 LET A1 = A1+D
70 NEXT N
80 PRINT "THE SUM ØF THE FIRST 20 TERMS IS" S
90 END
```

```
10 REM PRØGRAM USING THEØREM 2
20 INPUT A1, D
30 LET A2=A1
40 FØR N = 2 TØ 20
50 LET A2 = A2+D
60 NEXT N
70 LET S = N*(A1+A2)/2
80 PRINT "THE SUM ØF THE FIRST 20 TERMS IS" S
90 END
```

We can use the fact that $a_n=a_1+(n-1)d$ to rewrite the above program as follows:

```
10 REM PRØGRAM USING THEØREM 2
20 INPUT A1,D
30 LET A2 = A1 + 19*D
40 LET S = 20*(A1+A2)/2
50 PRINT "THE SUM ØF THE FIRST 20 TERMS IS" S
60 END
```

19. (a) $7\frac{1}{2}, 6$ (b) $12\frac{1}{2}, 10$ (c) $9, 6\sqrt{2}$ (d) $\frac{\sqrt{2}+\sqrt{3}}{2}\approx 1.573, \sqrt{\sqrt{6}}\approx 1.$
In each case the arithmetic mean is greater.

20. $(\sqrt{a}-\sqrt{b})^2 > 0$ since $a \neq b$
$a + b - 2\sqrt{ab} > 0$, $a+b > 2\sqrt{ab}$, $\frac{a+b}{2} > \sqrt{ab}$.

21. Yes. The common difference is the sum of the common differences of the original two sequences.

22. No. A counterexample may be used here.
$1,3,5,7,\ldots \quad d = 2$
$1,4,7,10,\ldots \quad d = 3$
But $1,12,35,70,\ldots$ is not arithmetic since $12-1 \neq 35-12$.

23. No. Using the sequences in the answer to Exercise 22, the "quotient" sequence is
$1, \frac{3}{4}, \frac{5}{7}, \frac{7}{10}, \ldots,$ and it is not arithmetic.

<u>Section 1.6 (pp.89-90)</u>

1. (a) $\frac{1}{2^n} < \frac{1}{10}$; $n = 4$ Emphasize that for <u>all</u> $n \geq 4$, s_n is within .1 of 1
(b) $\frac{1}{2^n} < \frac{1}{100}$; $n = 7$ (c) $\frac{1}{2^n} < \frac{1}{10000}$; $n = 14$

2. (a) $\frac{1}{2}(3^n-1) > 10$, $3^n-1 > 20$, $3^n > 21$; $n \geq 3$
(b) $3^n > 201$; $n \geq 5$ (c) $3^n > 2001$; $n \geq 9$

3. $-1 < r < 1$, or $|r| < 1$ 4. -4 5. $\frac{4}{3}$ 6. $\frac{1}{2}$ 7. 16

8. no sum 9. $\frac{1}{3}$ 10. $\frac{1}{2}$ 11. no sum 12. $2,1,\frac{3}{2},\frac{5}{4},\frac{11}{8}$

13. 5,0,5,0,5 14. 3,6,9,12,15

15. $1, 1\frac{1}{2}, 1\frac{5}{6}, 2\frac{1}{12}, 2\frac{17}{60}$ (Students may be interested to know that this series, the <u>harmonic series</u>, has no sum. It can be made greater than any specified N.)

16. $.6+.06+.006+\ldots$; $a_1=.6$, $r=.1$ $S_n = .6\left[\frac{1-(.1)^n}{.9}\right]$
as <u>n</u> increases, S_n approaches $\frac{6}{10} \times \frac{10}{9} = \frac{2}{3}$

17. $.00\overline{9} = .009+.0009+\ldots$ This approaches $.009(\frac{1}{.9}) = .01$
Thus $.24\overline{9} = .24+.00\overline{9} = .24 + .01 = .25$

18. $.249\left(\frac{1-(.001)^n}{.999}\right) \rightarrow \frac{249}{999} = \frac{83}{333}$

19. $.23\left(\frac{1-(.23)^n}{.99}\right)$; $-1.\overline{23} = -1\frac{23}{99}$ 20. $4\frac{37}{990}$ 21. $\frac{5074920}{990000}$

22. $\frac{33}{100}$ 23. $.4\overline{9}$ 24. $.874\overline{9}$ 25. $1.1\overline{3}$ 26. $-1.58\overline{3}$

27.
```
10 INPUT A1, R
20 LET S = 0
30 FØR N = 1 TØ 30
40 LET S = S+A1
50 LET A1 = A1*R
60 NEXT N
70 PRINT "THE SUM ØF THE FIRST 30 TERMS IS" S
80 END
```

28.
```
10 LET Q = 1
20 LET S = 0
30 FØR N = 1 TØ 20
40 LET S = S+Q/N
50 LET Q = -Q
60 PRINT N,S
70 NEXT N
80 END
```

This sequence does converge, as the printout suggests. (The limit is $\log_e 2 \approx 0.6931$.) Students may be interested to know that any series of type $\Sigma(-1)^{n+1}a_n$ converges provided that $|a_n|$ is decreasing and $a_n \to 0$ as $n \to \infty$

29.
```
10 LET S = 0
20 FØR N = 1 TØ 20
30 LET S = S+1/N
40 PRINT N,S
50 NEXT N
60 END
```

This is the famous <u>harmonic series</u>, and it does not converge, since the sum can be made as great as desired. For example, $S_8=1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})$ which is clearly greater than $1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}) = \frac{5}{2}$

In the same way the 2^kth term is greater than $\frac{k+2}{2}$. The secret is to "capture" enough terms each time to make each parenthetical expression greater than $\frac{1}{2}$.

<u>Review Exercises (pp.91-92)</u>

1. $-8, -4\frac{1}{2}, -1, 2\frac{1}{2}, 6, 9\frac{1}{2}$

2. $a_{100} = -8 + 3\frac{1}{2}(99) = 338\frac{1}{2}$

3. Divergent

4. $S_{20} = \frac{20}{2}(-8 + 62\frac{1}{2}) = 545$

5. $-8, -12\frac{1}{2}, -13\frac{1}{2}, -11, -5, 4\frac{1}{2}$

6. $S_{10} = \frac{10}{2}(-8 + 15\frac{1}{2}) = 37\frac{1}{2}$

7. 243, 81, 27, 9

8. $a_8 = 243(\frac{1}{3})^7 = \frac{1}{9}$

9. Converges to 0

10. $S_8 = 243\left[\frac{1-(\frac{1}{3})^7}{1-\frac{1}{3}}\right] = 364\frac{1}{3}$

11. 243, 324, 351, 360

12. $243\left[\frac{1-(\frac{1}{3})^n}{\frac{2}{3}}\right]$; the limit is $243 \times \frac{3}{2} = 364\frac{1}{2}$

13. $x + y = 40;\ xy = 144$

$x(40-x) = 144$

$x^2 - 40x + 144 = 0$ 4, 36

14. $.02[275 + 250 + 225 + \ldots + 25 + 0] =$
$.02[6(275) = 33.00$
Therefore the total paid is \$300 + \$ = \$333.

15. By the definition, addition is an operation on the set of real sequences. Associativity is guaranteed by the fact that addition of real numbers is associative. The identity element is the constant sequence 0,0,0,... The inverse of a sequence $a_1, a_2, a_3, \ldots$ is $-a_1, -a_2, -a_3, \ldots$

16. No. Any sequence in which a zero term appears has no inverse.

17. Yes - in fact, a commutative ring with unity.

CHAPTER 2

Section 2.1 (pp. 97-101)

1. $\overline{ACFKM} = 297$, $\overline{ACHLM} = 298$, $\overline{ACHKM} = 302$

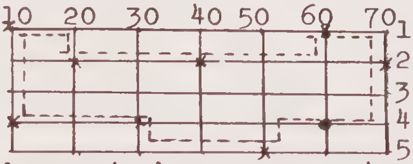
2. $\overline{ADGJ} = 232$

3. $\overline{MJEBA}$ is the reverse of the longest New York to Los Angeles route. West to East winds must be greatest on these segments, so it would be shortest going the west to east direction.

4. For example, rivers or railroads or highways or even mountains for transcontinental jets at 30,000 feet.

5. Cities often imply air traffic congestion and, of course, location of destination. Distances-entered on the map by a scale of miles-are a factor.

6. One of many minimal length paths is sketched below



The diagram does not show one way streets, particularly bad congestion spots, location of dumping spot (if different from depot). Students might well think of other important information.

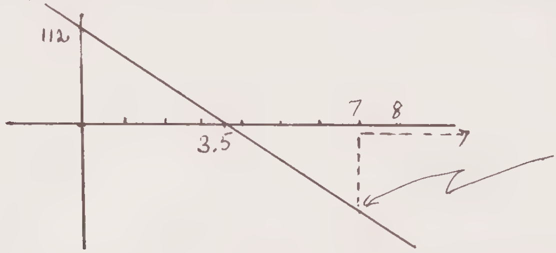
7. (a) On average quickest is No. 10 bus -- 12 minutes.
(b) Walking--15 minutes--since he <u>might</u> have to wait 8 or 10 minutes for the busses.
(c) Safety; rain or not; carrying packages; cost; healthful exercise, etc.

8. $h(8) = -56$ says that at 8 seconds after burnout the rocket has supposedly returned to earth and burrowed 56 feet into the ground. This emphasizes the idea that a model might be valid subject to certain limitations

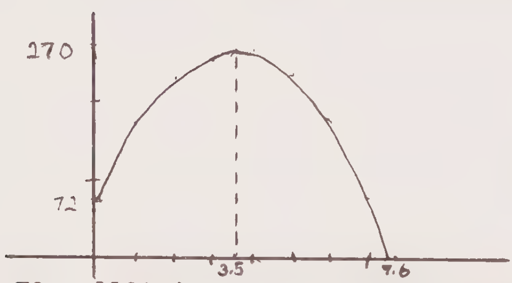
on applicability. This picks up on the exponential and curve situation where for a while the two have much the same shape. On a very high level, euclidean geometry is a good model of physical things in "small" chunks of the universe; for more macro-scopic models, non-euclidean parallel axioms fit experience better.

9. For time < 3.5 seconds the rocket is rising, $v(t) > 0$; for time > 3.5 seconds the rocket is falling, $v(t) < 0$.

10. The graph will look like that below. However, for $t > 7$ $v(t) = 0$ in practice!



11. The graph will look roughly like that below:



12. $h(t) = 72 + 112t$ is an unbounded increasing function which is positive for all $t > -\frac{9}{14}$.

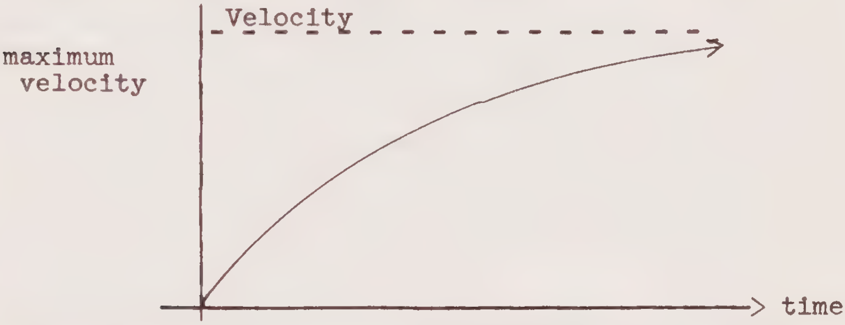
13. (a) 20 seconds
(b) -640 feet per second [about 436 miles per hour]
(c)

t	v(t)	h(t)
0	0	6400
2	-64	6336
4	-128	6144
6	-192	5824
8	-256	5376
10	-320	4800
12	-384	4096
14	-448	3264
16	-512	2304
18	-576	1216
20	-640	0

velocity is in feet per second
height is in feet

(d) 320 feet/second or about 218 miles per hour!

14. Of course no one could open a chute at 218 miles per hour. The model that has been used does not account for the retarding force of wind resistance--which actually makes a falling body asymptotically approach a terminal or limiting velocity. A better model of free fall is $h(t) = \frac{m \cdot g}{k}[1-e^{kt}]$ where m is mass of the falling body, g is gravitational constant acceleration (32 feet per second per second), k is a negative constant depending on shape and mass of falling object. Derivation of this equation involves simple differential equations, but it's graph is given below.



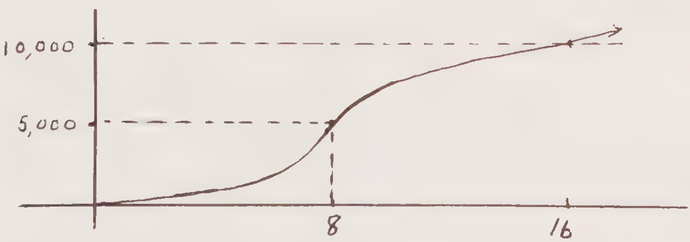
In the chapter on exponential functions it would be good to recall this--when students know the behavior of e^{kt} and $1-e^{kt}$ for various k.

15. The two shaped graphs are most likely--growth does not usually occur linearly, and the first exponential type (students can identify this as geometric) seems to increase without bound (unrealistic because food supply is usually limited).

16. The spread of rumors fits the disease model in text!

17. Note that from 1950-1970 growth was a geometric sequence with $a_{5n+5} \approx 1.25a_{5n}$. There are many special local factors that might invalidate the prediction--declining birth rate, major loss or gain of industry, etc. Lacking any special factors to consider, though, the method used is probably better than drawing numbers from a hat. Some sort of survey of pre-high school enrollments would help.

18.



In many ways this model makes sense for $0 \leq t \leq 16$, though it is unlikely that all 10,000 would become ill.

Section 2.2 (pp.104-106)

1. (a) 12480 lbs/sq.ft. (b) 18720 lbs/sq.ft. (c) 31200 lbs/sq.ft. (d) 1560 lbs/sq.ft. (e) 329,472 lbs/sq.ft.

2. (a) 15 feet (b) 75 feet (c) 1000 feet (d) 288 feet

3. $d(f) = k \cdot f \quad d(10) = .5 = k \cdot 10 \Rightarrow k = .05$
 $\therefore d(18) = .9$ inch

4. $a(r) = k \cdot r^2 \quad a(1) = 3.1416 = k \cdot 1^2 \Rightarrow k = 3.1416\ldots$
 $a(10) = 3.1416(10)^2 = 314.16$ square inches

5. 18.3 6. 33,000 feet or about 6.25 miles
 8800 feet or about 1.67 miles

7. 240 feet per second 8. \$20 9. \$8 10. \$1120/mo.

11. (a) increases (b) triples (c) is halved

12. multiplied by 9 13. multiplied by 8

14. $x = \frac{k}{y}$ or $xy = k$

15. $g = \frac{k}{d^2}$ (a) multiplied by $\frac{1}{9}$ (b) multiplied by 4

16.

m	157.5	3.75	1.88	.94	.47	.23	.12	.06	.03	.015
t	0 1	2	3	4	5	6	7	8	9	10

17. $m(t) = 8(\frac{1}{2})^{t/2}$ 18. see text

19. $o = 16\,p$ and ounces $= (\frac{\text{ounces}}{\text{pounds}})$ x pounds

20. $k = 1.6m$ and kilometers $= (\frac{\text{kilometers}}{\text{miles}})$ x miles

21. $k = 1.6m$ and $\frac{km}{hr} = \frac{km}{m} \times \frac{m}{hr}$

22. $k = 1.6m$ and $\frac{km}{gal} = \frac{km}{m} \times \frac{m}{gal}$

23. $d' = \frac{1}{16}d \quad \frac{\text{dollars}}{\text{ounce}} = \frac{\text{pound}}{\text{ounce}} \times \frac{\text{dollars}}{\text{pound}}$

24. $f = \frac{1}{144}i \quad (\text{feet})^2 = \frac{(\text{feet})^2}{(\text{inch})^2}(\text{inches})^2$

Section 2.3 (pp.109-110)

1. Many possibilities:
$$T_c + T_L = 200 \quad T_c = 200 - T_L$$
$$P_c + P_L = 800 \quad P_c = 800 - P_L$$
$$T_c + P_c = 600 \quad T_c = 600 - P_c$$
$$T_L + P_L = 400 \quad T_L = 400 - P_L$$
$$C = 3T_c + 6T_L + 4P_c + 8P_L$$

2. $T_L = 400 - P_L;\ T_c = -200 + P_L;\ P_c = 800 - P_L$

3. $C = 5000 + P_L$

4. P_L must be at least 200 since $T_c = P_L - 200$; but C is minimum when P_L is minimum. Thus let $P_L = 200$; C = \$5200

5. $T_c = 0$, $T_L = 200$, $P_c = 600$, $P_L = 200$

6. 15,700 mph 7. 10¢ per paper and 2 hours not related to \$40 minimum.

8. 133 papers per day

9. $(2x)(2y) = xy + 100 \Leftrightarrow 3xy = 100$
still need ratio of x and y or one of x or y

10. $2\omega + 2(2\omega) = 240 \Rightarrow \omega = 40,\ \ell = 80$

11. Error could be 0, of one measurement is short by .5 and the other long by .5; or error could be at most 2 inches.
Note: error in area is usually greater--depending on actual length and width because
$(\ell \pm .5)(\omega \pm .5) - \ell\omega = \pm .5\omega \pm .5\ell \pm .25$

12. Let ω = current wind and s = speed of plane in still air

$$\begin{cases}(s+\omega)(4.5) = 2800\\(s+\omega-25)(5.4) = 2800\end{cases}$$

Note: If you try to solve these simultaneously you will find no solution. The pilot is mistaken about his time estimates.

13. Let m_1 = model 1 and m_2 = model 2.

$$\begin{aligned}800\,m_1 + 1100\,m_2 &\le 24000\\2000\,m_1 + 3500\,m_2 &\le 70000\\m_1 \ge 0 \quad m_2 &\ge 0\end{aligned}$$

maximize $1500\,m_1 + 2900\,m_2$
Note: The solution to this linear program is $(m_1,m_2) = (0,20)$ with profit \$58,000, 22,000 man hours used, and \$70,000 capital required.

14. $4x^2 + 9y^2 = 36$ Determine a,b by substituting for x,y and solving the two equations in a^2,b^2.

<u>Section 2.4 (pp.113-115)</u>

1. $D_1 = 2$ 2. $D_2 = .5$ 3. $D_3 = 4$ 4. $D_4 = 6$

5. $D_5 \approx .08$ 6. (a) $D_1 = 3$ (b) $D_3 = 9$

7. Probably D_3 since the positive and negative deviations in D_1 cancel each other out, suggesting a better fit than is really the case.

8. (a) $D_1 = 4$ (b) $D_3 = 4$
Since all deviations are positive in this case, absolute value has no effect.

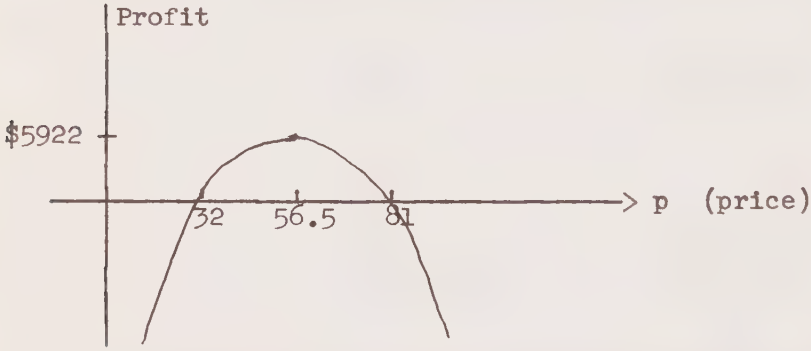
9. dollar cost = 280g + 2200, so

$$\text{cost} = 280(-p + 85) + 220 \left(\frac{\text{dollars}}{1000\text{gal}}\right)(1000\text{gal}) + \text{dollars}$$
$$= -280p + 26000$$

$$\text{dollar revenue} = \text{price x sales}$$
$$= \frac{p}{100} \text{ x } (1000g) \quad \frac{\text{dollars}}{\text{gallon}} \text{ x gallons}$$
$$= 10p(-p + 85)$$
$$= -10p^2 + 850p$$

$$\text{profit} = \text{revenue} - \text{cost} = -10p^2 + 1130p - 26000$$

10. To find maximum profit (without calculus), graph profit, observe symmetry, find zero points, and maximum is midway between them.



At 56.5¢/gallons, sales predicted are 28,500 gals. This yields revenues \$16,102, costs \$10,180, profit \$5,922

11. The model $g = -p + 85$ is based on pre-energy crisis price data, clearly.
   1. Under ordinary conditions buyer resistance at the higher price would probably cause sales to drop in a non-linear way.
   2. When demand exceeds supply, the station is not free to pump as much gasoline as might be used to meet optimum profit.
   3. The linear cost function is probably not reasonable because employee costs increase in steps as each additional service worker is hired.

12. Z represents cost of water, labor, and pumping power. Unit of Z is dollars per inch.

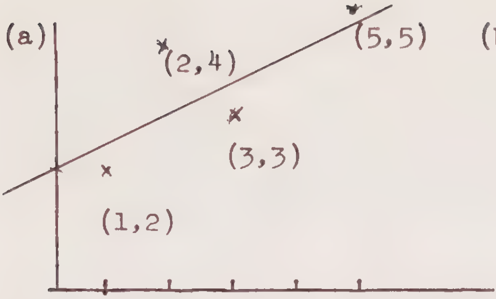
13. revenue $40(h) = 4\omega + 160$

14. $4\omega + 160 = 2\omega + 175$

$$\omega = 7.5$$

15. 40 mph. Solve by finding zeros of the quadratic and selecting the average of these mph. figures. The gasoline consumption is 24 mpg.

16. (a)



(b) $\Sigma(y_i - f(x_i))^2 = 1.75$

17.(a)

```
 05 PRINT "SLØPE"  "INTERCEPT   SUM SQUARE DEV."
 10 DIM X(4), Y(4)
 20 FØR I = 1 TØ 4
 25 READ X(I), Y(I)
 30 NEXT I
 40 DATA 1,2,2,4,3,3,5,5
 50 FØR M = 0 TØ 2 STEP .2
 60 FØR B = 0 TØ 5 STEP .2
 70 LET S = 0
 80 DEF FNG(X) = M*X + B
 90 FØR J = 1 TØ 4
100 LET S = S + (Y(J) - FNG(X(J)))↑2
110 NEXT J
120 PRINT M, B, S
130 NEXT B
140 NEXT M
150 END
```

Note: This will yield 250 lines of printout! Emphasizing the value of sharpening estimates in advance of computation!

(b) This search suggests y = .6x + 1.8 with sum of square deviations 1.56 (See 20 below).

18. $\overline{x}$ and $\overline{y}$ are means;

```
 10 READ N
 20 DIM X(100), Y(100)
 30 FØR J = 1 TØ N
 40 READ X(J), Y(J)
 50 NEXT J
 60 FØR I = 1 TØ N
 70 LET S = S + X(I)
 80 LET Q = Q + X(I)↑2
 90 LET U = U + Y(I)
100 LET P = P + X(I)*Y(I)
110 NEXT I
120 LET A = (P - S*(U/N))/(Q - N*(S/N)↑2)
130 LET B = U/N - A*(S/N)
140 PRINT "SLØPE ="; A, "INTERCEPT ="; B
150 DATA N X1 Y1 X2 Y2 etc _ _ Xn Yn
160 END
```

19. Slope = 2.7   Intercept = -.5

20. Slope = .628571   Intercept 1.77143;
Compare this to the match y = .6x + 1.8 suggested by exploration in 17.

Section 2.5 (pp. 120-123)

1. .98 2. .72 3. .891 4. .9639 5. .9801

6. ① $A \cup B$ ② $A \cap B$ ③ $A \cap (B \cup C)$ ④ $(A \cap C) \cup (B \cap D)$ ⑤ $(A \cup B) \cap (C \cup D)$

7. Answers will, of course, vary. But two students might get same output if the random number generator starts at the same value in a formula each time it is called. This depends on machine software traits.

8. Answers will vary.

9. Using all blocks of ten digits from the table in the text, Section 2.5, one gets $\frac{2}{40}$ successes. The binomial probability formula with p = .4 predicts

$$P(7 \text{ or more strikes}) = \binom{10}{7}(.4)^7(.6)^3+\binom{10}{8}(.4)^8(.6)^2 +\binom{10}{9}(.4)^9(.6)^1 + \binom{10}{10}(.4)^{10}$$
$$\approx .042 + .011 + .004 + .0001$$
$$\approx .047$$

The new method seems better since with P(success) = .4, old method, it is highly unlikely to get 7 or more successes.

10. The figures should be in the neighborhood of $.3(676) \approx 202.8$, again using the complete random digit list given in the text section 2.5. Note that this gives only 10 usable digit quadruples. Further testing is warranted, but the area will still be much the same.

11. P(o or 2) in toss of fair die.

12. 10 —— 110 loop counts forty samples
    20 sets up a counter for boys in each sample
    40 - 80 loop generates "children" counting in 70 if result is > .5 (boy).
    90 checks to see if the count of boys in the sample just completed is at least 8 —— increasing the S counter if it is.
    120 prints success samples/total samples.

13. Change
```
40 FØR J = 1 TØ 5
90 IF B > 1 THEN 110
120 PRINT "FRACTIØN 4 ØR MØRE = "; S/40
```

14. (a) Look at blocks of 3 digits--all even → HHH, otherwise a failure.
    (b) Look at blocks of 2 digits in which both entries are not 0 and less than 7.
    (c) Look at pairs of digits, HH <—> 0,1,2,3 in both places.
    (d) Look at blocks of 3 digits ranging from 001 to 365. Take samples of 20 such blocks and count success if one or more matches in a block.

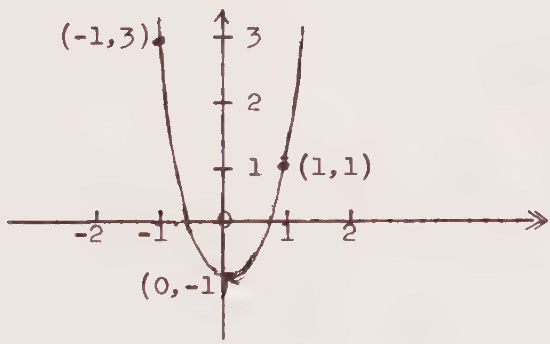
(e)
```
 05 DIN P(20)
 10 FØR I = 1 TØ 50
 20 FØR J = 1 TØ 20                 take sample from
 30 LET P(J) = INT(365*RND(Z))      1,2,...,365
 40 NEXT J
 50 FØR K = 1 TØ 19
 60 FØR L = 1 TØ 20
 70 IF P(K) = P(L) THEN 110
 80 NEXT L                          search for match
 90 NEXT K
100 GØ TØ 120
110 LET M = M+1
120 NEXT I
130 PRINT M/50  }  print ratio of matches
140 END
```

15. Here is a BASIC program for the Monte Carlo method
```
10 FØR I = 1 TØ 50
20 LET X = 6*RND(Z)
30 LET Y = 9*RND(Z)
40 IF Y > -(X↑Z) + 6*X THEN 60
50 LET A = A+1
60 NEXT I
70 PRINT (A/50)*54
80 END
```
Actual area is $\int_0^6 -x^2+6x = -\frac{x^3}{3} + 3x^2\Big]_0^6 = 36$

CHAPTER 3

Section 3.2 (pp.136-138)

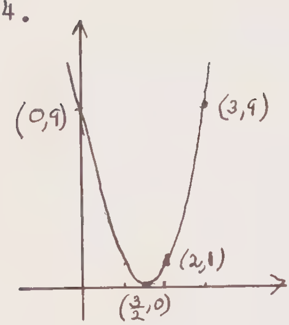
1.



2. $-1 < x < 0$ and $0 < x < 1$

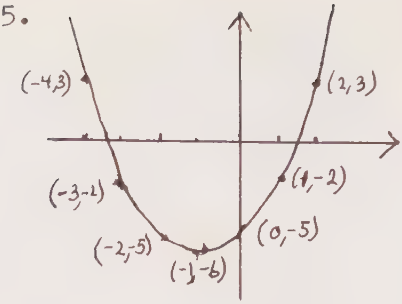
3.
```
10 INPUT A,B
20 PRINT "X=", "Y="
30 FØR X = A TØ B STEP .05
40 LET Y = 3*X*X-X-1
50 PRINT X,Y
60 NEXT X
70 END
```
The solutions, to the nearest tenth, are -.4 and .8.

4.



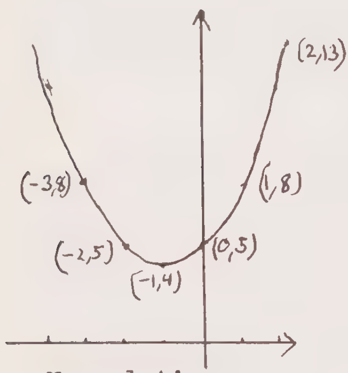
One solution

5.



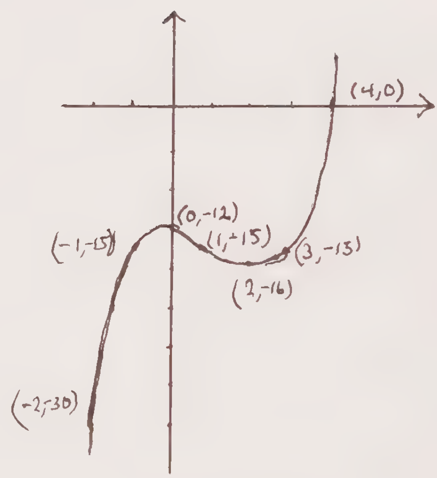
Two solutions

6.



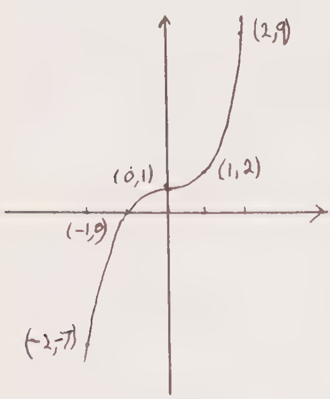
No solutions

7.



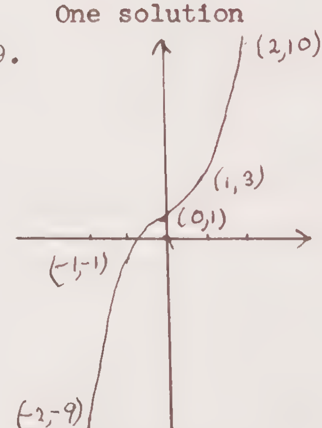
One solution

8.



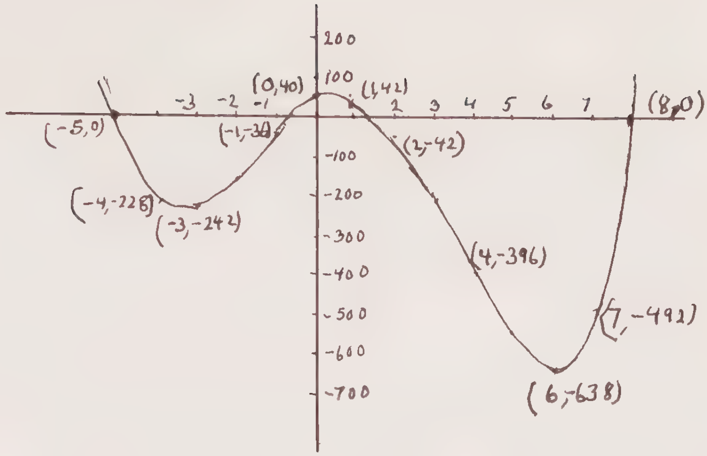
One solution

9.



One solution

10.



Four solutions

11.
```
10 INPUT A,B
20 PRINT "X=","Y="
30 FØR X = A TØ B STEP .05
40 LET Y = X↑4-4*X↑3-38*X↑2+43*X+40
50 PRINT X,Y
60 NEXT X
70 END
```

The solutions are -5, 8, and (approximately) -0.62 and 1.62. Note the relationship between this exercise and No. 10; i.e., $(x+5)(x-8)(x^2-x-1)=x^4-4x^3+38x^2+43x+40$

12.
```
10 INPUT A,B
20 PRINT "X=", "Y="
30 FØR X = A TØ B STEP .05
40 LET Y = X*X*X-X*X+2*X-1
50 PRINT X,Y
60 NEXT X
70 END
```

The only real solution is (to the nearest tenth) 0.6.

13. In the first case, the function is not "continuous" throughout the interval between x_1 and x_2; thus the graph is able to "jump" from positive values to negative without intersecting the x-axis.

In the second case, the graph is not a function, and so it is able to "snake" around, crossing the x-axis outside the interval between x_1 and x_2.

14. Yes, there may be any number of such *x*'s

15.
```
10 REM APPRØX SØL QUAD EQ
20 READ A, B, C
30 PRINT "X=", "Y="
40 FØR X = A TØ B STEP C
50 LET Y = X↑2 + 2*X - 1
60 IF ABS(Y) > = 10 THEN 80
70 PRINT X,Y
80 NEXT X
90 DATA -5,5,1
100 END
```

16. $x = 4$ or $x = -4$

17. $x = \sqrt{13}$ or $x = -\sqrt{13}$

18. $x = \sqrt{n}$ or $x = -\sqrt{n}$
$(n \geq 0)$

19. $x = 2+\sqrt{k}$ or $x = 2-\sqrt{k}$
$(k \geq 0)$

20. $x = \dfrac{-b}{2a} + \dfrac{\sqrt{b^2+4ac}}{2a}$ or $x = \dfrac{-b}{2a} - \dfrac{\sqrt{b^2-4ac}}{2a}$
For these to exist as real number solutions we would have to have $b^2-4ac \geq 0$.

*21. The approximate solutions are 4.7 and -1.7.

```
 10 DEF FNF(X) = (X-3)*X-8
 20 READ A,B,S
 30 DATA -5,5,.5
 40 READ C,D,N
 50 DATA -11,32,43
 60 LET H = (D-C)/N
 70 PRINT "Y-AXIS:FRØM" C; "TØ"D;"IN STEPS ØF"H
 80 PRINT
 90 FØR I = 0 TØ N
100 PRINT "-";
110 NEXT I
120 PRINT
130 DEF FNR(X) = INT(X+.5)
140 LET YO = FNR(-C/H)
150 FØR X = A TØ B STEP S
160 LET Y = FNF(X)
170 LET Y1 = FNR((Y-C)/H)
180 FØR I = 0 TØ N
190 IF I = Y1 THEN 250
200 IF I = YO THEN 230
210 PRINT " ";
220 GØ TØ 260
230 PRINT ".";
240 GØ TØ 260
250 PRINT "*"
260 NEXT I
270 PRINT "  "; X
280 NEXT X
290 END
```

Section 3.3 (pp.142-143)

1. 49+48 = 97; two
2. 49-48 = 1; two
3. 49-48 = 1; two
4. 1-4 = -3; none
5. 1+4 = 5; two
6. 1+4 = 5; two
7. 4+24 = 28; two
8. 4-24 = -20; none
9. 0+4 = 4; two
10. 0-4 = -4; none
11. 400-400 = 0; one
12. $\frac{1}{4} + 12 = 12\frac{1}{4}$; two
13. $4+1\frac{1}{2} = 5\frac{1}{2}$; two
14. 2+12 = 14; two
15. $2+4\sqrt{3}$; two
16. 9-8 = 1; two

17.
```
10 PRINT
20 INPUT A, B, C
30 IF A = 0 THEN 160
40 LET D = B↑2 -4*A*C
50 IF D < 0 THEN 140
60 IF D = 0 THEN 110
70 LET X1 = (SQR(D)-B)/2/A
80 LET X2 = (-SQR(D)-B)/2/A
90 PRINT "THE SØLUTIØNS ARE" X1; "AND" X2
100 GØ TØ 10
110 LET X1 = -B/2/A
120 PRINT "THE ØNLY SØLUTIØN IS" X1
130 GØ TØ 10
140 PRINT "NØ REAL SØLUTIØNS"
150 GØ TØ 10
160 END
```
(Note: This program will continue to ask for more data-i.e., another equation to solve-until A = 0, at which time it will stop.)

The real solutions to the equations in Exercises 1 through 16 are as follows:

(1) $\frac{-7\pm\sqrt{97}}{6}$ (2) $-\frac{4}{3}, -1$ (3) $\frac{4}{3}, 1$ (4) none (5) $\frac{-1\pm\sqrt{5}}{2}$

(6) $\frac{1\pm\sqrt{5}}{2}$ (7) $\frac{2\pm\sqrt{7}}{2}$ (8) none) (9) 1,-1 (10) none

(11) $\frac{5}{2}$ (12) $-\frac{3}{2}, 2$ (13) $\frac{-4\pm\sqrt{22}}{3}$ (14) $\frac{-\sqrt{2}\pm\sqrt{14}}{2}$

(15) $\frac{-\sqrt{2}\pm\sqrt{2+4\sqrt{3}}}{2}$ (16) $-\sqrt{2}, -\frac{1}{2}\sqrt{2}$

*18.
```
10 PRINT
20 INPUT A,B,C
30 IF A = 0 THEN 160
40 LET D = B*B-4*A*C
50 IF D < 0 THEN 140
60 IF D = 0 THEN 110
70 LET X1 = (SQR(D)-B)/2/A
80 LET X2 = (-SQR(D)-B)/2/A
90 PRINT "THE SØLUTIØNS ARE" X1; "AND" X2
*100 GØ TØ 10
120 PRINT "THE ØNLY SØLUTIØN IS" X1
130 GØ TØ 10
140 PRINT "NØ REAL SØLUTIØNS"
150 GØ TØ 10
160 IF B = 0 THEN 190
170 LET X1 = -C/B
180 GØ TØ 120
190 END
```
(Note: The above program will stop when <u>both</u> A and B are zero.)
```
*110 LET X1 = -B/2/A
```

19. $x = \frac{-b+\sqrt{D}}{2a}$ or $x = \frac{-b-\sqrt{D}}{2a}$ $\qquad (D = b^2 - 4ac)$

$$\Downarrow$$

$x + \frac{b}{2a} = \frac{\sqrt{D}}{2a}$ or $x + \frac{b}{2a} = -\frac{\sqrt{D}}{2a}$

$$\Downarrow$$

$$(x + \frac{b}{2a})^2 = \frac{D}{4a^2} = \frac{b^2-4ac}{4a^2}$$

$$\Downarrow$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2-4ac}{4a^2}$$

$$\Downarrow$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Downarrow$$

$$ax^2 + bx + c = 0$$

20. $(2x-3)(x+5) = 0 \Leftrightarrow x = \frac{3}{2}$ or $x = -5$
21. $(5x-4)^2 = 0 \Leftrightarrow x = \frac{4}{5}$
22. $(3x-1)(x+7) = 0 \Leftrightarrow x = \frac{1}{3}$ or $x = -7$
23. $(2x-7)(2x+7) = 0 \Leftrightarrow x = \frac{7}{2}$ or $x = -\frac{7}{2}$
24. $(5x-2)(x-3) = 0 \Leftrightarrow x = \frac{2}{5}$ or $x = 3$
25. $(3+2x)(2-x) = 0 \Leftrightarrow x = -\frac{3}{2}$ or $x = 2$

*26. The formula is valid in Z_7 since Z_7 is a field. However this means that in order for a solution to exist the discriminant must be a perfect square in Z_7, and the only such squares are 0,1,2, and 4. Thus

$$x^2 + 3x + 1 = 0$$

has no solution in Z_7 since $\sqrt{5}$ is without meaning. (The situation is analogous to that of the real numbers, where the discriminant must be positive--i.e., the square of a real number.)

The formula is not valid in Z_6 since Z_6 is not a field. (Not every nonzero element has a multiplicative inverse.) Specifically, in the derivation of the formula, multiplication by $\frac{1}{a}$ cannot be assured in Z_6. As one example, 3 and 5 are the only solutions of

$$2x^2 + 5x + 3 = 0$$

in Z_6. However, if you blindly apply the "quadratic formula" you get

$$\frac{1 \pm \sqrt{1}}{4}$$

Since both 1 and 5 are square roots of 1, and since $\frac{2}{4}$ designates both 2 and 5, you get <u>three</u> solutions - 3, 2, and 5. (2 of course is <u>not</u> a solution.)

27. The factoring procedure works in Z_7 since the principle

$$ab = 0 \Leftrightarrow a = 0 \text{ or } b = 0$$

applies there (as in any field). However, this principle (and hence the factoring procedure) is not valid in Z_6.

Examples: In Z_7, $x^2+3x+2 = (x+2)(x+1)$
$(x+2)(x+1) = 0 \Leftrightarrow x=5$ or $x=6$
$\{5,6\}$ is the solution set of
$x^2+3x+2 = 0$

In Z_6, $x^2+3x+2 = (x+2)(x+1)$
Mistakenly applying the zero product principle would yield 4 and 5 as the only solutions of $x^2+3x+2 = 0$, whereas the actual solution set is $\{1,2,4,5\}$.

Section 3.4 (pp.148-150)

1. $4I-3J$ 2. $4I+3J$ 3. $0I+J$ 4. $-I+0J$ 5. $\underline{r}I+\underline{s}J$

6. $\begin{pmatrix} 2 & -13 \\ 13 & 2 \end{pmatrix}$ 7. $\begin{pmatrix} -22 & 14 \\ -14 & -22 \end{pmatrix}$ 8. $\begin{pmatrix} a+c & -(b+d) \\ b+d & a+c \end{pmatrix}$

9. $\begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}$ 10. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ 11. $-3I+6J$ 12. $6I-44J$

13. $10I+35J$ 14. $-35I+10J$ 15. $-I+0J$

16. $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} a+c & -(b+d) \\ b+d & a+c \end{pmatrix}$

17. $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}$

18. $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ \frac{-b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (and the product commutes)

19. $\begin{pmatrix} \frac{5}{29} & \frac{2}{29} \\ \frac{-2}{29} & \frac{5}{29} \end{pmatrix}$ (Note that this means the inverse of $5I + 2J$ is $\frac{5}{29}I - \frac{2}{29}J$.)

20. $\begin{pmatrix} \frac{3}{25} & \frac{4}{25} \\ \frac{-4}{25} & \frac{3}{25} \end{pmatrix}$, or $\frac{3}{25}I - \frac{4}{25}J$ 21. $\begin{pmatrix} \frac{2}{29} & \frac{-5}{29} \\ \frac{5}{29} & \frac{2}{29} \end{pmatrix}$

22. $\begin{pmatrix} -12 & 3 \\ -3 & -12 \end{pmatrix}$; $\begin{pmatrix} \frac{12}{153} & \frac{3}{153} \\ \frac{-3}{153} & \frac{12}{153} \end{pmatrix}$

23. $\begin{pmatrix} -12 & -3 \\ 3 & -12 \end{pmatrix}$; $\begin{pmatrix} \frac{12}{153} & \frac{-3}{153} \\ \frac{3}{153} & \frac{12}{153} \end{pmatrix}$ 24. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

25. $\begin{pmatrix} 12 & 3 \\ -3 & 12 \end{pmatrix}$; $\begin{pmatrix} \frac{-12}{153} & \frac{3}{153} \\ \frac{-3}{153} & \frac{-12}{153} \end{pmatrix}$

26. $-aI-bJ$; $\frac{a}{a^2+b^2}I - \frac{b}{a^2+b^2}J$

Section 3.5 (pp156-158)

1. Two groups $(G,*)$ and $(H,\circ)$ are isomorphic if and only if there is a one-to-one correspondence f between G and H, and for all $a,b \in G$ $f(a*b) = f(a)\circ f(b)$.

2. Although there exists a one-to-one correspondence between $Z_4\backslash\{0\}$ and Z_3, there is no isomorphism here. (Z_3+) is a closed system-a group, in fact-but in $(Z_4\backslash\{0\},\cdot)$, there is no operation since $2\cdot 2 = 0$ is not in the set.

3. The correspondence is $a \longleftrightarrow 2^a$ for all integers $\underline{a}$. That is, $f(a) = 2^a$.
Furtherfore $f(a+b) = 2^{a+b} = 2^a\cdot 2^b = f(a)\cdot f(b)$.
Students will see this kind of isomorphism again when they study logarithms.

4. $\begin{pmatrix} -1 & 5\frac{1}{2} \\ -5\frac{1}{2} & -1 \end{pmatrix}$; $(2-\frac{1}{2}i) + (-3-5i) = -1-5\frac{1}{2}i$

5. $\begin{pmatrix} -5 & 5 \\ -5 & -5 \end{pmatrix}$; $(3+i)(-2-i) = -5-5i$

6. $\begin{pmatrix} -5 & -3 \\ 3 & -5 \end{pmatrix}$; $(-4+4i) + (-1-i) = -5+3i$

7. $\begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}$; $(3-2i)(3+2i) = 13$

8. $0;\begin{pmatrix} 5 & -2 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} -5 & 2 \\ -2 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

9. $6-17i;\ \begin{pmatrix} 16 & 12 \\ -12 & 16 \end{pmatrix} + \begin{pmatrix} -10 & 5 \\ -5 & -10 \end{pmatrix} = \begin{pmatrix} 6 & 17 \\ -17 & 6 \end{pmatrix}$

10. $-35+4i;\ \begin{pmatrix} -8 & -3 \\ 3 & -8 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -35 & -4 \\ 4 & -3 \end{pmatrix}$

11. $-5+12i;\ \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -5 & -12 \\ 12 & -5 \end{pmatrix}$

12. $8+10i$; $-1+34i$ 13. $6+3i$; $26+18i$ 14. 4; $-20+10i$

15. 24; 148 16. $(\sqrt{2}+3) + (4-\sqrt{2}i)$; $6\sqrt{2} + 10i$

17. $i^a = i \Leftrightarrow a \equiv 1 \pmod 4$
$i^b = -1 \Leftrightarrow a \equiv 2 \pmod 4$
$i^c = -1 \Leftrightarrow a \equiv 3 \pmod 4$
$i^d = 1 \Leftrightarrow d \equiv 0 \pmod 4$
Thus, $i^{513} = i$, $i^{514} = i^2 = -1$, $i^{1000} = i^0 = 1$

18. $-3+2i$ 19. $1-2i$ 20. $-10+15i$ 21. $3-i$

22. $10+2\sqrt{2}i$ 23. $(a-c)+(b-d)i$

24. $(a+bi)(a-bi) = a^2-b^2i^2 = a^2-b^2(-1) = a^2+b^2$
which is real since a and b are real.

25. $\frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2-3i}{13} = \frac{2}{13} + \frac{-3}{13}i$

$(2+3i)(\frac{2}{13} + \frac{-3}{13}i) = \frac{4}{13} - \frac{6}{13}i + \frac{6}{13}i - \frac{9}{13}i^2 = \frac{4}{13} + \frac{9}{13} = 1$

26. $\frac{1}{2+6i} \cdot \frac{2-6i}{2-6i} = \frac{2-6i}{40} = \frac{1}{20} + \frac{-3}{20}i$

27. $\frac{3}{13} + \frac{2}{13}i$ 28. $\frac{-4}{17} - \frac{1}{17}i$ 29. $\frac{-2}{29} + \frac{5}{29}i$

30. $\frac{2+3i}{5-7i} \cdot \frac{5+7i}{5+7i} = \frac{-11}{74} + \frac{29}{74}i$

$(5-7i)(\frac{-11}{74} + \frac{29}{74}i) = \frac{-55}{74} + \frac{145}{74}i + \frac{77}{74}i + \frac{203}{74}$

$= \frac{148}{74} + \frac{222}{74}i = 2+3i$

31. $\frac{5+2i}{3-i} \cdot \frac{3+i}{3+i} = \frac{13}{10} + \frac{11}{10}i$ 32. $\frac{2}{13} - \frac{23}{13}i$ 33. $-8-6i$

34. $\frac{1}{4} + \frac{1}{4}i$ 35. $x = \frac{3-i}{2+i} = 1-i$ 36. $\frac{5}{13} - \frac{1}{13}i$

37. $-1-4i$ 38. $-1-i$ 39. $\frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$

40. $\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd)}{c^2+d^2} + \frac{(bc-ad)}{c^2+d^2}i$

Section 3.6 (pp.160-161)

1.(a) -25 (b) -25 2.(a) -7 (b) -7

3.(a) $-r$ (b) $-r$ 4.(a) $3i$ (b) $\sqrt{13}i$ (c) $\sqrt{18}i = 3\sqrt{2}i$
(d) $\sqrt{r}i$ (e) $|a|i$

*5. No. If it did then for example we would have
$(\sqrt{-4})(\sqrt{-4}) = \sqrt{(-4)(-4)} = \sqrt{16} = 4$
This is not correct, since by our definition
$(\sqrt{-4})(\sqrt{-4}) = (2i)(2i) = 4i^2 = -4$
Thus the principle $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is valid only for positive radicands.

6. $1 \pm i$ 7. $1 \pm \sqrt{3}$ 8. $-1 \pm \sqrt{6}$ 9. $-1 \pm 2i$

10. ± 4 11. $\pm 4i$ 12. $\frac{1\pm 2i}{5}$ 13. $\frac{1\pm\sqrt{6}}{5}$

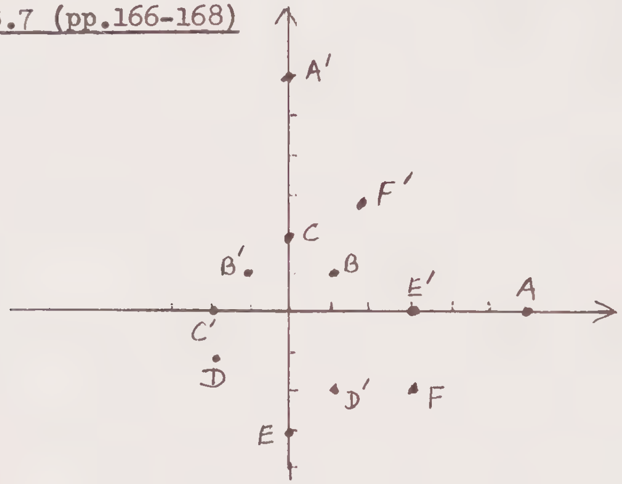
14. $\frac{-1\pm\sqrt{3}i}{2}$ 15. $\frac{1\pm\sqrt{3}i}{2}$

16.
```
10 READ A,B,C
20 DATA a,b,c (a,b,c are coefficients in ax²+bx+c=0)
30 LET D = B↑2-4*A*C
40 IF D > 0 THEN 120
50 LET X1 = -B/2/A
60 IF D = 0 THEN 100
70 LET I = ABS(SQR(-D))/2/A
80 PRINT "THE SØLUTIONS ARE"X1 "+"I"I AND"X1 "-"I "I"
```

```
 90 GØ TØ 150
100 PRINT "THE ØNLY SØLUTIØN IS" X1
110 GØ TØ 150
120 LET X1 = (SQR(D)-B)/2/A
130 LET X2 = (-SQR(D)-B)/2/A
140 PRINT "THE SØLUTIØNS ARE" X1; "AND" X2
150 END
```

Section 3.7 (pp.166-168)

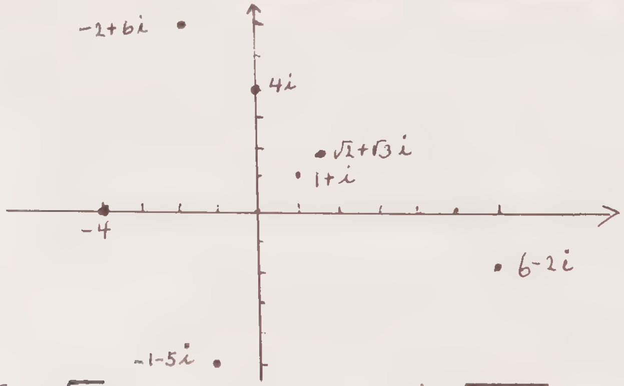
1. - 6.



7. $6i$ 8. $-1+i$ 9. -2 10. $1-2i$ 11. 3   12. $2+3i$

13. Under a 90° rotation the image point of the point associated with $a+bi$ is that point associated with $i(a+bi)$

14.



15. $\sqrt{13}$ 16. $2\sqrt{13}$ 17. 3   18. 3   19. $\sqrt{r^2+t^2}$

20. $a^2+b^2 = 36$ and $b = 2a-1$. So $(a^2)+(2a-1)^2 = 36$

$$5a^2-4a-35 = 0$$

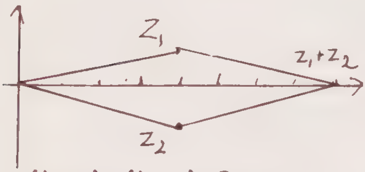
$$a = \frac{2+\sqrt{179}}{5} \text{ and } b = \frac{-3+\sqrt{179}}{5}$$

or

$$a = \frac{2-\sqrt{179}}{5} \text{ and } b = \frac{-3-\sqrt{179}}{5}$$

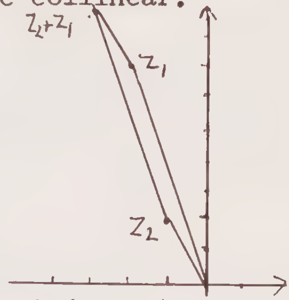
21. The slope of the segment joining (a,b) and (0,0) is $\frac{b}{a}$; this is also the slope of the segment joining (a+c,b+d) and (c,d). Similarly, the segments joining (c,d) and (0,0) and (a+c,b+d) and (a,b) have the same slope and so are parallel. Since the opposite sides are parallel, the quadrilateral is a parallelogram. If (c,d) = (ka,kb) the points are collinear.

22.



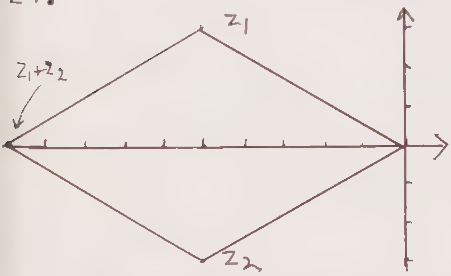
$(4+i)+(4-i)=8+0i$

23.



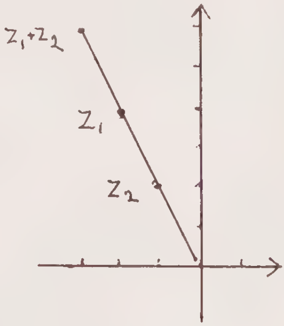
$(-2+7i)+(-1+2i)=-3+9i$

24.



$(-5+3i)+(-5-3i)=-10+0i$

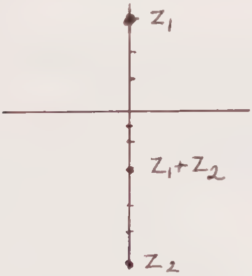
25.



The points are colli- near

$(-2+4i)+(-1+2i)=-3+6i$

26.



The points are collinear

$3i+(-5i) = -2i$

27. It is always true that $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$. This is in fact the triangle inequality, and the "=" holds only when $Z_2 = kZ_1$, so that Z_1, Z_2, and $Z_1 + Z_2$ are collinear.

28. Always true $[a^2 + b^2 = a^2 + (-b)^2]$

29. Always true $[(a+c) - (b+d)i = (a-bi) + (c-di)]$

30. Always true $[a^2 + b^2 \geq 0]$.

31. Always true $[|(a+bi)^2|$ and $|a+bi|^2$ are both equal to $(a^2 + b^2)^2]$.

Section 3.8 (pp.172-173)

1. $AB = \sqrt{(3-1)^2+(1-0)^2} = \sqrt{5}$

   $A'B' = \sqrt{(3-2)^2+(11-3)^2} = \sqrt{65}$ (Note: $\sqrt{65} = \sqrt{13}\sqrt{5}$)

   $BC = 1$

   $B'C' = \sqrt{(11-9)^2+(3-6)^2} = \sqrt{13}$

   $AC = 2$

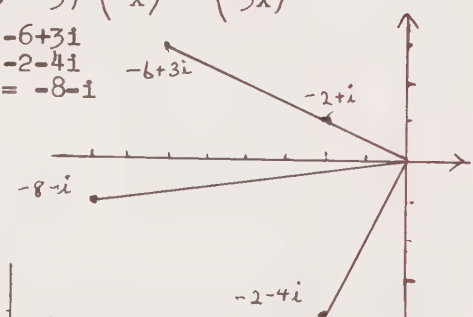
   $A'C' = \sqrt{(6-2)^2+(9-3)^2} = \sqrt{52} = 2\sqrt{13}$

   Thus $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'} = \dfrac{1}{\sqrt{13}}$

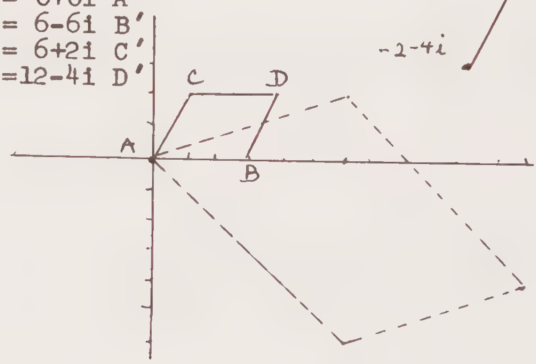
2. $$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3x \\ 3y \end{pmatrix} = \begin{pmatrix} -3y \\ 3x \end{pmatrix}$$

   $$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} -3y \\ 3x \end{pmatrix}$$

3. $f(-2+i) = (3)(-2+i) = -6+3i$
   $g(-2+i) = (2i)(-2+i) = -2-4i$
   $h(-2+i) = (3+2i)(-2+i) = -8-i$



4. $(2-2i)(0+0i) = 0+0i$ A′
   $(2-2i)(3+0i) = 6-6i$ B′
   $(2-2i)(1+2i) = 6+2i$ C′
   $(2-2i)(4+2i) = 12-4i$ D′



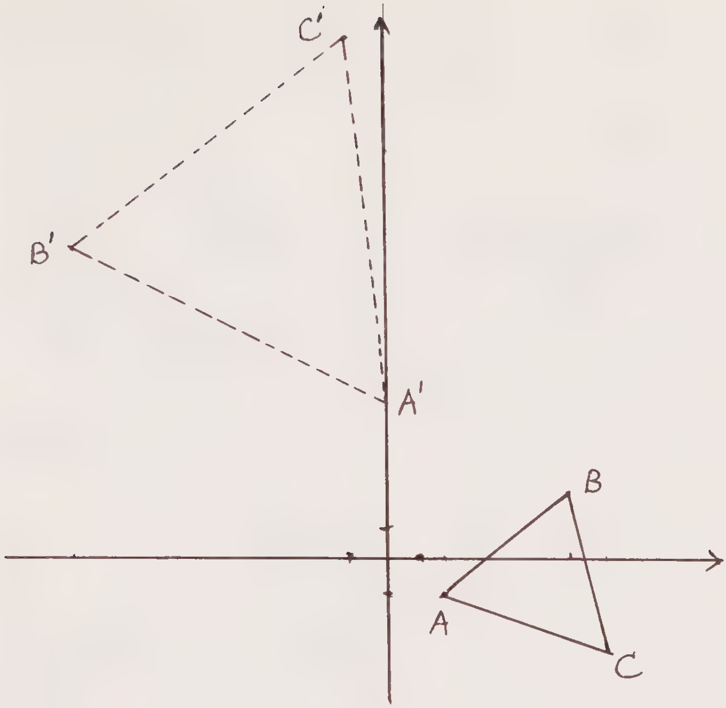
5. (a) $f(-6+\frac{3}{2}i) = (5-3i)(-6+\frac{3}{2}i) = \dfrac{-51}{2} + \dfrac{51}{2}i$

   (b) $$\begin{pmatrix} 5 & 3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} -6 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} -\frac{51}{2} \\ \frac{51}{2} \end{pmatrix}$$

6. $$\begin{pmatrix} 8 & 5 \\ -5 & 8 \end{pmatrix}\begin{pmatrix} -2 \\ -6 \end{pmatrix} = \begin{pmatrix} -46 \\ -38 \end{pmatrix} = A' \qquad \begin{pmatrix} 8 & 5 \\ -5 & 8 \end{pmatrix}\begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 30 \\ 48 \end{pmatrix} = C'$$

   $$\begin{pmatrix} 8 & 5 \\ -5 & 8 \end{pmatrix}\begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix} = \begin{pmatrix} \frac{115}{2} \\ 3 \end{pmatrix} = B' \qquad \begin{pmatrix} 8 & 5 \\ -5 & 8 \end{pmatrix}\begin{pmatrix} -4 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{-69}{2} \\ 16 \end{pmatrix} = D'$$

7.



$(a+bi)(2-i) = 0+5i$ So $2a + b = 0$, $b = -2a$

$$2b - a = 5$$
$$-4a - a = 5$$
$$a = -1,\ b = 2$$

Therefore the desired number is $-1+2i$

8. Reflection in x-axis: $(a,b) \longrightarrow (a,-b)$

9. Triangle inequality: The sum of two sides of a triangle is greater than the third side. The equality holds when $Z_2=k\cdot Z_1$, and the three points Z_1, Z_2, and Z_1+Z_2 are collinear.

10. Translation $T_{3,2}$: $(x,y) \longrightarrow (x+3,\ y+2)$

11. Translation $T_{a,b}$: $(x,y) \longrightarrow (x+a,\ y+b)$

12. Identity mapping: $(x,y) \longrightarrow (x,-y) \longrightarrow (x,y)$.

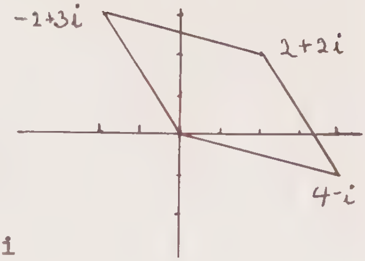
Review Exercises (pp.177-178)

1. (a) real, rational (b) nonreal
   (c) real, irrational (d) real, irrational
   (e) nonreal (f) nonreal
   (g) real, irrational (h) real, rational
   (i) real, rational (j) real, irrational

2. $\frac{5\pm\sqrt{23}i}{6}$ 3. $\frac{5\pm\sqrt{73}}{6}$ 4. $\pm\sqrt{3}i$ 5. $0,-\frac{5}{3}$ 6. $23-2i$

7. $8-2i$ 8. $-2+6i$ 9. $\frac{7}{41}+\frac{22}{41}i$ 10. $i^{431} = i^3 = -i$

11. $\frac{323}{2}+18i$ 12.



13. $(3,0) \to (6,9)$
$(4,2) \to (2,16)$
$(0,-2) \to (6,-4)$

14. $\begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$; $(5-3i)+(-2+i) = 3-2i$

15. $\begin{pmatrix} -2 & 5 \\ -5 & -2 \end{pmatrix}$; $(4-3i)-(6+2i) = -2-5i$

16. $\begin{pmatrix} 15 & 15 \\ -15 & 15 \end{pmatrix}$; $(-3-6i)(1+3i) = 15-15i$

17. $\begin{pmatrix} -\frac{1}{4} & -\frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & -\frac{1}{4} \end{pmatrix}$; $\frac{1}{8}(1+\sqrt{3}i)^2 = -\frac{1}{4} + \frac{1}{4}\sqrt{3}$

18. $\frac{2+3i}{5+2i} = 2+3i\left(\frac{1}{5+2i} \cdot \frac{5-2i}{5-2i}\right) = \frac{16}{29} + \frac{11}{29}i$

$$\frac{1}{5+2i} \cdot \frac{5-2i}{5-2i} = \frac{5}{29} + \frac{-2}{29}i$$

So the inverse of $\begin{pmatrix} 5 & -2 \\ 2 & 5 \end{pmatrix}$ is $\begin{pmatrix} \frac{5}{29} & \frac{2}{29} \\ \frac{-2}{29} & \frac{5}{29} \end{pmatrix}$

19. Yes. The table below makes it clear that it is isomorphic to $(Z_4,+)$.

$\cdot$	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

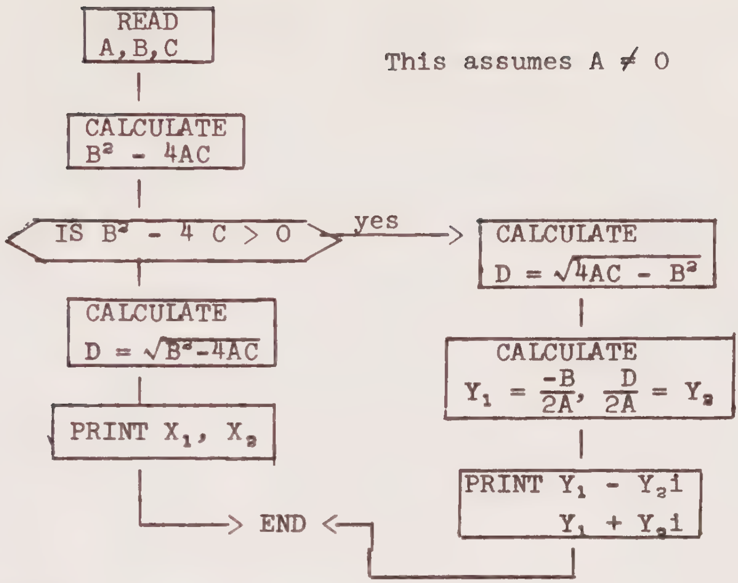
MAINTAINING SKILLS AND UNDERSTANDING I

BASIC (pp.180-181)

1. LET Y = X↑2+3+3*C (or possibly C3 intended)
2. FØR X = 6 TØ -3 STEP .1
3.

```
 10 DIM C(3,3)
 20 MAT READ C     or simply 15 MAT READ C(3,3)
 30 MAT C = (3)*C   parentheses needed
 40 FØR I = 1 TØ 3
 50 FØR J = 1 TØ 3
 60 PRINT C(I,J);
 70 NEXT J
 80 NEXT I
 90 DATA 5,5,3,2,1,6,-3,10,4.28
100 END
    RUN
```

4.



5. There are several ways to plan the investigation of f and Σf. One is given below:

```
10 FØR N = 1 TØ 20
20 LET X = 1/(N*(N+1))
30 PRINT X
40 LET X = S+X
50 NEXT N
60 PRINT S
```

[This gives print of first 20 terms and $\sum_{n}^{20} f(n)$. To get only $\sum_{n=1}^{k} f(n)$ for larger k, delete 30 and change line 10 to read FØR N = 1 TØ K]

6. Again, various programs are possible.

```
 10 PRINT "N"; "APPRØXIMATIØN"; "LIMIT"
 20 LET L = (SQR(S)-1)/2
 30 LET X = 1
 40 LET Y = 1
 50 FØR N = 1 TØ 30
 60 PRINT N; Y/X; L
 70 LET Z = Y
 80 LET Y = X+Y
 90 LET X = Z
100 NEXT N
110 END
```

To look at larger values of r replace lines 50 and 100 by:

```
 50 READ V
 55 IF N < V THEN 70
 65 GØ TØ 50
100 LET N = N+1
105 GØ TØ 55
```

```
106 DATA (List values of N for which you want ratio
          printed.)
```

7. Use the BASIC plotting program with the following special values.

```
(a) 10 DEF FNF (X) = (.9)↑X
    20 DATA 0, 20, 1
    50 DATA 0, 1, 40
(b) 10 DEF FNF (X) = (.5)↑X
(c) 10 DEF FNF (X) = (.1)↑X
```

8.

```
10 LET S = 1
20 LET T = 1
30 FØR N = 1 TØ 20
40 LET T = T*(1/N)
50 LET S = S+T
60 NEXT N
70 PRINT S
80 END
```

9.

```
 10 INPUT A,B,C,D
 20 LET S1 = A+C
 30 LET S2 = B+D
 40 PRINT "Z1+Z2="; S1; "+"; S2; "I"
 50 LET D1 = A-C
 60 LET D2 = B-D
 70 PRINT "Z1-Z2="; D1; "+"; D2; "I"
 80 LET P1 = A*C-B*D
 90 LET P2 = A*D+B*C
100 PRINT "Z1*Z2"; P1; "+"; P2; "I"
110 LET V = SQR(C*C+D*D)
120 PRINT "ABS(Z2) ="; V
130 LET K = C*C+D*D
140 PRINT "1/Z2="; C/K; "-"; D/K; "I"
*150 LET Q1 = (A*C+B*D)/K
170 PRINT "Z1/Z2="; Q1; "+"; Q2; "I"
                  *160 LET Q2 = (B*C-A*D)/K
```

10.

```
 10 INPUT A,B,C,D,E,F
 20 LET Z1 = E-C
 30 LET Z2 = F-D
 40 LET K = A*A+B*B
 50 LET L = Z1*Z1+Z2*Z2
 60 IF K < > 0 THEN 120
 70 IF L < > 0 THEN 100
 80 PRINT "TRUE FØR ALL X"
 90 GØ TØ 150
100 PRINT "TRUE FØR NØ X"
110 GØ TØ 150
120 LET X1 = (Z1*A+Z2*B)K
130 LET X2 = (Z2*A-Z1*B)/K
140 PRINT "X="; X1; "+"; X2; "I"
150 END
```

11.

```
10 INPUT A,B,C
20 LET D = B↑2-4*A*C
30 IF D < 0 THEN 80
40 LET X1 = (-B+SQR(D))/(2*A)
```

```
 50 LET X2 = (-B-SQR(D))/(2*A)
 60 PRINT "X1="; X1; "X2="; X2
 70 GØ TØ 130
 80 LET D = -D
 90 LET X1 = -B/(2*A)
100 LET X2 = SQR(D)/(2*A)
110 PRINT "X1="; X1; "+"; X2; "I"
120 PRINT "X2="; X1; "-"; X2; "I"
130 END
```

Sequences and Series (pp.181-185)

Snowflake curve. Perimeter is infinite. $P = \lim 3a(\frac{4}{3})^{n-1}$. A geometric sequence with ratio greater than 1. Area is finite. A geometric sequence with ratio $\frac{4}{9}$.

$A = \frac{a^2\sqrt{3}}{12} + \frac{a^2\sqrt{3}}{12} \cdot \frac{4}{5} = \frac{19a^2\sqrt{3}}{60}$

1. See textbook. 2. $\frac{1}{1\cdot2}, \frac{1}{2\cdot3}, \frac{1}{3\cdot4}, \ldots, \frac{1}{10\cdot11}, \ldots$

3. 1,8,27,...,1000,... 4. $1, -\frac{1}{2}, \frac{1}{3}, \ldots, -\frac{1}{10}, \ldots$

5. 3,7,13,...,111,... 6. $a, 2a^2, 3a^3, \ldots, 10a^{10}, \ldots$

7. $1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{100}, \ldots$ 8. $a_n = 2+(n-1)3$

9. $a_n = 3\cdot2^{n-1}$ 10. $a_n = (-1)^{n-1}\cdot\frac{1}{2^{n-1}}$

11. $a_n = \frac{1}{2}n(n+1)$ 12. $a_n = 2n+1$

13. $a_n = \frac{1}{6}(n)(n+1)(2n+1)$ 14. See textbook

15. 1, 2, 4, 8, ... 16. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

17. 2, 6, 18, 54, ... 18. $1, x^2, x^3, \ldots$

19. $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, \ldots$ 20. See textbook

21. $r < 1$ implies $\frac{1}{r} > 1$. Hence $\frac{1}{r^2}, \frac{1}{r^3}, \ldots$ are > 1; sequence diverges.

22. $a_n = C\cdot r^{n-1}$ 23. 1,1,1,1,...; $d = 0$, $r = 1$.

24. Converges. 25. Converges, $|r| < 1$

26. Neither. (1 and 3 are accumulation points.)

27. Converges to 2 28. Converges to 0

29. Converges to 1 30. Divergent

31. (24) $a_n = 1$; (25) $a_n = (\frac{1}{2})^{n-1}$; (28) $a_n = \frac{1}{n^2}$
(29) $a_n = 1 - \frac{1}{(n+1)^2}$; (30) $a_n = n(n+2)$

32. (24) Converges to 1; any term; (25) Converges to 0; 11th term $\frac{1}{2^{10}}$; (29) Converges to 1; 31st term $1-\frac{1}{1024}$

33. 6   34. $6 = \sqrt[3]{216}$ 35. $\sqrt[4]{2^4 \cdot 3^4} = 6$

36. $a_4 = \sqrt{18 \cdot 162} = 54$ 37. (a) $ka, kar, \ldots, kar^{n-1}$ has ratio r

(b) $\frac{1}{k}a, \frac{1}{k}ar, \ldots, \frac{1}{k}ar^{n-1}$ has ratio r

38. $(a_n) = a, ar, \ldots, ar^{n-1}$, $(b_n) = b, bs, bs^2, \ldots, bs^{n-1}$

$(a_n)(b_n) = ab, abrs, abr^2s^2, \ldots, abr^{n-1}s^{n-1}$

$\therefore$ (a)(b) is geometric

39. (a) $a_k = 4k-3$; $a_{k+1} = (4k-3)+4 = 4(k+1)+3$

(b) $S_k = k(2k-1)$; $S_{k+1} = k(2k-1)+4(k+1)+3$

$= (k+1)[2(k+1)-1]$

40. (a) $a_k = a+(k-1)d$; $a_{k+1} = a+(k-1)d+d = a+[(k+1)-1]d$

(b) $S_k = \frac{k}{2}[2a+(k-1)d]$; $S_{k+1} = \frac{k}{2}[2a+(k-1)d]+a+kd$,

$S_{k+1} = \frac{k+1}{2}[2a+\{(k+1)-1\}d]$

41. $S_k = \frac{k(k+1)(2k+1)}{6}$; $S_{k+1} = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{(k+1)(k+2)(2k+3)}{6}$

42. $S_k = \frac{k}{k+1}$; $S_{k+1} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$

43. $S_k = \frac{k}{2k+1}$; $S_{k+1} = \frac{k}{2k+1} + \frac{1}{4(k+1)^2-1} = \frac{k+1}{2k+3}$

44. $S_k = \frac{a(1-r^k)}{1-r}$; $S_{k+1} = \frac{a(1-r^k)}{1-r} + ar^k = \frac{a(1-r^{k+1})}{1-r}$

45. (a) $\frac{1-r^n}{1-r} = 1+r+\ldots+r^{n-1}$ (b) $\frac{a(1-r^n)}{1-r} = a+ar+ar^2+\ldots+ar^{n-1}$

46. See Exercise 40.   47. $a_n = 1+(n-1)2 = 2n-1$; $a_{50}=99$

48. $a_n = x+(n-1)y$; $a_{50}=x+49y$ 49. $a_n = 2n$; $a_{50}= 100$

50. $a_n = x+(n-1)3d$; $a_{50} = x+147d$ 51. AM = 6

52. AM = 9   53. AM = $\frac{n+1}{2}$

54. Assume $a \geq b$; $(a-b) \geq 0 \rightarrow a^2 - 2ab + b^2 \geq 0$

$\rightarrow a^2 + 2ab + b^2 \geq 4ab \rightarrow \frac{(a+b)^2}{4} \geq ab$

$\rightarrow \frac{a+b}{2} \geq \sqrt{ab}$

55. The AM of two numbers is $\geq$ their GM

56. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$

57. Reciprocals are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$. This is arithmetic with $d = -\frac{1}{6}$.

58. Let r_1 and r_2 be given; $\frac{1}{r_1}$, $\frac{1}{r_2}$ are reciprocals; their AM is $\frac{\frac{1}{r_1}+\frac{1}{r_2}}{2} = \frac{r_1+r_2}{2r_1r_2}$. The reciprocal of this is $\frac{2r_1r_2}{r_1+r_2}$ or the harmonic mean.

59. $AM = \frac{r_1+r_2}{2}$, $HM = \frac{2r_1r_2}{r_1+r_2}$; $AM \cdot HM = r_1r_2$; $GM = \sqrt{r_1r_2}$

60. $a+k, a+d+k, a+2d+k, \ldots$ is arithmetic with difference d
$a-k, a+d-k, a+2d-k, \ldots$ is arithmetic with difference d
$ak, ak+kd, ak+2kd, \ldots$ is arithmetic with difference kd

61. No limit   62. 2   63. No limit; each group in parenthesis is $> \frac{1}{2}$.

64. Geometric, $a = 4$, $r = -\frac{1}{2}$, $\text{Lim} = \frac{8}{3}$

65. Geom. $a = \frac{1}{4}$, $r = \frac{1}{4}$, $\text{Lim} = \frac{1}{3}$.

66. No limit (oscillating)   67. Geom. Lim = 16

68. $\frac{13}{3}$ 69. $\frac{16}{11}$ 70. $\frac{257}{111}$ 71. $-\frac{23}{9}$

72. $a_n = 10,000(1.015)^{n-1}$, $n = 1, 2, \ldots, 9$

73. $100 \cdot 2^{12}$; No

74. (a) $3,6,9,\ldots,399$ $n_1 = 133$
$n_1+n_2-n_3 = 171$, divisible by 3 or 7
(b) $7,14,21,\ldots399$ $n_2 = 57$
(c) $21,42,63,\ldots399$ $n_3 = 19$
divisible by 3 <u>and</u> 7

75. (a) $16,48,80,\ldots,16+(n-1)32$
(b) $a_{10} = 16+9+32 = 304$ $S_{10} = \frac{10}{2}(32+9\cdot2+) = 1600$

76. $20,19,18,\ldots, 7$; $n = 14$, $S = 189$

77. Sequence of length of sides is $1, \frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{\sqrt{2}}{4}, \frac{1}{4}, \ldots$
(Geom. $r = \frac{\sqrt{2}}{2}$)
(a) Side of 10^{th} square $\frac{\sqrt{2}}{32}$; Area is $\frac{2}{1024}$ or $\frac{1}{528}$
(b) $4, 2\sqrt{2}, 2, \sqrt{2}, \ldots$ (c) Geometric
(d) $\lim \frac{4}{1-\frac{\sqrt{2}}{2}} = \frac{8}{2-\sqrt{2}} \approx 13.65$

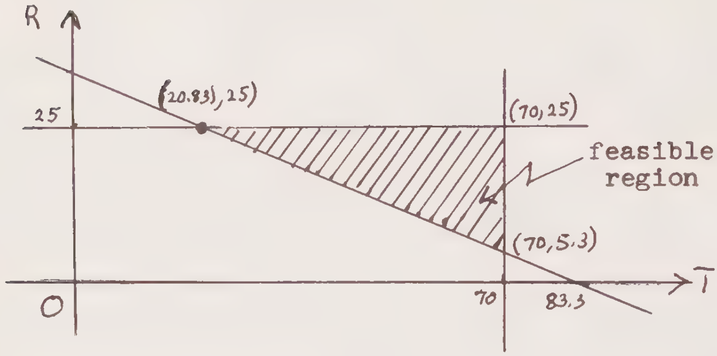
78. A geometric sequence.

<u>Mathematics and the Real World (pp.186-187)</u>

1. Linear Program: R = rail cars and T = trucks

$$R \le 25$$
$$T \le 70$$
$$15R + 6T \ge 500$$

Minimize $750R + 350T$; $T = 21$, $R = 25$ gives
Minimum Cost, \$26,100



2. $V = k \cdot I$
$6 = k \cdot 1.2$ $\quad V = 5 \cdot 5 = 25$ volts
$k = 5$

3. $I = \dfrac{k \cdot P_1 \cdot P_2}{d}$; $\quad 50 = k \cdot \dfrac{(20,000)(35,000)}{70}$

$$k = .000005 \text{ or } 5 \cdot 10^{-6}$$

The model predicts 27,777 calls per day between Washington and Chicago. Of course, the extrapolation from small to large cities has been assumed to be valid. It is not hard to envision factors that would make Chicago/Washington interaction more vigorous <u>per capita</u> than for smaller cities.

4. Exact answer is very hard to come by. It is about .37.
   (1) One simulation would be to get a deck of cards and do the dealing of 5 cards many times.
   (2) Another is to look at groups of five pairs of random digits 00-12 Heart, 13-25 Diamond, 26-38 Club, 39-51 Spade. Ignore 52-99 and repetitions. For instance, a draw of 11, 23, 07, ~~75~~, 50, ~~86~~, ~~23~~, 03 would have 3 hearts. Compute ratio of at least 3 in a suit to number of 5 card groups.

5. Simulate by looking at blocks of ten random digits. Each digit represents a toss with 0, 1, 2 standing for a six face. The block 03544 61823 represents 3 sixes and 7 other faces up; this is a failure in the search for 5 or more sixes in ten tosses. Exact probability is binomial P (at least 5 sixes in ten tosses) =

$$= \binom{10}{5}(.3)^5(.7)^5 + \binom{10}{6}(.3)^6(.7)^4 + \binom{10}{7}(.3)^7(.7)^3$$

$$+ \binom{10}{8}(.3)^8(.7)^2 + \binom{10}{9}(.3)^9(.7) + \binom{10}{10}(.3)^{10}$$

This is approximately .149

6. There are several reasonable ways to approach this problem. The difficulty is defining what is meant by

evidence of improvement over the season.
Suppose you test the following question: What is the probability that a team that is really a .500 team (2 wins 2 losses over the first half of the season) would win its last four games?
The answer here is $(.5)^4$, assuming games independent, or about .0625. As another approach one might take the team's overall winning percentage, .75 and check the probability that such a team would win 4 in a row; $(.75)^4 \approx .32$.
It is now up to the fan to decide whether the performance is remarkable enough to conclude improvement took place.

7. Actual area is $\pi \cdot 3 \cdot 4 \approx 37.7$
To use Monte Carlo methods, generate pairs of random digits from $-4 \le x \le 4$ and $-3 \le y \le 3$ and check to see if (x,y) is within ellipse.
Simplify by taking $0 \le x \le 4$ and $0 \le y \le 3$ and checking if $\sqrt{9 - \frac{9}{16}x^2} \gtrless y$. This will give an estimate of .25 of the area. Sample Solution:

Random digits	Point	In	Out
2925	(2.9, 2.5)		√
0217	(.2, 1.7)	√	
4007	(4, .7)		√
3511	(3.5, 1.1)	√	
3012	(3.0, 1.2)	√	

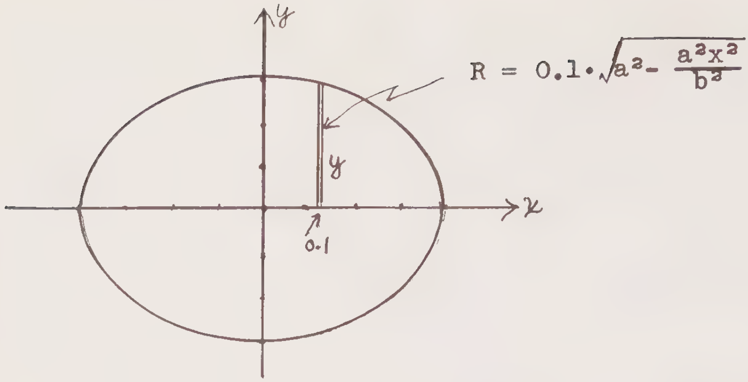
$\therefore$ estimate sector area $\frac{3}{5} \cdot 12 = 7.2$

Total area estimate $4 \cdot 7.2 = 28.8$

Taking more sample points should improve accuracy of estimate.

**8. Two solutions are given, one by calculus style approximation and the other by Monte Carlo simulation. Each works with $\frac{1}{4}$ of the ellipse first.

```
05 INPUT A,B
10 FØR X = .1 TØ B STEP .1
20 LET Y = SQR(A↑2-(A↑2)*(X↑2)/B↑2)
30 LET R = Y*.1
40 LET S = S+R
50 NEXT X
60 PRINT "AREA IS ABØUT"; 4*S
70 END
```



For Monte Carlo, program the technique of 7.

```
10 INPUT A,B,C
20 FØR N = 1 TØ C
30 LET X = B*RND(Z)
40 LET Y = A*RND(Z)
50 LET R = SQR(A↑2-(A↑2)*(X↑2)/B↑2)
60 IF  R > Y THEN 80
70 LET I = I+1
80 NEXT N
90 PRINT "AREA IS ABØUT"; (4*A*B)*(I/N)
```

A simple formula to use for checking results is

$$\text{area} = \pi \cdot a \cdot b$$

To see the reasonableness, look at what happens to unit circle if it is stretched to fit over a$\times$b ellipse.

9. Many factors can apply here: population density, transportation facilities, age of various population areas, projected growth zones, location of current facilities, etc. Your class can probably come up with a long list.

10. Main factor is likelihood of loss due to flood compared to dollar cost of the dike. For instance, if dike costs amortize at $25,000 per year for ten years and flood loss now is $200,000 with probability .05 each year, expected flood loss without dike is roughly (.05 x 200,000) = $10,000 per year.
Dike doesn't seem worth cost; however, company must weigh risk of flood the first year!
To do a thorough analysis of loss one must consider probability of flood in several of the ten years too. But the key factors are cost of protection; potential loss, and probability of loss.

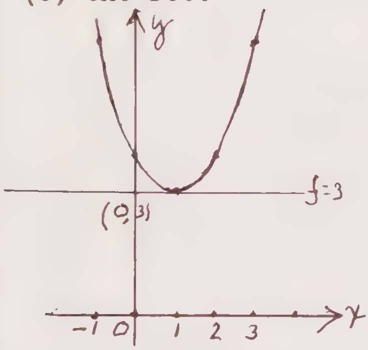
11. Factors to consider include: Most efficient route in terms of time to travel, which phone boxes fill more quickly, which phone boxes get broken into most frequently, etc.

Quadratic Functions, Equations, and Complex Numbers (pp.188-191)

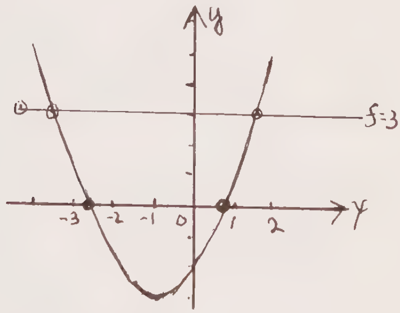
1. Let x be the number of phones in excess of 100, and make a table of profits.

x	No. of phones	Profit from each phone	Total Profit	Mathematical Explanation
0	100	10.00	1000	No.of phone = $(100+x)$
10	110	9.50	1045	Profit from each = $(100+x)(10-.05x)$
20	120	9.00	1080	$f = 1000+5x-.05x^2$
30	130	8.50	1150	Rewrite in completed square form
40	140	8.00	1120	$f= -.05(x^2-100x)+1000$ $= -.05(x^2-100+2500)+1125$
50	150	7.50	1125*	$= -.05(x-50)^2+1125$
60	160	7.00	1120	Maximum at $x = 50$; total profit=$1125
100	200	5.00	1000	

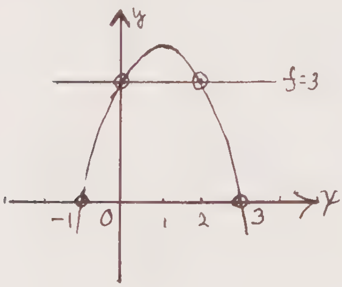
2. (b) complex
   (c) one root



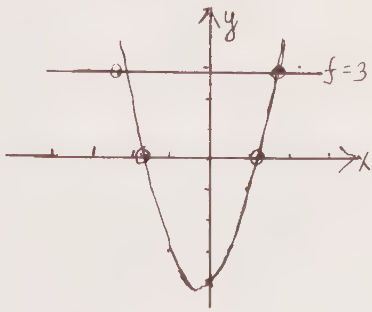
3. (b) two real roots
   (c) two real roots



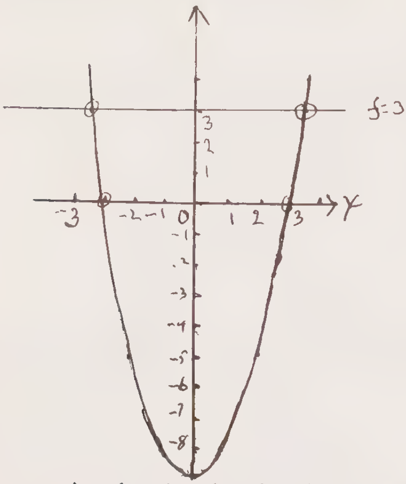
4. (b) two real roots
   (c) two real roots



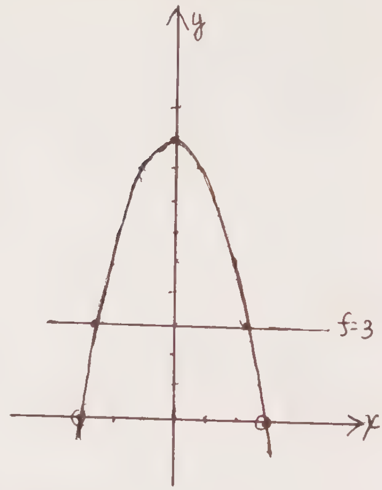
5. (b) two real roots
   (c) two real roots



6. (b) Two real roots
   (c) Two real roots

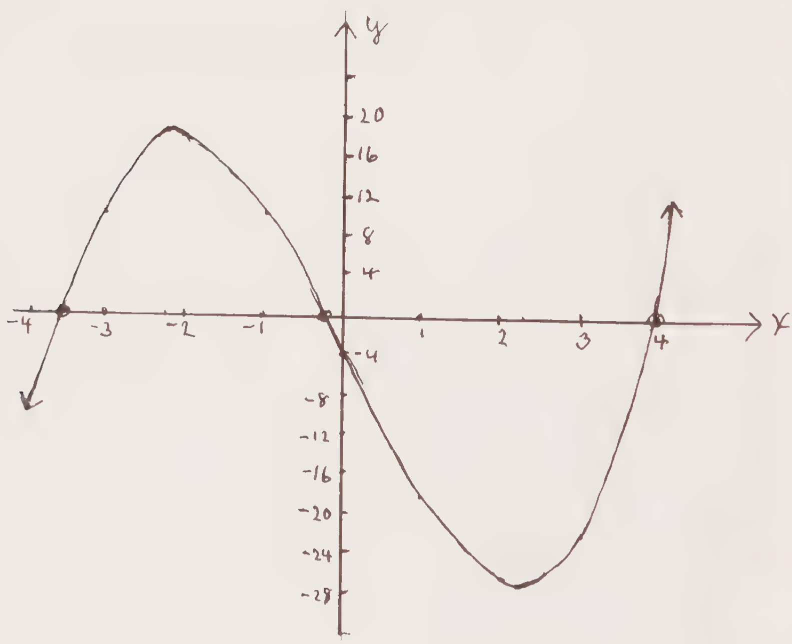


7. (b) Two real roots
   (c) Two real roots



8.

x	-4	-3	-2	-1	0	1	2	3	4	5
f(x)	-8	14	18	10	-4	-18	-26	-22	0	46



(b) $x^3 - 15x - 4 = (x-4)(x^2 + 4x + 1) = 0$

Solve $x^2 + 4x + 1 = 0 \therefore x = 4, -0.27, -3.73$

9. $-1 \pm \sqrt{5}$ 10. $1, \frac{1}{3}$ 11. 3 12. $\frac{2 \pm 11i}{5}$

13. $0, \frac{8}{3}$ 14. $\pm \frac{\sqrt{30}}{2}$ 15. $\frac{3 \pm 3\sqrt{5}}{4}$ 16. $\frac{5 \pm \sqrt{35}i}{2}$

17. (a) Multiply by $4a$; add b^2-4ac to each member;
$2ax + b = \pm\sqrt{b^2-4ac}$: Solve for x

(d) $16x^2 + 40x + 25 = 24 + 25$; $4x + 5 = \pm 7$; $\{-3, \frac{1}{2}\}$

18. (a) $(x-p)^2 = q^2$ $x_1 = p+q$, $x_2 = p-q$

(b) $r_1+r_2 = p$ $r_1r_2 = p^2-q^2$

(c) $x = p$, $x-p = 0 \therefore -q^2$ is minimum when $x = p$

19. For $x = +a$ or $-a$ F has same value $\therefore x = 0$ is line of symmetry.

20. $x^2 - 4x - 5 = (x-2)^2-3^2$ if $x \to \overline{X}+2$;
$x^2 - 4x - 5 \to \overline{X}^2-3^2$; $\overline{X} = 0$ or $x-2 = 0$ is line of symmetry.

21. $ax^2 + bx + c = a[(x+\frac{b}{2a})^2 - \frac{b^2-4ac}{4a^2}]$; $x = -\frac{b}{2a}$ is line of symmetry.

22. (a) See Ex. 21. if $x = -\frac{b}{2a}$ the curve has a minimum of $-(\frac{b^2-4ac}{4a})$

(b) See Ex. 21. if $a < 0$, $x = -\frac{b}{2a}$ is line of symmetry, $\frac{b^2-4ac}{4a}$ is maximum.

23. (a) 768 ft. (b) 16 sec. (c) at 4 sec. and 12 sec.
(d) in 8 sec; 1024 ft.

24. $x-y = 16$; $\frac{12}{y} = \frac{6}{6-x}$; $y = 12-2x$; $12x-2x^2 = 16$;
$x = 4$ or 2, $y = 4$ or 8 respectively.

25. $3\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} -2\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ 26. $4\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} +5\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$

27. $3\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ 28. $4\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ 29. $3+8i$ 30. $-5+2i$

31. $-1-i$

32. $$\left.\begin{array}{l} R_1=7+5x \quad R_2=1+2x \quad R_3=R_1+R_2=8+7x \\ F+G=4x^2+7x+12 \quad R_{F+G}=8+7x \\ R_1\cdot R_2=10x^2+19x+7;\ R_{R_1R_2}=-3+19x \\ R_{F\cdot G} = -3+19x \end{array}\right\} \begin{array}{l} R_{R_1+R_2} = R_{F+G} \\ \\ R_{R_1\cdot R_2} = R_{F\cdot G} \end{array}$$

33. The remainders of polynomials, modulo x^2+1 have rules for addition and multiplication exactly the same as for complex numbers with x replacing i.

34. $S = -2+8i$, $P = -27+8i$

35. $S = (a+e)+(b+f)i$ $P = (ae-bf)+(af+be)i$

36. $S = (2\sqrt{3}+3)+i(\sqrt{3}-1)$ $P = 7\sqrt{3}+3i$

37. $S = 2\sqrt{2}$ $P = 4$ 38. $S = -2.5-2.4i$ $P = 5.84-28.32i$

39. a^2+b^2 40. $\frac{3-\sqrt{3}i}{12}$ 41. $\frac{1+6i}{74}$ 42. $\frac{6-\sqrt{2}i}{38}$ 43. $-i$

44. $-3-2i$ 45. $-\frac{15}{13}+\frac{11}{13}i$ 46. $\frac{10+11i}{17}$ 47. $\frac{7+4i}{2}$

48. $\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$ 49. $9i$ 50. $40\sqrt{2}i$ 51. -12

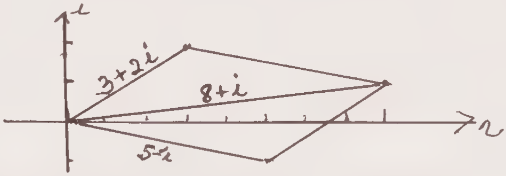
52. 24 53. $-16i$ 54. 5; $2+i$, $-2-i$ 55. 25; $4+3i$, $-4-3i$

56. 13; $3+2i$, $-3-2i$

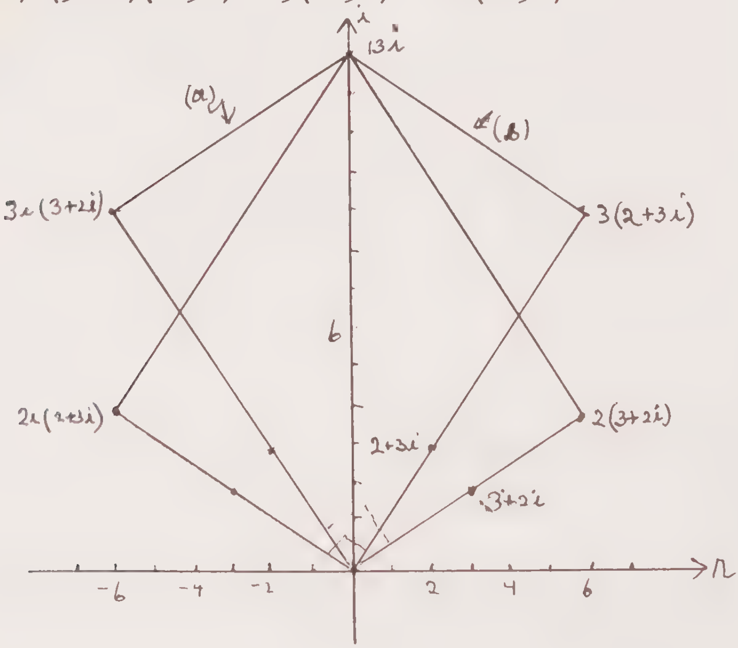
57. Given numbers $a+bi$, $c+di$; conjugates are $a-bi$, $c-di$; sum of numbers $(a+c)+(b+d)i$. Sum of conjugates is $(a+c)-(b+d)i$ $\therefore$ sum of conjugates equals conjugate of sum.

58. Follow procedure of Ex. 57.

59. $8+i$

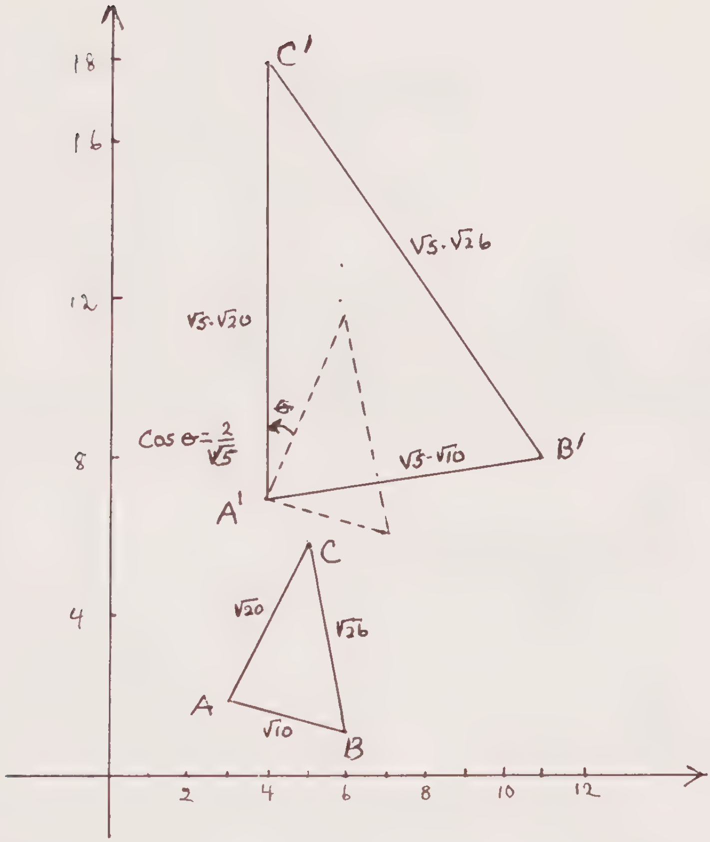


60. (a) $(2+3i)(3+2i) = 2(3+2i) + 3i(3+2i)$
(b) $(3+2i)(2+3i) = 3(2+3i) + 2i(2+3i)$



61. $2+i = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$; $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} A = A'$; $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} B = B'$;

$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} C = C'$ $\sqrt{5}\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \Delta ABC = \Delta A'B'C'$



CHAPTER 4

Section 4.1 (pp. 198-201)

1. $12x^8$ 2. $-10r^3t^4$ 3. $-28x^6y^6$ 4. x^9 5. $16x^6$
6. $2c^4-8c^3+6c^2$ 7. $16x^7$ 8. $16x^6+4x^7$ 9. $-15c^2t+6c^3t^2$
10. $2x^4-5x^2-12$ 11. $r^6+4xr^3+4x^2$ 12. m^3+7m^2n
13. x^8 14. x^8+x^4 15. x^8+x^4+1 16. $x^6+3x^4+3x^2+1$
17. 1 18. $\frac{9}{4}$ 19. $-\frac{1}{27}$ 20. $-\frac{1}{9}$ 21. 0 22. $\frac{3}{10}$

23. $\frac{25}{16}$ 24. $\frac{1}{9}$ 25. undefined 26. $12a^3c^2$ 27. $\frac{24}{x^2}$

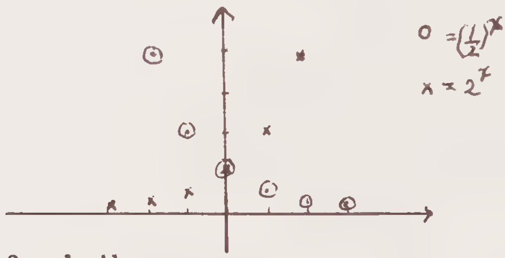
28. $-4r^3t+12r^2-\frac{4r}{t}$ 29. $\frac{1}{x^2}+\frac{1}{x}+1$ 30. $\frac{6}{a^2}+\frac{1}{a}-12$ 31. $\frac{1}{4y^6}$

32. $\frac{1}{x^2+6x+9}$ 33. $4x^6y^2$ 34. x^6+x^4 35. $\frac{xy}{y+x}$

36. (a) $P(t) = 2560\cdot 4^t$ (b) 655,360 (c) 10

37. Discussion 38. (a) 400 (b) probably none; 204,800

39.



40. (a) Yes, for both

(b) From $3^{x_1}=3^{x_2} \rightarrow 3^{x_1-x_2}=1 \rightarrow x_1-x_2=0 \rightarrow x_1 = x_2$

41. (a) 1 move (b) 3 moves; 7 moves

42. (a) $2^{64}-1$ = 18, 446, 744, 073, 709, 551, 615
(b) About 213, 503, 982, 334, 601 days
$\approx$ 584, 942, 417, 355 years
$\approx$ 584, 942, $417\frac{1}{3}$ centuries
(c) No!

43. Case 1 $(m,n > 0)$

$$b^m\cdot b^n = \underset{m\text{ factors}}{(b\cdot b\cdot\ldots\cdot b)}\cdot\underset{n\text{ factors}}{(b\cdot b\cdot\ldots\cdot b)}$$

$$= \underset{m+n\text{ factors}}{b\cdot b\cdot\ldots\cdot b}$$

$$= b^{m+n}$$

Case 2 $(m,n < 0)$

$$b^m\cdot b^n = \frac{1}{b^{-m}}\cdot\frac{1}{b^{-n}}$$

$$= \frac{1}{b^{-(m+n)}} \quad \text{(because } -m, -n > 0\text{)}$$

$$= b^{m+n}$$

Case 3 $(m > 0,\ n < 0)$

$$b^m\cdot b^n = b^m\cdot\frac{1}{b^{-n}}$$

$$= b^{m-(-n)} \quad \text{(because } m, -n > 0\text{)}$$

$$= b^{m+n}$$

Case 4 ($m = 0, n \neq 0$)

$b^m \cdot b^n = b^0 \cdot b^n = b^n = b^{0+n} = b^{m+n}$

44. Case 1 ($m > 0$)

$$(ab)^m = \underbrace{(ab)(ab)(ab)\ldots(ab)}_{m \text{ factors of form } (ab)}$$
$$= \underbrace{(a \cdot a \cdot \ldots \cdot a)}_{m}\underbrace{(b \cdot b \cdot \ldots \cdot b)}_{m}$$
$$= a^m b^m$$

Case 2 ($m < 0$)

$$(ab)^m = \frac{1}{(ab)^{-m}} = \frac{1}{a^{-m}b^{-m}} \text{ (because } -m > 0)$$
$$= \frac{1}{a^{-m}} \cdot \frac{1}{b^{-m}} = a^m \cdot b^m$$

Case 3 ($m = 0$)

$$(ab)^m = (ab)^0 = 1 \text{ (for } ab \neq 0)$$
$$= a^0 \cdot b^0 \text{ (for } a \neq 0,\ b \neq 0)$$
$$= a^m \cdot b^m$$

Section 4.2 (pp. 206-207)

1. 8   2. $\frac{1}{3}$ 3. 5   4. $\frac{4}{5}$ 5. $\frac{1}{2}$ 6. 2   7. -4 8. $\frac{9}{4}$

9. 33   10. 4   11. 2   12. $\frac{3}{28}$ 13. 3   14. $x^{\frac{1}{4}}$ 15. $r^{\frac{2}{3}}t^{\frac{2}{3}}$

16. $(x^2+y^2)^{\frac{1}{2}}$ 17. $3^{\frac{1}{5}}r^{\frac{6}{5}}$ 18. $r(3+r)^{\frac{1}{5}}$ 19. $m^{\frac{4}{3}}$ 20. $(x+y)^{\frac{2}{3}}$

21. $(r^3+t)^{\frac{4}{3}}$ 22. $(r^3+t)^{\frac{5}{3}}$ 23. No   24. Yes

25. No (try $x = 4$)

In exercises 25-27, symbols refer to the principal, or positive real root.

26. $\sqrt[4]{9} = 9^{\frac{1}{4}} = (3^2)^{\frac{1}{4}} = 3^{\frac{1}{2}} = \sqrt{3}$

27. $\sqrt[6]{125} = 125^{\frac{1}{6}} = (5^3)^{\frac{1}{6}} = 5^{\frac{1}{2}} = \sqrt{5}$

28. $\sqrt[8]{4} = 4^{\frac{1}{8}} = (2^2)^{\frac{1}{8}} = 2^{\frac{1}{4}} = \sqrt[4]{2}$

29. (a) For $x = -4$ to 4 in steps of .25, a table of values for $(x, F(x))$ where $F(x) = 160 \cdot 2^x$.
    (b) sketch

30. Take $x = -3$, $y = 3$, and $n = 2$
    $(-3)^2 = (3)^2$, but $-3 \neq 3$

31. In each part let $x = \frac{m}{n}$, $y = \frac{p}{q}$, where $m, p \in Z$ and $n, q \in Z^+$.

(a) As in the discussion following Theorem 1, because we want to show

$$(b^{\frac{m}{n}})^{\frac{p}{q}} = b^{\frac{mp}{nq}}, \quad \text{consider}$$

$$((b^{\frac{m}{n}})^{\frac{p}{q}})^{nq} \quad \text{and} \quad (b^{\frac{mp}{nq}})^{nq}$$

By using Lemma 2 a number of times,

$$((b^{\frac{m}{n}})^{\frac{p}{q}})^{nq} = ((b^{\frac{pm}{n}})^{\frac{1}{q}})^{nq} = (b^{\frac{pm}{n}})^{n} = b^{pm} = (b^{\frac{mp}{nq}})^{nq}$$

By Lemma 1,

$$(b^{\frac{m}{n}})^{\frac{p}{q}} = b^{\frac{mp}{nq}}$$

(b) $\frac{b^x}{b^y} = b^x \cdot \frac{1}{b^y} = b^x \cdot b^{-y}$ (by Lemma 2)

$= b^{x+-y}$ (by Part (1) of Theorem 1)

$= b^{x-y}$

(c) $((ab)^{\frac{m}{n}})^n = (ab)^m = a^m b^m = (a^{\frac{m}{n}})^n \cdot (b^{\frac{m}{n}})^n$

$= (a^{\frac{m}{n}} \cdot b^{\frac{m}{n}})^n \therefore (ab)^{\frac{m}{n}} = a^{\frac{m}{n}} \cdot b^{\frac{m}{n}}$

(d) $(\frac{a}{b})^x = (a \cdot b^{-1})^x = a^x \cdot b^{-x}$ (by C)

$= \frac{a^x}{b^x}$

32. (a) Assume not; i.e. $b^{\frac{1}{n}} = 1$ or $b^{\frac{1}{n}} < 1$.

If $b^{\frac{1}{n}} = 1$, $b = 1$, but this contradicts hypothesis.

Suppose $b^{\frac{1}{n}} < 1$; since $0 < b^{\frac{1}{n}} < 1$

$(b^{\frac{1}{n}})^2 < 1$ (Don't prove this sequence of steps, although you could use mathematical induction.)

$(b^{\frac{1}{n}})^3 < 1$

$\vdots$

$(b^{\frac{1}{n}})^n < 1$

or $b < 1$. But this also contradicts the hypothesis, so $b^{\frac{1}{n}} > 1$.

(b) Use the fact that if $x > 1$, $x^m > 1$ for m a positive integer. In particular

$$b^{\frac{1}{n}} > 1 \quad \text{(By part (a))}$$

Hence $$(b^{\frac{1}{n}})^m > 1 \Rightarrow b^{\frac{m}{n}} > 1$$

Section 4.3 (pp.212-214)

1. (a) 3.32 (b) 8.83 (c) 1.82 (d) 1 (e) 36.49

2. $x^{2\sqrt{3}}$ 3. $-10r^{1+\sqrt{2}}$ 4. $x^{2\sqrt{3}}-y^{2\sqrt{5}}$ 5. 2 6. 4

7. $9c^{2\sqrt{3}}+6c^{\sqrt{3}}+1$ 8. $2x^{2\sqrt{2}}-yx^{\sqrt{2}}-15y^2$ 9. 8

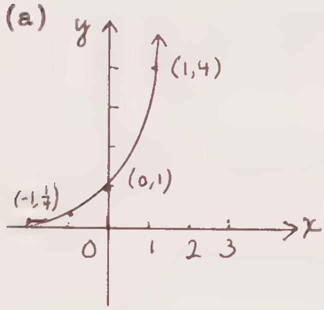
10. $x = \sqrt{2}$. This can be seen by:

If $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$, $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$, $x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Take the positive root

11. (a) Graph approaches (gets closer to) negative direction of x-axis and positive direction of y-axis.
    (b) Discussion

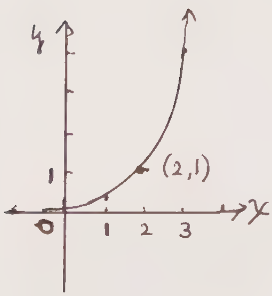
12. (a)

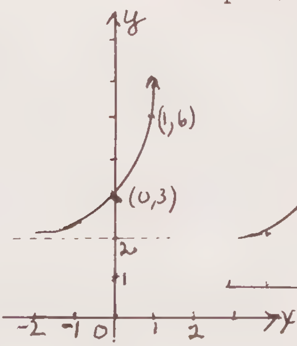


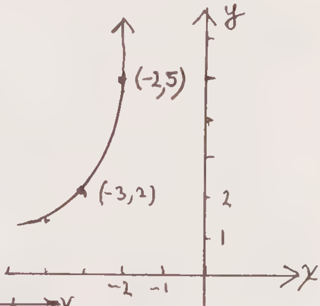
(b) reflect in y-axis

(c) reflect in x-axis

(d) The translations
$(x,y) \to (x+2,y)$
$(x,y) \to (x,y+2)$, and
$(x,y) \to (x-3,y+1)$
acting on the graph of $y=4^x$ will produce the graphs of $y = 4^{x-2}$, $y-2 = 4^x$, and $y-1 = 4^{x+3}$, respectively.







13. (a) Consider $f(\frac{m}{n}) = 2^{\frac{m}{n}}$, $m \in Z$, $n \in Z^+$

Now $2^{\frac{m}{n}} = (2^{\frac{1}{n}})^m$; but $2^{\frac{1}{n}} = \sqrt[n]{2} > 0$

by agreement (see text). The $\sqrt[n]{2}$ represents the positive n^{th} root of 2. The rest is easy.

For $m > 0$, $m = 0$, or $m < 0$,

$$(2^{\frac{1}{n}})^m > 0 \text{ because } 2^{\frac{1}{n}} > 0.$$

(b)

Statement	Reason
i. $x_2 - x_1 > 0$	i. $x_1 < x_2$
ii. $2^{x_2 - x_1} > 1$	ii. This is an application of Exercise 32, Section 4.2
iii. $2^{x_2} > 2^{x_1}$	iii. Multiply both sides in ii. by positive 2^{x_1}
iv. $f(x_1) < f(x_2)$	iv. $f(x_1) = 2^{x_1}$, $f(x_2) = 2^{x_2}$

14. Strictly increasing for $a > 1$, strictly decreasing for $0 < a < 1$, constant for $a = 1$.

15. Let $g: x \to a^x$, where $a > 0$, $a \neq 1$. To prove that g is 1-1 requires that we show that for any $x \neq y$ in the domain of g, $g(x) \neq g(y)$.

Proof. If $x \neq y$, then one of x and y is the smaller; say x. That is, $x < y$. But by Exercise 14,

if $x < y$, $a^x < a^y$ (g is strictly increasing)

or $a^x > a^y$ (g is strictly decreasing),

hence $g(x) \neq g(y)$.

When $a = 1$, g is not 1-1 for $g(x) = 1$ for all $x \in R$.

16. $P(t) = 3000 \cdot (4)^{\frac{t}{2}}$. $P(1) = 6000$, $P(5) = 96{,}000$. Major assumption: growth remains unchecked

17. (a) $P(t) = (3.5)(2)^{\frac{t}{24}}$

(b) In 1982, $P(12) = (3.5)(2)^{\frac{1}{2}} \approx 4.9$ million

In 2006, $P(36) = (3.5)(2)^{\frac{3}{2}} \approx 9.8$ million

Note how the population in 2006 can be predicted from that in 1982--there is a difference of 24 years; if the population growth pattern remains stable, the population should double.

18. Research question.

In Exercises 19-24, the BASIC programs will all be variations of the type of graph-plotting program given in Section 6 of the Prologue. A sample program for Exercise 20 will be given:

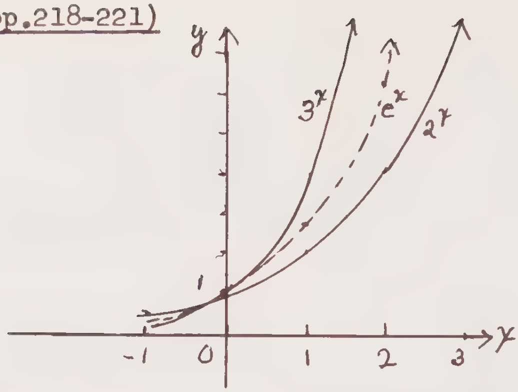
```
  10 DEF FNF(X) = 2↑(-1*(X↑2))
  20 READ A, B, S
  30 DATA -3, 3, .2
  40 READ C, D, N
  50 DATA 0, 1, 50
  60 LET H = (D-C)/N
  70 PRINT "Y-AXIS: FRØM "C;"TØ"D;"IN STEPS ØF"H
  80 PRINT
  90 FØR I = 0 TØ N
 100 PRINT "-"
 110 NEXT I
 120 DEF FNR(X) = INT(X+.5)
 130 FØR X = A TØ B STEP S
 140 LET Y = FNF(X)
 150 LET Y1 = FNR((Y-C)/H)
 160 FØR I = 0 TØ N
 170 IF I = Y1 THEN 200
 180 PRINT " ";
 190 GØ TØ 210
 200 PRINT "*";
 210 NEXT I
 220 PRINT "  ";
 230 PRINT X
 240 NEXT X
 250 END
```

```
RUN
PLØT

Y-AXIS: FRØM 0 TO 1  IN STEPS OF .02
                                              -2.8
                                              -2.6
                                              -2.4
                                              -2.2
                                              -2.
*                                             -1.8
   *                                          -1.6
        *                                     -1.4
             *                                -1.2
                    *                         -1.
                          *                   -.800001
                                 *            -.600001
                                      *       -.400001
                                          *   -.200001
                                           *  -8.94070E-07
                                          *    .199999
                                      *        .399999
                                 *             .599999
                          *                    .799999
                    *                          .999999
             *                                1.2
        *                                     1.4
   *                                          1.6
*          and so on up to 3. the graph is symmetrical
```

Section 4.4 (pp.218-221)

1.

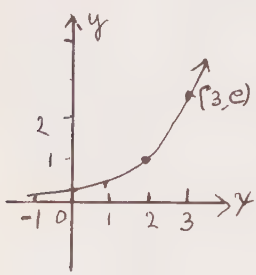


2.

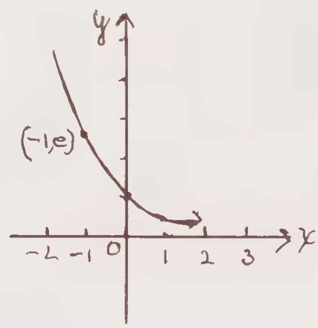
x	-3	-2.5	-2	-1.5	-0.5	0	.5	1.5	2.0	2.5	3.0
e^x	.05	.08	.14	.22	.61	1	1.65	4.48	7.39	12.18	20.09

In Exercises 3-6, the graphs are all congruent to the graph of $y = e^x$ because each graph is a translation and/or a reflection of the graph of $y = e^x$.

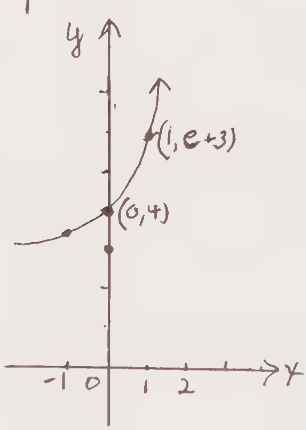
3.



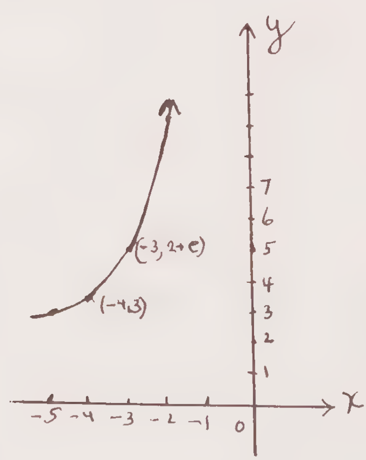
4.



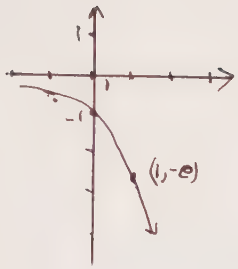
5.



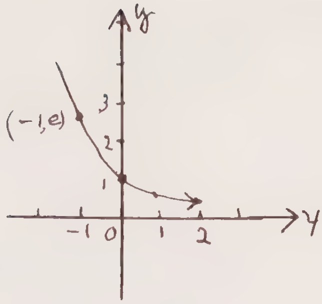
6.



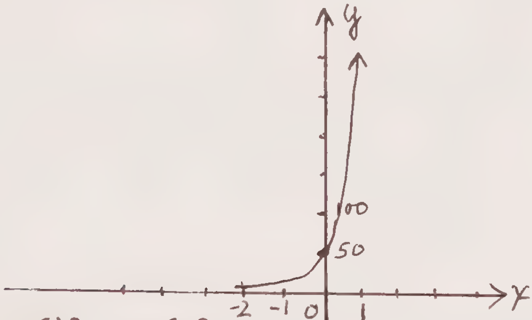
7. (a)



(b)



8. (a) Range = $\{g(x) > 0 : g(x) \in R\}$
(b) strictly increasing
(c) $0 < g(x) < 50$
(d)



9. (a) $5000(1.06)^2 = 5,618$
(b) $5000(1.03)^4 = 5627.54$

10. $A(t) = 5000e^{.06t}$; $A(2) = 5000e^{.12}$. But $e^{.12}$ is not in the tables. Since .12 is $\frac{2}{5}$th of the way between .10 and .15,

$$e^{.10} \approx 1.1052$$
$$e^{.12} \approx x$$
$$e^{.15} \approx 1.1618$$

$\frac{2}{5}(1.1618 - 1.1052) = .0226$, and

$x = 1.1052 + .0226 = 1.1278$. Therefore

$A(2) = 5000(1.1278) = \$5,639$.

11. (a) 73.89 (b) $N(-1) \approx .183$ and $N(\frac{3}{2}) \approx 4034.3$
(c) 0.425 (d) -0.35

12. From $164 = 80e^{3k}$, $e^{3k} = 2.05$. The table shows that $.70 < 3k < .75$.

$$e^{.70} = 2.0138 \qquad \frac{362}{1032}(.05) \approx .02$$
$$e^{3k} = 2.05$$
$$e^{.75} = 2.1170$$

So $3k \approx .72$ and $k \approx .24$

13. (a) Verify
(b) $1800k = -0.67$
$k = -0.00037$
(c) $100e^{-.00037t} = 20.19$
$-.00037t = -1.6$
$t \approx 4,324\frac{1}{3}$ years

14. $A_o e^{3k} = .10A_o$; $e^{3k} = .10$, so

$e^{-2.3} = .1003$
$e^{3k} = .10$ 93 96 $\qquad \frac{93}{96}(.1) \approx .096$
$e^{-2.4} = .0907$

So $3k \approx -2.304$ and $k \approx -.768$

15. (a) $f(x+y) = f(x) \cdot f(y)$
(b) $f(0) = 1$
(c) $f(-x) = \frac{1}{f(x)}$

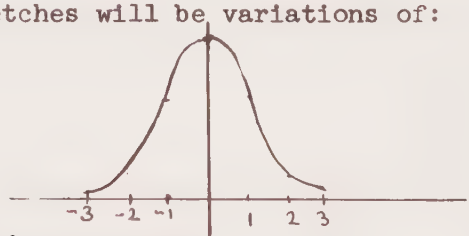
16. In $(R,+)$, 0 is the identity and each element $x \in R$, has an inverse $-x \in R$. In $(R^+,\cdot)$, 1 is the identity and each positive real number x has an inverse $\frac{1}{x} \in R^+$.

17. $f: x \to e^x$, $x \in R$ is a 1 - 1 mapping of R <u>onto</u> R^+ (from Section 4.3). By Exercise 15, since

$$f(x+y) = e^{x+y} = e^x \cdot e^y = f(x) \cdot f(y),$$

f preserves the underlying structure of $(R,+)$ and $(R^+,\cdot)$; hence f is an isomorphism.

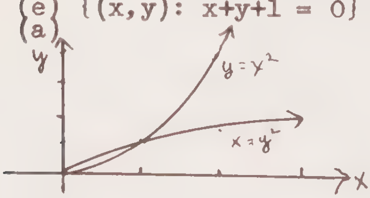
18. All sketches will be variations of:



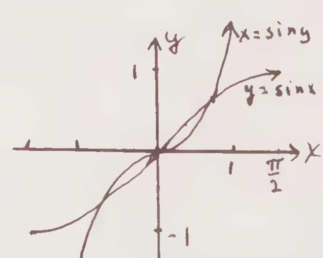
Section 4.5 (pp.225-226)

1. (a) $x = 2y-4$ or the function given by $y = \frac{1}{2}x+2$
(b) $3y+2x-6 = 0$
(c) $g^{-1}: x \to \frac{1}{3}x-\frac{1}{3}$
(d) $\{(2,1),(5,3),(4,4),(0,-1)\}$
(e) $\{(x,y): x+y+1 = 0\}$

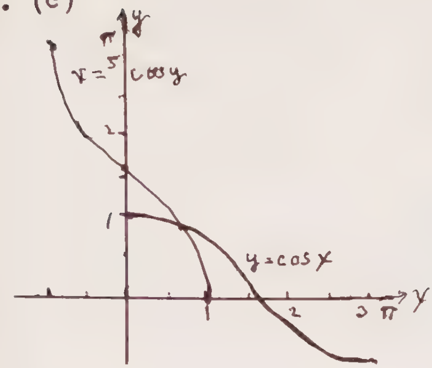
2. (a)



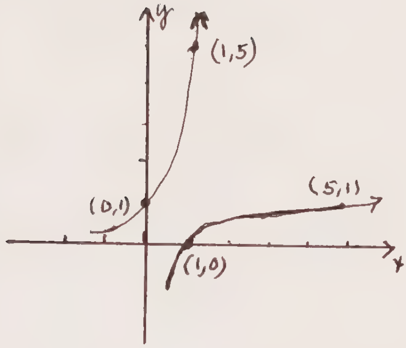
(b)



2. (c)

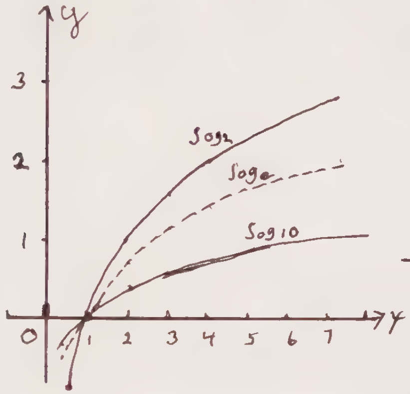


(d)

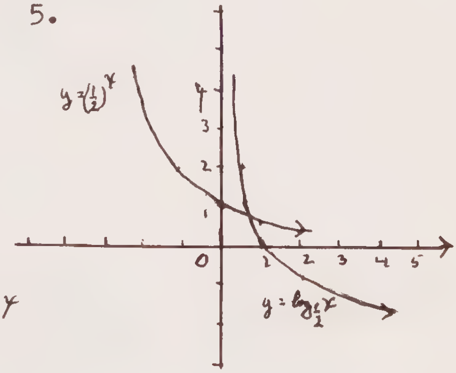


3. (a) 2 (b) $\frac{1}{2}$ (c) 0 (d) $\frac{1}{2}$ (e) $\frac{2}{3}$ (f) $^{-}\frac{1}{3}$
   (g) 2 (h) r (i) 1 (j) 9 (k) 5 (ℓ) a

4.



5.



6. (a) I and IV (b) No (c) Yes at (1,0) (d) Yes

7. $\frac{5}{2} + -2 = \frac{1}{2}$ 8. $\frac{1}{2} - (-\frac{1}{2}) + 2 = 3$ 9. 1

10. (a) $r(0,0) = (0,0)$; $r(1,0) = (0,1)$; $r(0,2) = (2,0)$
    $r(\sqrt{3},\sqrt{3}) = (\sqrt{3},\sqrt{3})$

    (b)
    (c) Under a reflection in the line y = x, the image of the graph of a function is the graph of its composition inverse, assuming it exists.

11. Not an identity (Try x = 2 and y = 4)

12. An identity 13. An identity

14. Not an identity (Try x = b and y = b)

Section 4.6 (pp.232-234)

1. (a) a (b) $\frac{11}{6}$ (c) 2 (d) $\frac{3}{2}$ (e) -1 (f) $\frac{-1}{3}$
   (g) 0 (h) 100

2. (a) .6767 (b) $-2+.3365 = -1.6635$ (c) 2.3365
(d) -2.1938 (e) 3 (f) 1.8189 (g) 0.5119
(h) -3.3251 (i) 4.1461

3. (a) 754 (b) 8.31 (c) .0424 (d) 2.20×10^{24}
(e) .01 (f) .000061

4. (a) $n \approx 8.16$ (b) $n \approx .066$ (c) $n \approx 12.21$
(d) $n \approx .0007$

5. $x = \frac{3}{2}$ 6. $t = \frac{\log 14}{2\log 3} \approx \frac{1.1461}{.9452} = 1.20$

7. $n = \frac{\log 9.82 - \log 3}{\log 3} \approx 1.08$

8. 8 years 9 months. (Solve $10{,}000(1.02)^{4t} = 20{,}000$)

9. $\frac{-1}{3}$

10. The equation is: $2 \cdot 10^7 = 10^7 \cdot e^{3t}$.
The solution: $t \approx .23$ days

11. Solve $\frac{1}{2}c = c \cdot e^{-5t}$ for t. $t \approx .14$ units

12. (a) $t = 11\frac{1}{2}$ years (b) $r \approx 5\frac{1}{2}\%$

13. $0 = \log_b 1$

$= \log_b \frac{1}{y} \cdot y$

$= \log_b \frac{1}{y} + \log_b y$

$\therefore \log_b \frac{1}{y} = -\log_b y$

<u>Review Exercises (pp.235-236)</u>

1. True 2. True 3. True 4. True 5. True

6. False: $\log 143 = 2 + \log 1.43$

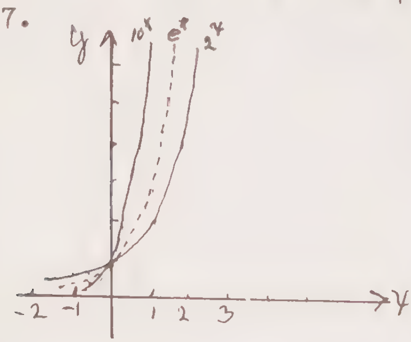
7. False: Range $= \{f(x): f(x) > 0\}$

8. True 9. True 10. True 11. 243 12. 5 13. $\frac{9}{4}$

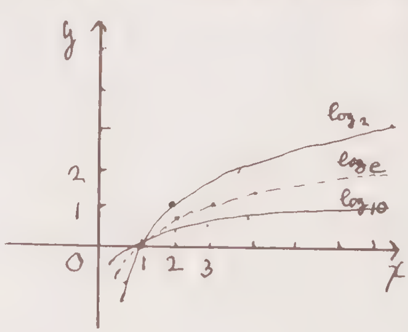
14. $\frac{5}{2}$ 15. $\frac{1}{9}$ 16. 2 17. $\frac{5}{2}$ 18. -1 19. 1 20. 100

21. $-15x^{\frac{5}{6}}$ 22. $x-y$ 23. $\frac{x^{11}}{4}$ 24. $3^{2\pi+1}$ 25. $x+1$ 26. x^5

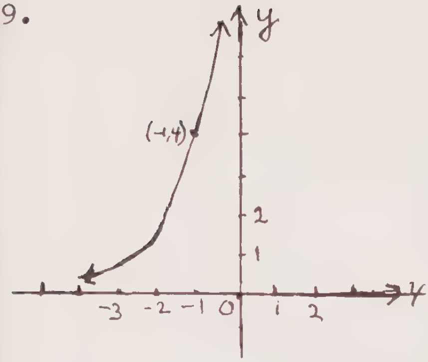
27.



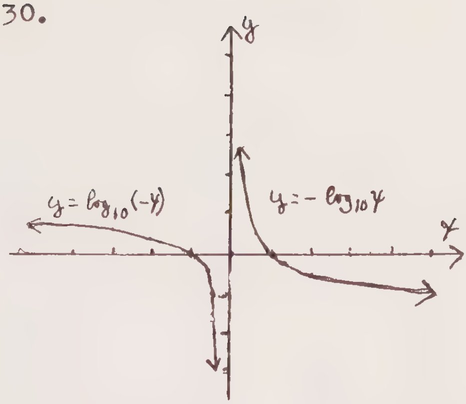
28.



29.



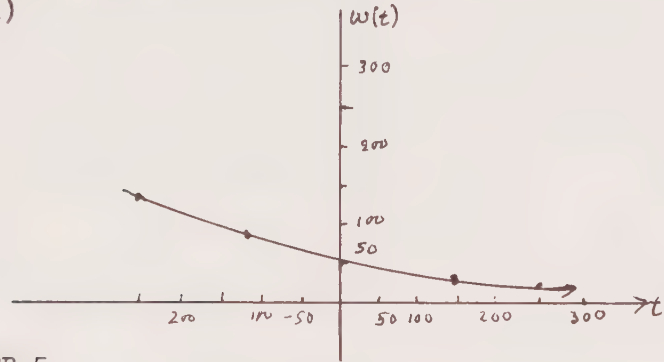
30.



31. (a) $x = \frac{1}{2}$ (b) $x = {}^-\frac{1}{2}$ (c) $x \approx 1.49$

32. $r = .5$ 33. $t = -.5\overline{3}$ 34. $n = 7.05$

35. (a) 50 watts (b) 18.4 watts; 11.6 watts
(c) about 172 days
(d)



CHAPTER 5

Section 5.1 (pp. 241-245)

1. 0 2. 180° 3. 90° 4. $\frac{\pi}{4}$ 5. $\frac{3\pi}{4}$ 6. 120°

7. $\frac{5\pi}{6}$ 8. 210° 9. $\frac{3\pi}{2}$ 10. $\frac{5\pi}{3}$ 11. 120° 12. 225°

13. $\frac{7\pi}{4}$ 14. 0° 15. 157.5° 16. B 17. E 18. F

19. A 20. D

21.

radian measure	cosine	sine
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

radian measure	cosine	sine
$\frac{\pi}{2}$	0	1
$\frac{\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{-\sqrt{3}}{2}$	$\frac{1}{2}$
π	-1	0
$\frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$
2π	1	0

22. The unit circle is $\{(x,y): x^2+y^2 = 1\}$

23. The point corresponding to $2\pi-\theta$ is symmetric in x-axis to the point corresponding to θ. $\therefore$ the x-coordinates must be identical and cosines are identical.

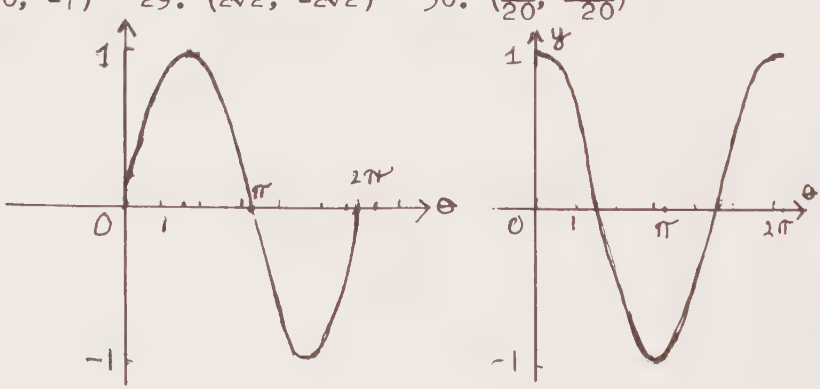
24. $0 \leq \theta \leq \frac{\pi}{2}$, the points corresponding to θ and $\frac{\pi}{2}-\theta$ are symmetric in the line $y = x$. $\therefore$ the y-coordinate of one equals the x-coordinate of the other.

25. The points corresponding to θ and $\pi-\theta$ are symmetric about the y-axis. $\therefore$ y-coordinates are identical.

26. $(-\frac{5}{2}, \frac{5\sqrt{3}}{2})$ 27. $(\frac{1}{2}\cos 23^\circ, \frac{1}{2}\sin 23^\circ) \approx (.46, .20)$

28. $(0, -7)$ 29. $(2\sqrt{2}, -2\sqrt{2})$ 30. $(\frac{\sqrt{2}}{20}, \frac{-\sqrt{2}}{20})$

31.



Section 5.2 (pp.250-252)

1. $(-1,0)$ 2. $(0,-1)$ 3. $(0,1)$ 4. $(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2})$

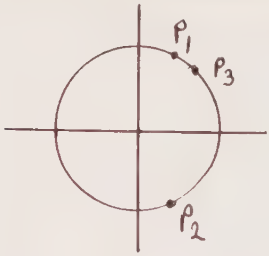
5. $(\frac{-\sqrt{2}}{2},\frac{-\sqrt{2}}{2})$ 6. $(\frac{\sqrt{3}}{2},\frac{1}{2})$ 7. $(\frac{\sqrt{3}}{2},\frac{1}{2})$ 8. $(\frac{\sqrt{3}}{2},-\frac{1}{2})$

9. $(-\frac{\sqrt{3}}{2},\frac{1}{2})$ 10. $(-\frac{\sqrt{3}}{2},-\frac{1}{2})$ 11. $\theta = 2n\pi$ (n integral)

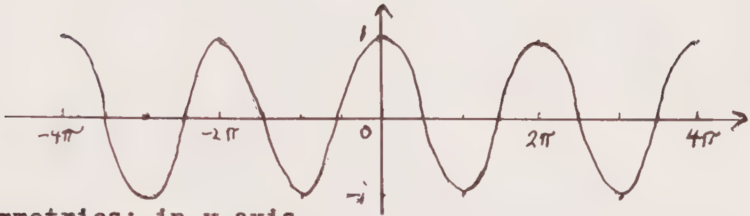
12. $\theta = \frac{3\pi}{2} + 2n\pi$ (n integral) 13. $\theta = 2n\pi - \frac{\pi}{6}$ (n integral)

14. $\theta = n\pi$ (n integral)   15. $\theta = 2n\pi$ (n integral)

16.

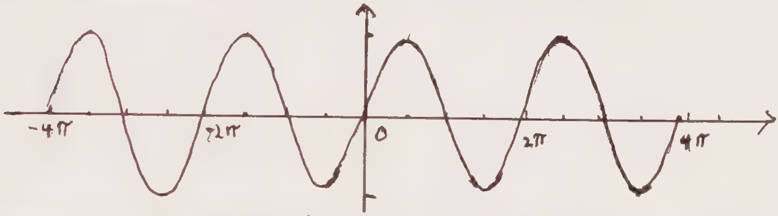


17.



symmetries: in y-axis
If allow domain $\mathbb{R}$, then line symmetry in $x = k\pi$, point symmetry in $\left((2k+1)\frac{\pi}{2}, 0\right)$, translation $2k\pi$.

18.



symmetry = in point (0,0)
If allow domain $\mathbb{R}$ then symmetries $\begin{cases} \text{lines } x = (2k+1)\frac{\pi}{2} \\ \text{points } (k\pi, 0) \\ \text{translation } 2k\pi \end{cases}$

19. Range of each function is $[-1,1]$; that is $\{y : -1 \leq y \leq 1\}$

20. $\cos\theta \begin{cases} \max\ 2k\pi \\ \min\ (2k+1)\pi \end{cases}$ $\sin\theta \begin{cases} \max\ \frac{\pi}{2}+2k\pi = (4k+1)\frac{\pi}{2} \\ \min\ \frac{3\pi}{2}+2k\pi = (4k+3)\frac{\pi}{2} \end{cases}$

21. $Q = (\cos\theta, \sin\theta)$
$B = (OB\cos\theta, OB\sin\theta)$
$AB = OB \cdot \sin\theta$
$AB \div OB = \sin\theta$
$OA = OB \cdot \cos\theta$
$OA \div OB = \cos\theta$

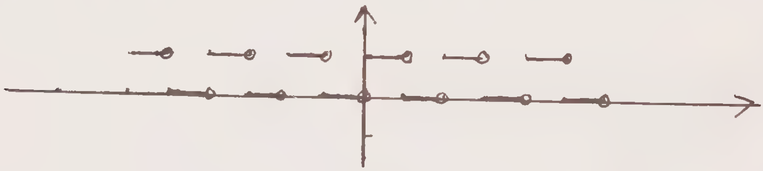
22. $OB = 16$
$OA = 8\sqrt{3}$

23. $XY = \frac{10\sqrt{3}}{3}$
$XZ = \frac{20\sqrt{3}}{3}$

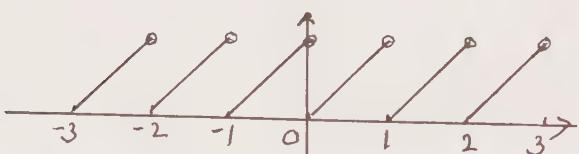
24. $NP = 19.25$
$MP = 7.45$

Section 5.3 (pp.254-258)

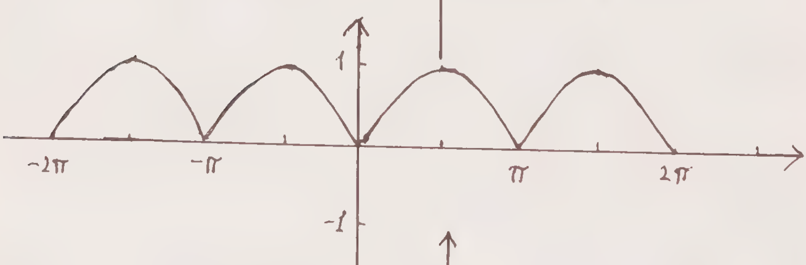
1. $\omega(\frac{\pi}{2}) = (0,1)$ and $\omega(-\frac{\pi}{2}) = (0,-1)$
   In each case $0^2 + 1^2 = 1$
2. $\omega(\frac{3\pi}{4}) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $\omega(\frac{-3\pi}{4}) = (\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})$
   $(\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2 = \frac{1}{2} + \frac{1}{2} = 1$
3. $\omega(\frac{7\pi}{6}) = (\frac{-\sqrt{3}}{2}, \frac{-1}{2})$ and $\omega(\frac{-7\pi}{6}) = (\frac{-\sqrt{3}}{2}, \frac{1}{2})$
4. $\omega(\frac{17\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $\omega(\frac{-17\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})$
5. $\omega(\frac{-2\pi}{3}) = (\frac{-1}{2}, \frac{-\sqrt{3}}{2})$ and $\omega(\frac{2\pi}{3}) = (\frac{-1}{2}, \frac{\sqrt{3}}{2})$
6.



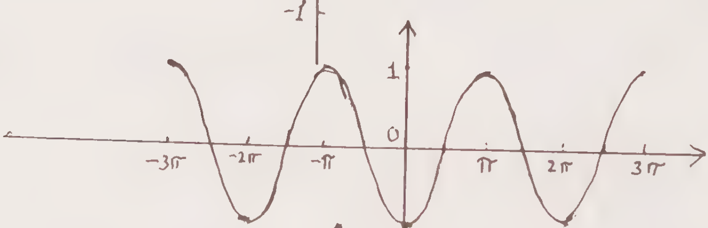
7.



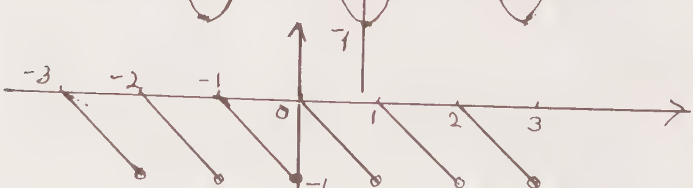
8.



9.



10.



11. A function f is periodic with period p iff p is the smallest number such that for all x, $f(x+p) = f(x)$.

Alternate: (x,y) in graph of f iff $(x+p,y)$ in graph of f.

12. period in (6) is 2
(7) is 1
(8) is π
(9) is 2π
(10) is 1

13. (a) $f \circ t = f$
(b) translation; translation of period moves graph onto itself.

14. (a) period 2 (b) period $2\frac{1}{2}$ (c) not periodic
(d) no apparent period

15. even; prof $f(-x) = (-x)^2 = x^2 = f(x)$

16. even 17. odd 18. odd 19. even 20. odd

21. even 22. even 23. even if n even, odd otherwise

24. even function symmetric in y-axis
odd function symmetric in (0,0)

25. All odd index coefficients must be zero if p(x) is to be even. All even index coefficients zero is necessary and sufficient for p(x) odd

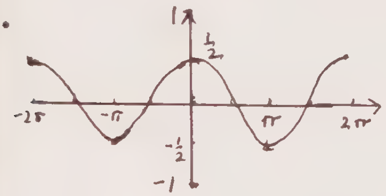
26. $10.6 \approx c$ 27. $8.06 \approx c$ 28. $2.12 \approx c$ 29. $6.95 \approx a$

30. $b \approx 1.84$, $m\angle A = 22\frac{1}{2}^\circ$, $m\angle C = 22\frac{1}{2}^\circ$

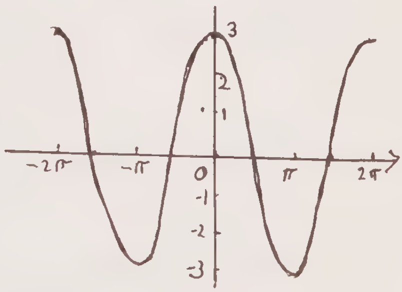
31. $m\angle A \approx 19^\circ$, $m\angle B \approx 26^\circ$

Section 5.4 (pp.262-266)

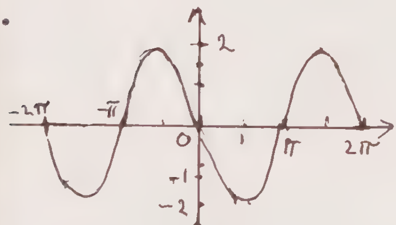
1.



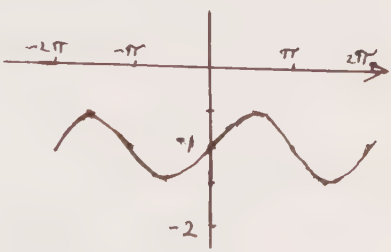
2.



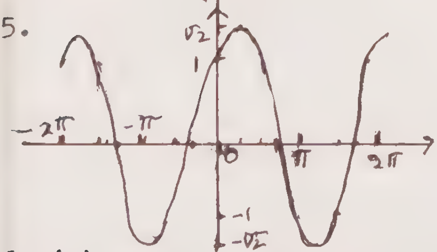
3.



4.



5.

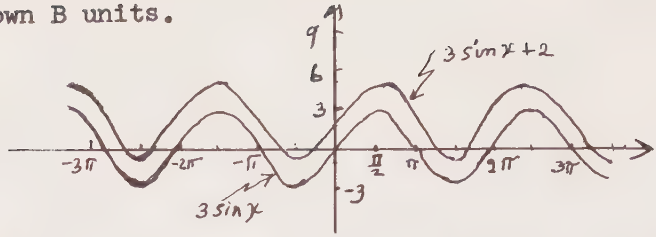


$\sin x + \cos x = \sqrt{2}\,\sin(x+\frac{\pi}{4})$

6. (a) amplitude A with graph translated up B units
(b) amplitude A, flip graph about x-axis, then slide

down B units.

7.



8. period = π
amplitude = 1
frequency = $\frac{1}{\pi}$

9. $p = \frac{2\pi}{3}$
$a = 1$
$\omega = \frac{3}{2\pi}$

10. $p = 6\pi$
$a = 4$
$\omega = \frac{1}{6\pi}$

11. $p = 2\pi$
$a = 3$
$\omega = \frac{1}{2\pi}$

12. $p = \frac{2\pi}{5}$
$a = 1$
$\omega = \frac{5}{2\pi}$

13. $p = 10\pi$
$a = 1$
$\omega = \frac{1}{10\pi}$

14. $p = \frac{\pi}{3}$
$a = 1$
$\omega = \frac{3}{\pi}$

15. $p = 3$
$a = 1$
$\omega = \frac{1}{3}$

16. 2π, $\frac{2\pi}{B}$ $\therefore$ period is $\frac{2\pi}{B}$

17. π, $\frac{\pi}{B}$ $\therefore$ period is $\frac{2\pi}{B}$

18. $\frac{2\pi}{B}$; $\frac{B}{2\pi}$

19. $|A|$

20. (a) $2\cos 2x$
amplitude = 2
period = π

(b) $\frac{1}{3}\sin\frac{x}{2}$
$a = \frac{1}{3}$
$p = 4\pi$

(c) $10(\sin\frac{1}{2}x) + 10$
$a = 10$
$p = 4\pi$

(d) $220\sin 4\pi x$

(e) $\frac{3}{2}(\cos\frac{1}{6}x)-\frac{1}{2}$

(f) $\frac{1}{2}(\cos\frac{\pi}{3}x)+\frac{1}{2}$

21. $\frac{5}{2}(\cos\frac{t}{2})+\frac{45}{2}$; rising and falling fastest midway between highs and lows

22. $h(t) = 6\sin\frac{\pi}{6}t$ start at P
$h(t) = 6\cos\frac{\pi}{6}t$ start at Q

23. $a(t) = 10\sin \pi t$

24. maximum $t = \frac{1}{240} + \frac{k}{60} = \frac{4k+1}{240}$
minimum $t = \frac{1}{80} + \frac{k}{60} = \frac{4k+3}{240}$
zero $t = \frac{k}{120}$

Section 5.5 (pp.269-271)

1. $\cos 75^\circ = \cos(45^\circ + 30^\circ)$
$= \cos 45^\circ \cdot \cos 30^\circ - \sin 45^\circ \cdot \sin 30^\circ$
$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$
$\approx .259$

2. $\sin 75^\circ = \sin(45^\circ+30^\circ)$
$= \sin 45^\circ\cos 30^\circ + \sin 30^\circ\cos 45^\circ$
$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$
$\approx .966$

3. $\sin \frac{7\pi}{12} = \sin(\frac{\pi}{3} + \frac{\pi}{4})$

$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot(+\frac{1}{2})$

$\approx .97$

4. $\cos \frac{11\pi}{12} = \cos(\frac{2\pi}{3} + \frac{\pi}{4})$

$= \cos\frac{2\pi}{3}\cos\frac{\pi}{4} - \sin\frac{2\pi}{3}\sin\frac{\pi}{4}$

$= (-\frac{1}{2})(\frac{\sqrt{2}}{2}) - (\frac{\sqrt{3}}{2})(\frac{\sqrt{2}}{2})$

$\approx -.97$

5. $\sin \frac{11\pi}{12} = \sqrt{1-\cos^2\frac{11\pi}{12}}$

$= \sqrt{1-(.97)^2}$

$\approx .26$

6. $\sin(\theta+\pi) = -\sin\theta$

Proof: $\sin(\theta+\pi) = \sin\theta\cos\pi + \sin\pi\cos\theta$

$= (\sin\theta)(-1) + (0)(\cos\theta)$

$= -\sin\theta$

7. $\cos(\theta+\pi) = \cos\theta\cos\pi - \sin\theta\sin\pi$

$= -\cos\theta$

8. $\sin(\frac{\pi}{2} - \theta) = \sin(\frac{\pi}{2}+(-\theta))$

$= \sin\frac{\pi}{2}\cos(-\theta) + \sin(-\theta)\cos\frac{\pi}{2}$

$= \cos(-\theta) = \cos\theta$

9. $\sin(\theta_1-\theta_2) = \sin(\theta_1+(-\theta_2))$

$= \sin\theta_1\cos(-\theta_2) + \sin(-\theta_2)\cos\theta_1$

$= \sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1$

10. $\cos(\theta_1-\theta_2) = \cos\theta_1\cos(-\theta_2) - \sin\theta_1\sin(-\theta_2)$

$= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$

11. $\cos 2\theta = \cos(\theta + \theta)$

$= \cos\theta\cos\theta - \sin\theta\sin\theta$

$= \cos^2\theta - \sin^2\theta$

12. $\sin 2\theta = 2\sin\theta\cos\theta$

13. $\cos(\frac{\pi}{2} - \theta) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$

$= \sin\theta$

14. $\cos(\pi - \theta) = \cos\pi\cos\theta + \sin\pi\sin\theta$

$= -\cos\theta$

15. $\cos(2\pi - \theta) = \cos-\theta$

$= \cos\theta$

16. $\cos m$ 17. $\sin x$ 18. $-\cos(p-q)$ 19. $-\sin(\theta_1 + \theta_2)$

20. $2\sin x \cos x + 1 = (\sin 2x) + 1$

21. $$\begin{aligned}\cos 15^\circ &= \cos(45^\circ-30^\circ)\\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &\approx .97\end{aligned}$$

22. $$\begin{aligned}\sin 15^\circ &= \sin(45^\circ-30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \sin 30^\circ \cos 45^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} \quad \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\\ &\approx .26\end{aligned}$$

23. $\cos -15^\circ = \cos 15^\circ \approx .97$

24. $\sin -15^\circ = -\sin 15^\circ \approx -.26$

25. $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ 26. $\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$ 27. $\begin{bmatrix} \frac{-\sqrt{6}+\sqrt{2}}{4} & \frac{\sqrt{2}+\sqrt{6}}{4} \\ -\frac{\sqrt{2}+\sqrt{6}}{4} & \frac{-\sqrt{6}+\sqrt{2}}{4} \end{bmatrix}$

28. (a) $\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$ (c) $\left(\frac{\sqrt{2}+\sqrt{6}}{4}, \frac{-\sqrt{6}+\sqrt{2}}{4}\right)$

29. (a) $PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

(b) $M\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = (x_1\cos\theta - y_1\sin\theta,\ x_1\sin\theta + y_1\cos\theta)$

$M\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (x_2\cos\theta - y_2\sin\theta,\ x_2\sin\theta + y_2\cos\theta)$

(c) Distance between MP and MQ =

$$\sqrt{((x_2-x_1)\cos\theta-(y_2-y_1)\sin\theta)^2+((x_2-x_1)\sin\theta+(y_2-y_2\cos\theta)^2}$$

$$= \sqrt{\begin{aligned}&(x_2-x_1)^2\cos^2\theta-2(x_2-x_1)(y_2-y_1)\cos\theta\sin\theta+(y_2-y_1)^2\sin^2\theta\\ &+(x_2-x_1)^2\sin^2\theta+2(x_2-x_1)(y_2-y_1)\cos\theta\sin\theta+(y_2-y_1)^2\cos^2\theta\end{aligned}}$$

$$= \sqrt{(x_2-x_1)^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}+(y_2-y_1)^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}}$$

30. (a) $M\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (b) $M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ (c) $M\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(d) Leave origin fixed; spin (1,0) to an angle of θ; spin point at angle $-\theta$ to (1,0), angle 0.

(e) If M is an isometry that agrees with rotation of θ about (0,0) for 3 non-collinear points, then M must be the same as the rotation.

31. (a) $\sin(x+y) = \sin x \cos y + \sin y \cos x$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$

(b) $f(x+y) = f(x)\cdot f(y) = 2^x \cdot 2^y$
(c) $f(x+y) = f(x)+f(y) = 5x+5y$

32. Many possibilities:

```
05 PRINT "SIN(X+Y)"; "FØRMULA"; "CØS(X+Y)"; "FØRMULA"
10 FØR X = -6.28 TØ 12.56 STEP .314
20 FØR Y = -6.28 TØ 12.56 STEP .314
30 LET S = SIN(X)*CØS(Y)+SIN(Y)*CØS(X)
40 LET C = CØS(X)*CØS(Y)-SIN(X)*SIN(Y)
50 PRINT SIN(X+Y); S; CØS(X+Y); C
60 NEXT Y
70 NEXT X
80 END
```

Section 5.6 (pp.273-274)

1. $\sin 22\frac{1}{2}^{\circ} = \sin\frac{45^{\circ}}{2} = \sqrt{\frac{1-\cos 45^{\circ}}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2} \approx .375$

2. $\cos 67\frac{1}{2}^{\circ} = \cos\frac{135^{\circ}}{2} = \sqrt{\frac{\cos 135^{\circ}+1}{2}}$

$$= \sqrt{\frac{\frac{-\sqrt{2}}{2}+1}{2}} \approx .375$$

$[\cos 67\frac{1}{2}^{\circ} = \sin(90^{\circ}-67\frac{1}{2}^{\circ})]$

3. $\sin 2x = 2\sin x \cos x = 2(\frac{3}{5})(\frac{4}{5}) = \frac{24}{25}$

4. $\cos 2x = \cos^2 x-\sin^2 x = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

5. $\cos\frac{1}{2}x = \sqrt{\frac{.8+1}{2}} \approx .95$ 6. $\sin\frac{1}{2}x = \sqrt{\frac{1-.8}{2}} \approx .32$

7. $\sin\frac{x}{y} = \sqrt{\frac{1-\cos\frac{x}{2}}{2}} \approx .16$ 8. $\sin 2y = 2(\frac{12}{13})(\frac{-5}{13}) = -\frac{120}{169}$

9. $\sin\frac{5\pi}{12} = \sin\frac{1}{2}(\frac{5\pi}{6}) = \sqrt{\frac{1-\cos\frac{5\pi}{6}}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \approx .965$

10. $\cos\frac{7\pi}{12} = \cos\frac{1}{2}(\frac{7\pi}{6}) = \sqrt{\frac{\frac{-\sqrt{3}}{2}+1}{2}} \approx .26$

11. If $2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sqrt{1-\cos^2\theta}$

Then $\sin\frac{\theta}{2}[2\sqrt{\frac{\cos\theta+1}{2}}] = \sqrt{1-\cos^2\theta}$

$$\sin\frac{\theta}{2} = \sqrt{\frac{2}{4}\cdot\frac{(1-\cos\theta)(1+\cos\theta)}{1+\cos\theta}} = \sqrt{\frac{1-\cos\theta}{2}}$$

12. $\cos^2 x = \cos^2 x-\sin^2 x = \cos^2 x-(1-\cos^2 x) = 2\cos^2 x-1$

13. $\frac{\sin 2x}{1-\cos 2x} = \frac{2\sin x\cos x}{1-\cos^2 x+\sin^2 x} = \frac{2\sin x\cos x}{2\sin^2 x} = \frac{\cos x}{\sin x}$

14. $\cos(\frac{\pi}{6}+y)-\cos(\frac{\pi}{6}-y) = [\frac{\sqrt{3}}{2}\cdot\cos y-\frac{1}{2}\sin y]-[\frac{\sqrt{3}}{2}\cos y+\frac{1}{2}\sin y]$

$= -\sin y$

$\cos(\frac{\pi}{2}+y) = \cos\frac{\pi}{2}\cos y - \sin\frac{\pi}{2}\sin y = -\sin y$

15. $\sin 6x = \sin 2(3x) = 2(\sin 3x)(\cos 3x)$

16. Many possibilities:

```
05 PRINT "P"; "SIN P"; "FØRMULA"; "CØS P"; "FØRMULA"
10 LET P = 3.1416
20 FØR N = 1 TØ 6
30 LET S = ((1-CØS(P))/2)↑.5
40 LET C = ((1+CØS(P))/2)↑.5
50 PRINT P/2; SIN(P/2); S; CØS(P/2); C
60 LET P = P/2
70 NEXT N
80 END
```

Section 5.7 (pp.277-278)

	a + bi	polar	trigonometric
1.	$-\frac{5}{2}+\frac{5\sqrt{3}}{2}i$	$[5, \frac{2\pi}{3}]$	$5\,(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$
2.	$-3i$	$[3, \frac{3\pi}{2}]$	$3\,(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$
3.	$3 + 4i$	$[5, 53^\circ]$	$5\,(\cos 53^\circ + i\sin 53^\circ)$
4.	$-2 + 2\sqrt{3}i$	$[4, \frac{2\pi}{3}]$	$4\,(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$
5.	$5.5 - 2.4i$	$[6, -23^\circ]$	$6\,(\cos -23^\circ + i\sin - 23^\circ)$
6.	$-.52 - 1.9i$	$[2, 255^\circ]$	$2\,(\cos 255^\circ + i\sin 255^\circ)$
7.	$-4 - 4i$	$[4\sqrt{2}, \frac{5\pi}{4}]$	$4\sqrt{2}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$

8. (a) $(6i)(1+i) = 6i-6$; $[6,\frac{\pi}{2}]\cdot[\sqrt{2},\frac{\pi}{4}] = [6\sqrt{2},\frac{3\pi}{4}]$

$= -6+6i$ $\qquad$ $= -6+6i$

(b) $(-3i)^2 = -9$; $[3,\frac{3\pi}{2}]^2 = [9,3\pi] = -9$

(c) $(\frac{3\sqrt{3}}{2}+\frac{3}{2}i)(-1+\sqrt{3}i) = \frac{-3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} - \frac{3i}{2} + \frac{9}{2}i$

$= -3\sqrt{3} + 3i$

$[3,\frac{\pi}{6}][2,\frac{2\pi}{3}] = [6,\frac{5\pi}{6}] = -3\sqrt{3} + 3i$

(d) $(4-4i)(3+3i) = (12+12)-12i+12i = 24$

$[4\sqrt{2},-\frac{\pi}{4}]\cdot[3\sqrt{2},\frac{\pi}{4}] = [24,0]$

9. $z^{-1} = [\frac{1}{r}, -\theta]$

10. $\frac{1}{4+4i} = \frac{1(4-4i)}{(4+4i)(4-4i)} = \frac{4-4i}{32} = \frac{1}{8} - \frac{i}{8} = [\frac{1}{4\sqrt{2}}, -\frac{\pi}{4}]$

$[4\sqrt{2}, \frac{\pi}{4}]^{-1} = [\frac{1}{4\sqrt{2}}, -\frac{\pi}{4}] = \frac{1}{8} - \frac{i}{8}$

11. $\frac{1}{2i} = -\frac{i}{2}$; $[2, \frac{\pi}{2}]^{-1} = [\frac{1}{2}, -\frac{\pi}{2}] = \frac{1}{2}\cdot(\cos{}^{-}\frac{\pi}{2} + i \sin{}^{-}\frac{\pi}{2})$

$= \frac{1}{2}(-i) = \frac{-i}{2}$

12. $\frac{1}{-3i} = \frac{i}{3}$; $[3, \frac{3\pi}{2}]^{-1} = [\frac{1}{3}, \frac{\pi}{2}]$

13. $\frac{1}{1+i} = \frac{1}{2} - \frac{i}{2}$; $[\sqrt{2}, \frac{\pi}{4}]^{-1} = [\frac{\sqrt{2}}{2}, \frac{-\pi}{4}] = \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = \frac{1}{2} - \frac{i}{2}$

14. $\overline{Z} = [r, -\theta]$ 15. $Z\cdot\overline{Z} = [r, \theta]\cdot[r, -\theta] = [r^2, 0]$

16. $Z_0\cdot Z = [3, \frac{\pi}{2}]\cdot[r, \theta] = [3r, \theta + \frac{\pi}{2}]$

$\therefore$ f is a stretch with factor 3 and rotation of $\frac{\pi}{2}(90^\circ)$ all about the origin.

17. stretch rotation $\frac{1}{2}$, $\frac{4\pi}{3}$ (240°)

18. stretch rotation r, θ

19. $|r(\cos\theta + i \sin\theta)| = |r\cos\theta + i\ r \sin\theta|$

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$
$$= \sqrt{r^2}\sqrt{\cos^2\theta + \sin^2\theta}$$
$$= \sqrt{r^2}\cdot 1$$
$$= |r|$$

20. $[r_1(\cos\theta_1 + i \sin\theta_1)][r_2(\cos\theta_2 + i \sin\theta_2)]$

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)]$

$= r_1r_2(\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2))$

21. translation 22. do stretch rotation and then translation.

Section 5.8 (pp.280-281)

1. $Z = 1 + \sqrt{3}i = [2, \frac{\pi}{3}]$

Thus $Z^2 = [4, \frac{2\pi}{3}] = 2 + 2\sqrt{3}i$

$Z^3 = [8, \pi] = -8$

$Z^{-1} = [\frac{1}{2}, \frac{-\pi}{3}] = \frac{1}{4} - \frac{\sqrt{3}}{4}i$

$Z^{-2} = [\frac{1}{4}, \frac{-2\pi}{3}] = -\frac{1}{8} - \frac{\sqrt{3}}{8}i$

2. $-6-6i = [6\sqrt{2}, \frac{5\pi}{4}]$

Thus $Z^2 = [72, \frac{\pi}{2}] = 72i$

$Z^3 = [432\sqrt{2}, \frac{7\pi}{4}] = 432 - 432i$

$Z^{-1} = [\frac{\sqrt{2}}{12}, \frac{-5\pi}{4}] = \frac{-1}{12} + \frac{i}{12}$ $Z^{-2} = [\frac{1}{72}, \frac{-\pi}{2}] = \frac{-i}{72}$

3. $-\frac{1}{2} = [\frac{1}{2},\pi]$ Thus $Z^2 = [\frac{1}{4},0] = \frac{1}{4}$ $Z^3 = [\frac{1}{8},\pi] = -\frac{1}{8}$

$Z^{-1} = [2,\pi] = -2$ $Z^{-2} = [4,0] = 4$

4. $\frac{1-i\sqrt{3}}{2} = [1,-\frac{\pi}{3}]$ Thus $Z^2 = [1,-\frac{2\pi}{3}] = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$Z^3 = [1,-\pi] = -1$ $Z^{-1} = [1,\frac{\pi}{3}] = \frac{1}{2}+\frac{\sqrt{3}}{2}i$

$Z^{-2} = [1,\frac{2\pi}{3}] = -\frac{1}{2}+\frac{\pi 3}{2}i$

5. $-2i = [2,\frac{3\pi}{2}]$ Thus $Z^2 = [4,\pi] = -4$ $Z^3 = [8,\frac{\pi}{2}] = 8i$

$Z^{-1} = [\frac{1}{2},\frac{\pi}{2}] = \frac{i}{2}$ $Z^{-2} = [\frac{1}{4},\pi] = -\frac{1}{4}$

6. $\{2i,-2i\} = \{[2,\frac{\pi}{2}],[2,\frac{3\pi}{2}]\}$

7. $\{-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i,\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\} = \{[1,\frac{3\pi}{4}],[1,\frac{7\pi}{4}]\}$

8. $\{-i,-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i,\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i,-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\}=\{[1,\frac{\pi}{4}],[1,\frac{3\pi}{4}],[1,\frac{5\pi}{4}],[1,\frac{7\pi}{4}]\}$

9. $\{2,-1+\sqrt{3}i,-1-\sqrt{3}i\} = \{[2,0],[2,\frac{\pi}{3}],[2,\frac{2\pi}{3}]\}$

10. $\{\sqrt{2}+\sqrt{6}i,-\sqrt{2}-\sqrt{6}i\} = \{[2\sqrt{2},\frac{\pi}{3}],[2\sqrt{2},\frac{4\pi}{3}]\}$

11. $\{1,-\frac{1}{2}+\frac{\sqrt{3}}{2}i,-\frac{1}{2}-\frac{\sqrt{3}}{2}i\} = \{[1,0],[1,\frac{2\pi}{3}],[1,\frac{4\pi}{3}]\}$

12. $\{[+1,\pi],[+1,\frac{\pi}{5}],[1,\frac{3\pi}{5}],[1,\frac{7\pi}{5}],[1,\frac{9\pi}{5}]\}$

13. $\{[\sqrt[3]{2},0],[\sqrt[3]{2},\frac{2\pi}{3}],[\sqrt[3]{2},\frac{4\pi}{3}]\}$

14. $\{0\} \cup$ solution set to No. 8

15. $\{0,-\frac{1}{2},-\frac{1}{4}+\frac{\sqrt{3}}{4}i,\frac{1}{4}-\frac{\sqrt{3}}{4}i\} = \{0,[\frac{1}{2},\frac{\pi}{3}],[\frac{1}{2},\pi],[\frac{1}{2},\frac{5\pi}{3}]\}$

16. $\{[1,\frac{2k\pi}{n}]:k=0,1,2,\ldots,n-1\}$

17. A commutative group isomorphic to $(Z_4,+)$

18. A commutative group isomorphic to $(Z_3,+)$

19. A commutative group isomorphic to $(Z_n,+)$ because absolute values are all 1 and under $\cdot$, angles add (mod 2π).

20. You generate an infinite set because the sets of roots are not closed under +. This is reasonable because looking for roots to $Z^n=1$ is essentially a multiplication problem.

Section 5.9 (pp.285-288)

1. $I = 1.5$ amps 2. $R = 15$ ohms 3. $E = 240$ volts

4. $E = 185$ volts 5. $R_1 = 15$ ohms

6. $I_1 = .8$ amps, $R_2 = 4$ ohms, $I = 3.8$ amps

7. $I_1 = 3$ amps, $R_2 = 1.5$, $R = .6$

8. $E = IR_1 + IR_2$

9. $E = IR \qquad I = \frac{E}{R} \qquad \frac{E}{R} = \frac{E}{R_1} + \frac{E}{R_2}$

$E = I_1R_1$ so $I_1 = \frac{E}{R_1}$ so $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$E = I_2R_2 \qquad I_2 = \frac{E}{R_2} \qquad \boxed{R = \frac{R_1R_2}{R_1+R_2}}$

10. $E_1 = 200$ volts, $R_2 = 1.2$ ohms, $E = 212$ volts

14. $\sin 120\pi x$ has period $\frac{1}{60}$ and $\sin x$ has period 2π. Otherwise the functions trace the same pattern of values.

16. rms $= \frac{70}{\sqrt{2}} \approx 50$ 17. $154 \sin 120\pi t$

18. $308 \sin 120\pi t$ 19. $\frac{120}{5} = 24$ ohms 20. 3 amps

21. 5 amps 22. $\frac{1}{3}$amps, $\frac{1}{2}$amps, $\frac{5}{6}$amps

23. 360 ohms, 240 ohms, 144 ohms

Section 5.10 (pp.291-292)

Function	Max $x =$	Min $x =$	Zero $x =$
1. $\sqrt{2}\sin(x + \frac{\pi}{4})$	$(8n+1)\frac{\pi}{4}$	$(8n+5)\frac{\pi}{4}$	$(4n+3)\frac{\pi}{4}$
2. $4 \sin(x + \frac{\pi}{3})$	$(12n+1)\frac{\pi}{6}$	$(12n+7)\frac{\pi}{6}$	$(3n+2)\frac{\pi}{3}$
3. $5 \sin(x + .65)$	$.92+2n\pi$	$4.06+2n\pi$	$2.49+n\pi$
4. $13 \sin(x+ 1.17)$	$.40+2n\pi$	$3.54+2n\pi$	$1.97+n\pi$
5. $13 \sin(x + 1.97)$	$2n\pi-.4$	$2.74+2n\pi$	$1.17+n\pi$
6. $25 \sin(x - .65)$	$2.22+2n\pi$	$-.92+2n\pi$	$.65+n\pi$
7. $8 \sin x$	$(4n+1)(\frac{\pi}{2})$	$(4n+3)(\frac{\pi}{2})$	$n\pi$
8. $\sqrt{2}\sin(mx + \frac{\pi}{4})$	$(\frac{8n+1}{m})(\frac{\pi}{4})$	$(\frac{8n+5}{m})(\frac{\pi}{4})$	$(\frac{4n+3}{m})(\frac{\pi}{4})$
9. $2 \sin(120\pi x + \frac{\pi}{3})$	$\frac{12n+1}{720}$	$\frac{12n+7}{720}$	$\frac{3n+2}{360}$
10. $27.7 \sin(120\pi x + .44)$	$\frac{1.13+2n\pi}{120\pi}$	$\frac{4.27+2n\pi}{120\pi}$	$\frac{2.70+n\pi}{120\pi}$

11. 15 volts 12. $\sqrt{96.5}$ volts ≈ 9.8 volts

13. $\sqrt{117}$ volts ≈ 10.8 volts 14. $\sqrt{585}$ volts ≈ 24.2 volts

15. 8 volts

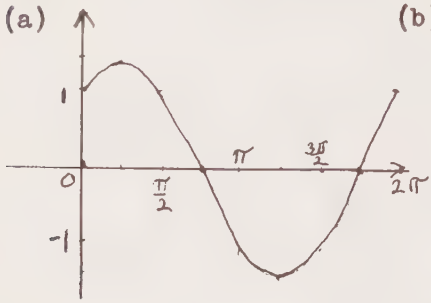
16. The function g is the same graph as f translated 2 units to the left; n is the same graph translated 1 unit to the right.

17. $\sin(120\pi x+\theta) = \sin(120\pi x + \frac{120\pi}{120\pi}\theta)$

$$= \sin 120\pi[x + \frac{\theta}{120\pi}]$$

18. Slide the graph of $\sin 120\pi x$ left (+) or right (-) a distance $\frac{\theta}{120\pi}$; the new function is $f(x \pm \frac{\theta}{120\pi})$ where $f(x) = \sin 120\pi x$.

19. (a)



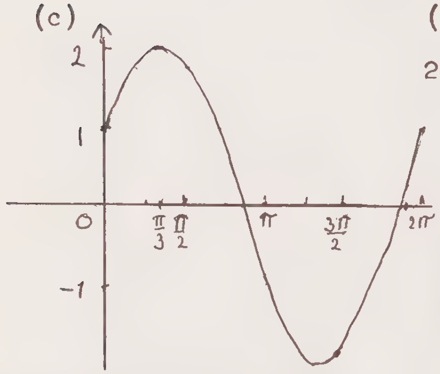
(b) $\sqrt{2}\left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right)$

is

$\sqrt{2}\cdot\frac{\sqrt{2}}{2}(\sin x+\cos x)$

Same as (a)

(c)



(d) same as (a)
same as (c)

20. Second and third columns should agree approximately.

One program that will work is:

```
10 PRINT "X";"3*SIN(X)+4*CØS(X)";"5*SIN(x+.64)"
20 FØR X = 0 TØ 6.28 STEP .2
30 PRINT X; 3*SIN(X)+4*CØS(X);5*SIN(X+.64)
40 NEXT X
50 END
```

Section 5.11 (pp.297-299)

1. 1 2. $\sqrt{3}$ 3. $-\frac{1}{\sqrt{3}}$ 4. ≈ 0 5. undefined 6. 0

7. $-\sqrt{3}$ 8. undefined 9. $\frac{1}{\sqrt{3}}$ 10. $\frac{1}{\sqrt{3}}$ 11. $\frac{2}{\sqrt{2}}$ 12. $\frac{2}{\sqrt{2}}$

13. 2 14. $\frac{2}{\sqrt{3}}$ 15. undefined 16. 1 17. -2

18. undefined 19. $\frac{3}{5}$ 20. $\frac{5}{13}$ 21. $y = x$ 22. $y = \sqrt{3}x$

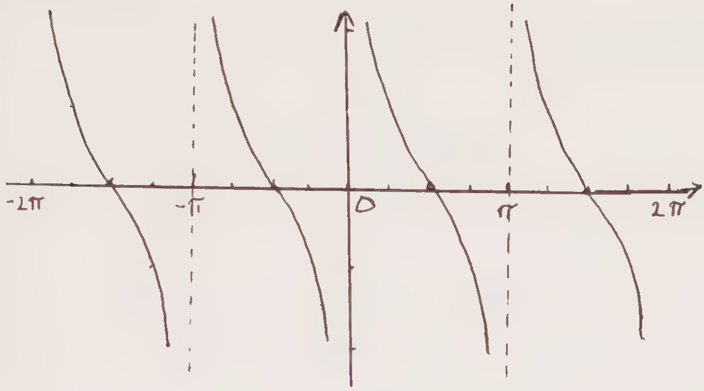
23. $y = 4.3x$ 24. $y = -.36x$ 25. $y = -\frac{\sqrt{3}}{3}x$

26. One program that will work:

```
10 PRINT "X"; "TAN(X)"
20 FØR X = -4.7 TØ 4.7 STEP .25
30 PRINT X; TAN (X)
40 NEXT X
50 END
```

Note: $-4.7 \approx -\frac{3\pi}{2}$ and $4.7 \approx \frac{3\pi}{2}$

27. The sketch will be as below:



28. Period is π

Proof: $\cot(x+\pi) = \dfrac{\cos(x+\pi)}{\sin(x+\pi)}$

$$= \frac{\cos x \cos\pi - \sin x \sin\pi}{\sin x \cos\pi + \sin\pi \cos x}$$

$$= \frac{-\cos x}{-\sin x}$$

$$= \cot(x)$$

To show π is minimum repeating unit, assume

$$\cot(x+p) = \cot(x)$$

Then $\dfrac{\cos x \cos p - \sin x \sin p}{\sin x \cos p + \sin p \cos x} = \dfrac{\cos x}{\sin x}$

or $\cos x \sin x \cos p - \sin^2 x \sin p$

$= \cos x \sin x \cos p + \cos^2 x \sin p$

or $0 = (\cos^2 x + \sin^2 x)\sin p$

or $0 = \sin p$

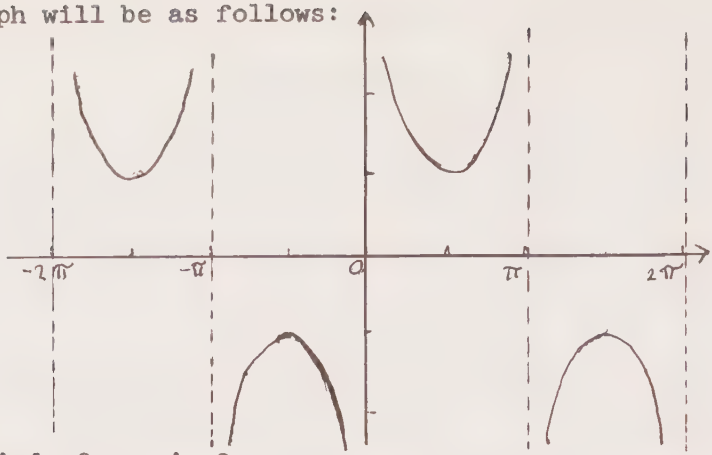
or $p = n \cdot \pi$

29. (a) $T_{n\pi}$ (b) $T_{n\pi}$

30. (a) symmetry about points $(n\pi, 0)$
translation $x \to x + n\pi$
(b) symmetry about points $((2n+1)\frac{\pi}{2}, 0)$
translation $x \to x + n\pi$

31. (a) odd (b) odd (c) even (d) odd

32. Graph will be as follows:



33. Period of csc is 2π

Proof: I. $\csc(x+2\pi) = \dfrac{1}{\sin(x+2\pi)} = \dfrac{1}{\sin x} = \csc x$

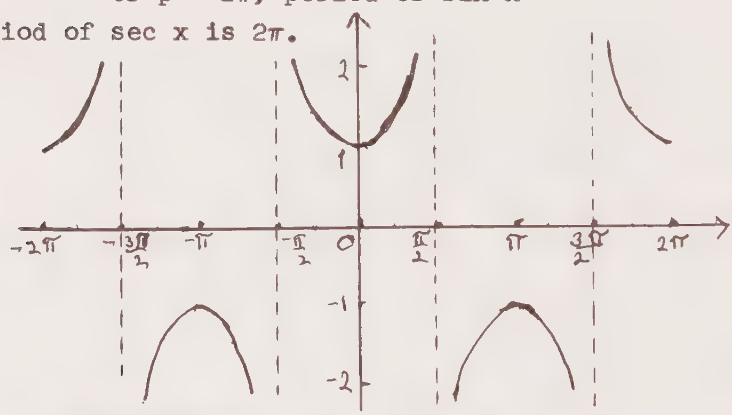
II. If $\csc(x+p) = \csc x$ for all x,

then $\dfrac{1}{\sin(x+p)} = \dfrac{1}{\sin x}$ for all x

or $\sin(x+p) = \sin x$ for all x

or $p = 2\pi$, period of sin x

34. Period of sec x is 2π.



35. $\cos\alpha = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$; $\tan\alpha = \dfrac{\text{opposite leg}}{\text{adjacent leg}}$;

$\cot\alpha = \dfrac{\text{adjacent leg}}{\text{opposite leg}}$; $\sec\alpha = \dfrac{\text{hypotenuse}}{\text{adjacent leg}}$;

$\csc\alpha = \dfrac{\text{hypotenuse}}{\text{opposite leg}}$

36. (a) $1 + \tan^2 x = 1 + \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{\cos^2 x+\sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

(b) $\cot(x+\pi) = \dfrac{\cos(x+\pi)}{\sin(x+\pi)} = \dfrac{\cos x\cos\pi-\sin x\sin\pi}{\cos x\sin\pi+\sin x\cos\pi} = \dfrac{-\cos x}{-\sin x}$

$= \cot x$

(c) $(\sin x)(\cos x)(\sec x)(\csc x) = (\sin x)(\cos x)\left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right)$

$$= 1$$

(d) $\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \cdot \sin x}$

$$= \frac{1}{\cos x \sin x} = (\sec x)(\csc x)$$

(e) $\tan 2t = \frac{\sin 2t}{\cos 2t} = \frac{2 \sin t \cos t}{\cos^2 t - \sin^2 t} = \frac{2 \sin t}{\cos t}\left[\frac{\cos^2 t}{\cos^2 t - \sin^2 t}\right]$

$$= 2 \tan t\left[\frac{1}{1 - \tan^2 t}\right]$$

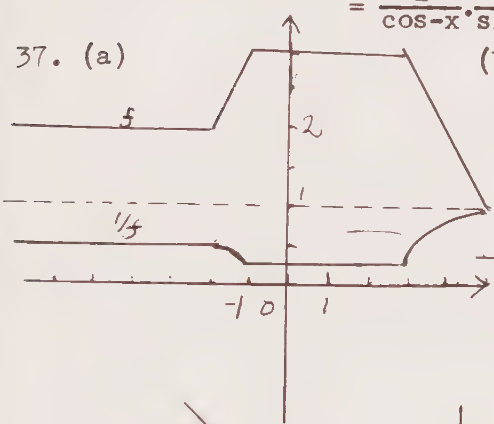
(f) $\csc(x + \frac{\pi}{2}) = \frac{1}{\sin(x + \frac{\pi}{2})} = \frac{1}{\sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2}}$

$$= \frac{1}{\cos x} = \sec x$$

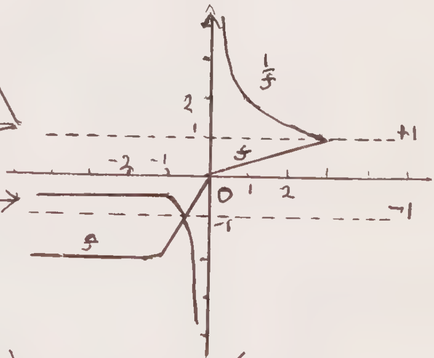
(g) $(\sec x)(\csc x) = \frac{1}{\cos x} \cdot \frac{1}{\sin x} = \frac{1}{\cos -x} \cdot \frac{-1}{-\sin x}$

$$= \frac{1}{\cos -x} \cdot \frac{-1}{\sin -x} = -1(\sec -x)(\csc -x)$$

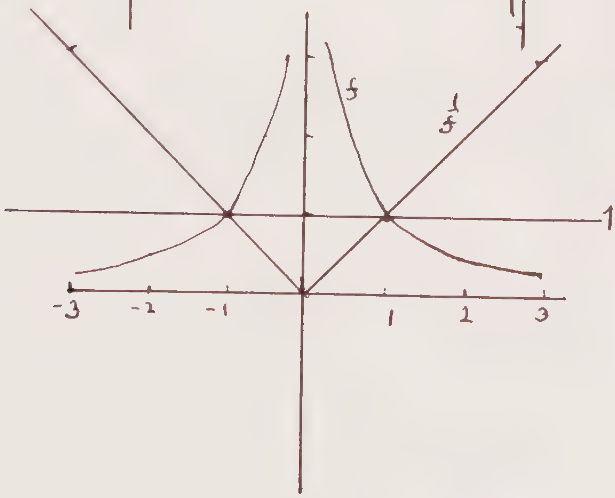
37. (a)



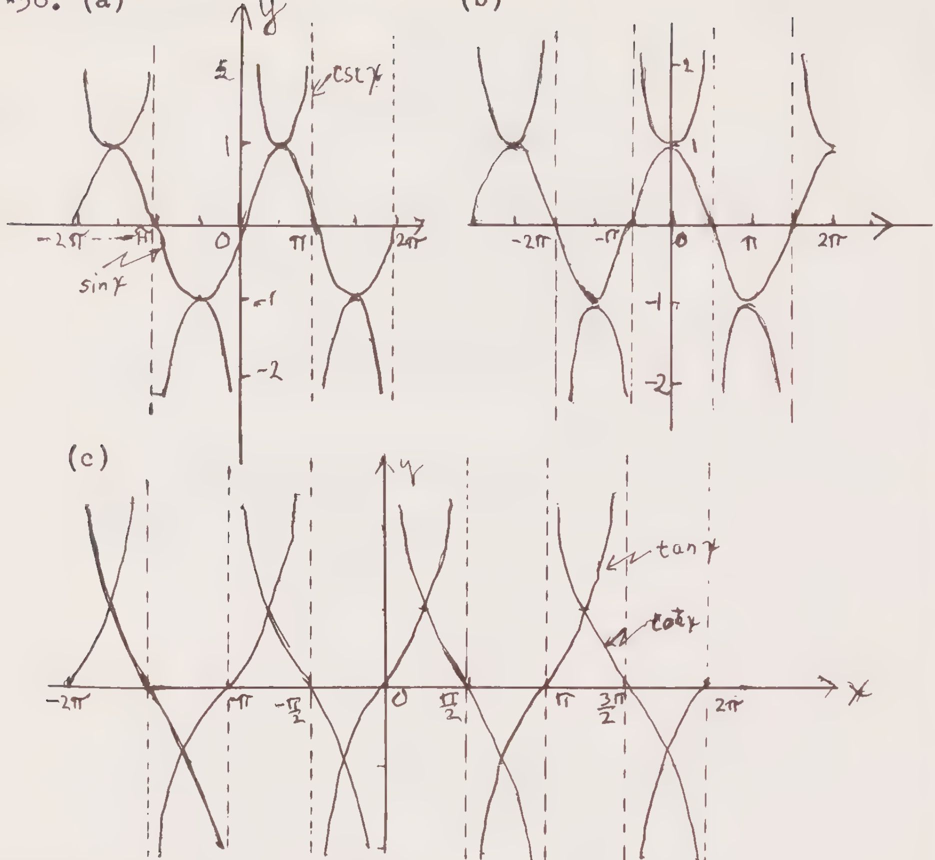
(b)



(c)



*38.



Section 5.12 (pp.303-305)

1. $\{x : x = \frac{\pi}{2} + 2n\pi\}$ 2. $\{x : x = \frac{\pi}{2} + 2n\pi\}$

3. $\{x : x = \frac{3\pi}{4} + n\pi\}$ 4. $\{x : x = n\pi\}$

5. $\{x : x = .435 + 2n\pi\} \cup \{x : x = 2.71 + 2n\pi\}$

6. $\{x : x = \pm\frac{\pi}{3} + 2n\pi\}$ 7. $\{t : t = \frac{12n\pm1}{720}\} = \{t : t = \pm\frac{1}{720} + \frac{n}{60}\}$

8. $\{t : t = \pm\frac{4\pi}{3} + 8n\pi\}$ 9. $\emptyset$

10. $\{x : x = .33 + 2n\pi\} \cup \{x : x = 2.81 + 2n\pi\}$ 11. $\emptyset$

12. $\{x : x = \frac{\pi}{3} + n\pi\}$ 13. $\{x : x = (2n+1)\pi\}$

14. $\{x : x = (2n+1)\frac{\pi}{2}\} \cup \{x : x = \frac{2\pi}{3} + 2n\pi\} \cup \{x : x = \frac{4\pi}{3} + 2n\pi\}$

15. $\{x : x = n\pi\} \cup \{x : x = \frac{\pi}{4} + n\pi\}$

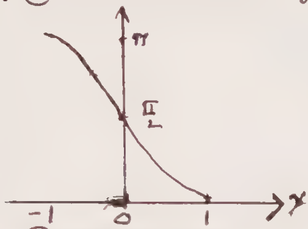
16. $\{x : x = \frac{\pi}{6} + 2n\pi\} \cup \{x : x = \frac{5\pi}{6} + 2n\pi\}$

17. $\{x : x = (2n+1)\frac{\pi}{2}\} \cup \{x : x = \frac{\pi}{4} + n\pi\}$

18. $\{x:x = 2n\pi\}\cup\{x:x = \frac{\pi}{2} + 2n\pi\}$
19. $\{x:x = n\pi\}\cup\{x:x = \pm\frac{\pi}{3} + 2n\pi\}$
20. $\{x:x = 1.37 + n\pi\}$

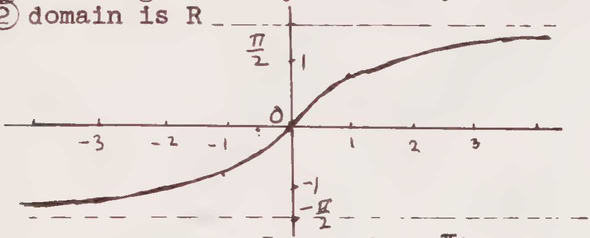
21. ① arc-cosine x = y iff cos y = x and $0 \le y \le \pi$

② domain is $[-1,1]$



arc-cosine

22. ① arc-tangent x = y iff tan y = x and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
② domain is R



23. $\sin x + \cos x = \sqrt{2}\ \sin\ (x + \frac{\pi}{4})$

$\therefore \sin x + \cos x = 1 \iff \sin(x + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$\iff x = 0, \frac{\pi}{2}$

Program for search would be:

```
05 LET R = 2↑.5
10 FØR X = -3.14 TØ 3.14 STEP .1
20 IF ABS(R*SIN(X+.785)-1) > .05 THEN 40
30 PRINT X, R*SIN(x+.785)
40 NEXT X
45 PRINT "DØNE"
50 END
```

Make more flexible with INPUT values for interval of search, step, and criterion for accuracy. Then one can zero in on a solution region that appears likely.

24. Change 20 and 30 as follows:

```
20 IF ABS(TAN(X)-5) > .05 THEN 40
30 PRINT X, TAN(X)
```

Also, delete 05

25. (a) $\approx .646$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{4}$ (e) $\frac{-\pi}{4}$

(f) $\frac{\pi}{3}$ (g) ≈ 1.177

26. (a) $t = 0, 4\pi, 8\pi, \ldots$
(b) $t = 2\pi, 6\pi, 10\pi, \ldots$
(c) $t = \pi, 3\pi, 5\pi, \ldots$
(d) middle

(c) A suitable program:

```
10 PRINT "TIME"; "DEPTH"
20 FØR X = 0 TØ 12.56
30 PRINT X; 6*CØS(.5*X)+15
40 NEXT X
50 END
```

27. (a) 106 volts (b) 75 volts; 127 volts

(c) max $t = \frac{1}{720} + \frac{60}{n}$; min $t = \frac{7}{720} + \frac{n}{60}$;

zero $t = \frac{1}{180} + \frac{n}{120}$

(d) $E(t) = 75$ when $t = \frac{1}{720} + \frac{n}{60}$ $t = \frac{1}{144} + \frac{n}{60}$

$E(t) = -75$ when $t = -\frac{1}{720} + \frac{n}{60} = -\frac{1}{144} + \frac{n}{60}$

28. (a) max, $t \approx 91$ days; min, $t \approx 273$ days; even, $t \approx 182$ and 365 days

(b) 14 hours at 42 days and 140 days
10 hours at 224 days and 323 days

(c) sunlight depends on latitude location on earth.

Review Exercises (pp.306-307)

1. (a) $\frac{2\pi}{3}, \frac{4\pi}{3}$ (b) $\frac{\pi}{3}, \frac{2\pi}{3}$ (c) $\frac{\pi}{3}, \frac{5\pi}{3}$ (d) $\frac{\pi}{8}, \frac{7\pi}{8}$

2. $\sin\frac{1}{2}(\theta + 4\pi) = \sin(\frac{1}{2}\theta + 2\pi) = \sin\frac{1}{2}\theta$

Any smaller period would imply contradiction as follows:

1 Suppose $\sin\frac{1}{2}(\theta+p) = \sin\frac{1}{2}\theta$ for all θ

2 Then in particular, for $\theta=0$, $\sin\frac{p}{2} = 0$ or $p = 2k\pi$

3 Can k be odd? No

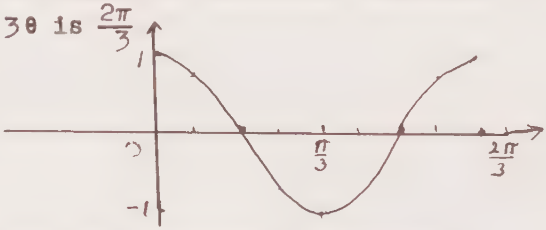
$\sin\frac{1}{2}(\theta+p) = \sin\frac{1}{2}\theta \Rightarrow \sin\frac{\theta}{2}\cos k\pi + \sin k\pi \cos\frac{\theta}{2} = \sin\frac{\theta}{2}$

If k odd, then $-\sin\frac{\theta}{2} = \sin\frac{\theta}{2}$ or $\sin\frac{\theta}{2} = 0$ for all θ

$\therefore$ k even and $p = 2\cdot(2\cdot k')\pi = 4k'\pi$

Informally, as θ ranges from 0 to 2π, $\frac{1}{2}\theta$ has only covered half the domain of the sine function.

3. Period of $\cos 3\theta$ is $\frac{2\pi}{3}$



4. $\cos 3x = \cos(2x+x) = \cos 2x\cos x - \sin 2x\sin x$
$= (\cos^2 x-\sin^2 x)\cos x - 2\sin^2 x\cos x$
$= \cos^3 x-3\cos x+3\cos^3 x = 4\cos^3 x-3\cos x$

5. $\sin 3x = \sin(2x+x) = \sin 2x\cos x + \sin x\cos 2x$
$= 2\sin x\cos^2 x + \sin x\cos^2 x - \sin^3 x$
$= -4\sin^3 x+3\sin x$

6. voltage 157 volts, frequency 80 cycles per second

7. (a) $[16, \frac{\pi}{4}]$ (b) $[18, \frac{11\pi}{6}]$

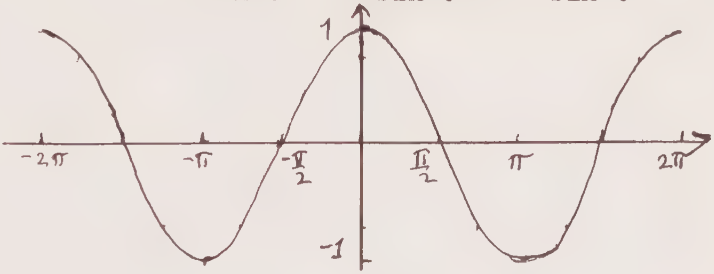
8. (a) $-5 + 5\sqrt{3}i$ (b) $8-8\sqrt{3}i$

9. Solutions in polar form: $[1,0]$, $[1,\frac{2\pi}{3}]$, $[1,\frac{4\pi}{3}]$

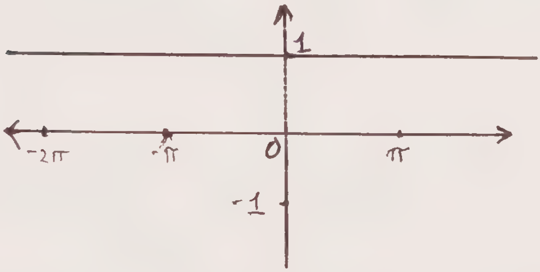
in rectangular form: 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

10. $1 + \cot^2\theta = 1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} = \csc^2\theta$

11. (a)



(b)



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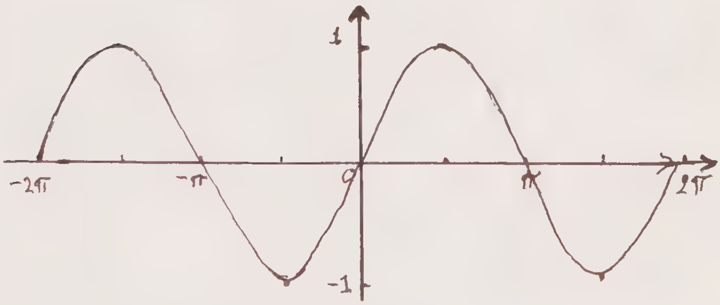
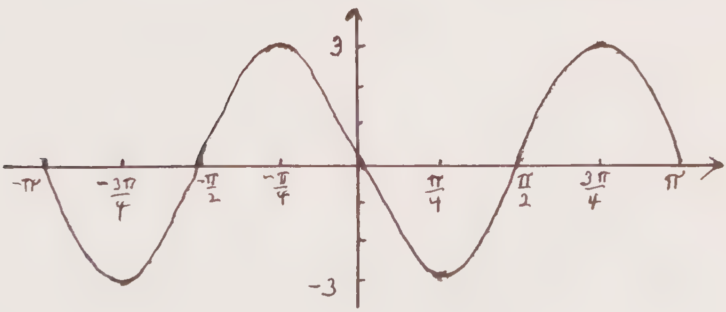
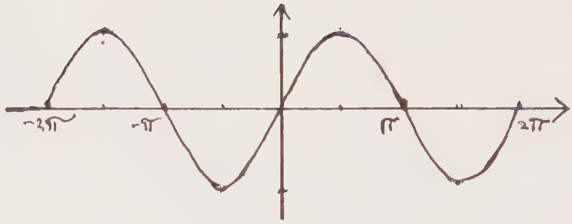


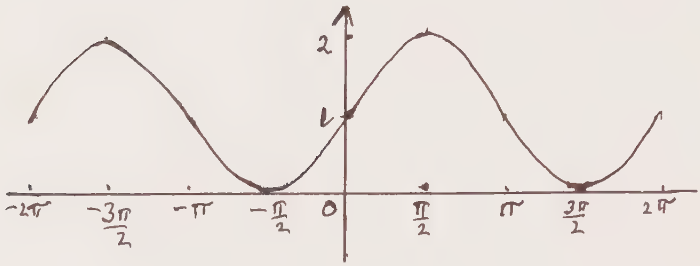
(d)



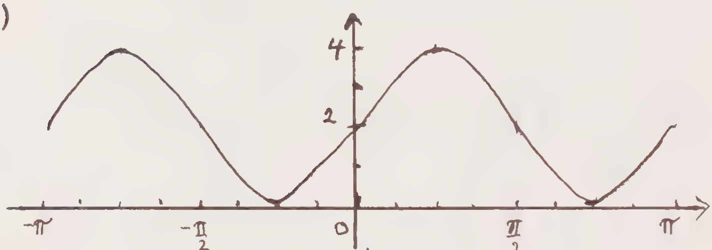
12. (a)



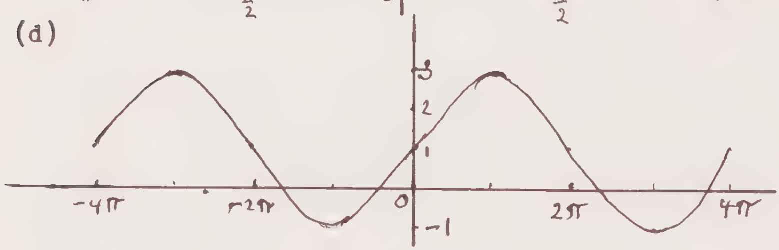
(b)



(c)

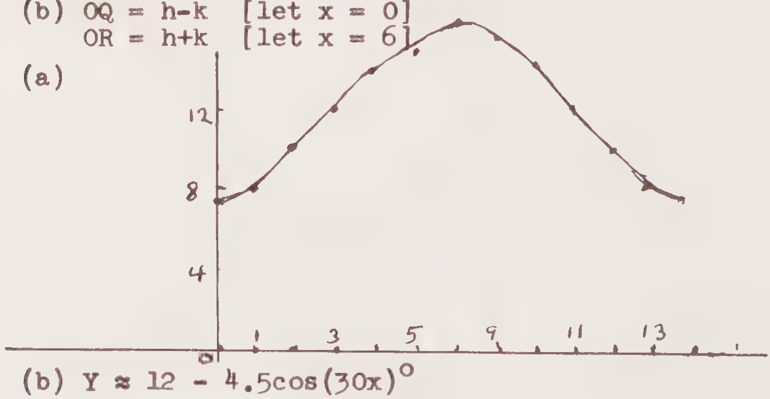


(d)



13. (a) OP = 12 since $12 \cdot 30 = 360^{\circ}$
(b) OQ = h-k [let x = 0]
OR = h+k [let x = 6]

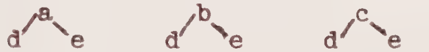
14. (a)



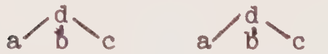
(b) $Y \approx 12 - 4.5\cos(30x)^{\circ}$

CHAPTER 6

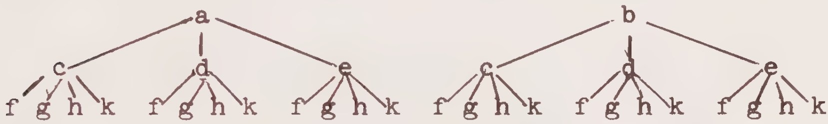
Section 6.2 (pp.316-319)

1. 

$\{(a,d),(a,e),(b,d),(b,e),(c,d),(c,e)\}$

2. 

$\{(d,a),(d,b),(d,c),(e,a),(e,b),(e,c)\}$

3. 

$\{(a,c,f),(a,c,g),(a,c,h),(a,c,k),(a,d,f),(a,d,g),$
$(a,d,h),(a,d,k),(a,e,f),(a,e,g),(a,e,h),(a,e,k),$
$(b,c,f),(b,c,g),(b,c,h),(b,c,k),(b,d,f),(b,d,g),$
$(b,d,h),(b,d,k),(b,e,f),(b,e,g),(b,e,h),(b,e,k)\}$

4. 

5. $6\cdot5\cdot4 = 120$ 6. $8\cdot7\cdot6\cdot5\cdot4 = 6720$ 7. $9\cdot8\cdot7\cdot6 = 3024$

8. $\frac{6\cdot5\cdot4}{3\cdot2\cdot1} = 20$ 9. $\frac{8\cdot7\cdot6\cdot5\cdot4}{5\cdot4\cdot3\cdot2\cdot1} = 56$

10. There are three elements to be assigned one-to-one to 5 elements. The first element can be assigned in any of 5 ways, then the second to any of the remaining unassigned elements, and finally the third element can be assigned in any of the three remaining ways. Thus $n = 5 \times 4 \times 3 = (5)_3 = 60$

11. $2\cdot4\cdot3\cdot2 = 48$

12. $(n)_r = n\cdot(n-1)\cdot\ldots\cdot(n-r+1) = n(n-1)(n-2)\ldots(n-r+1)\frac{(n-r)!}{(n-r)!}$
$= \frac{n!}{(n-r)!}$

13. $(10)_3 = 10\cdot9\cdot8 = 720;\ (6)_6 = 6\cdot5\cdot4\cdot3\cdot2\cdot1 = 720$

14. Use formula in Example 22, with $r = n$.
$(n)_n = \frac{n!}{0!}$ but $(n)_n = n! \therefore 0! = 1$

15. $(n)_{n-1} = n(n-1)\ldots2 = (n(n-1)\ldots2)\cdot1 = n!$

16. $\binom{10}{5} = \frac{10\cdot9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4\cdot5} = 252$ 17. $\frac{52\cdot51\cdot50\cdot49\cdot48}{1\cdot2\cdot3\cdot4\cdot5} = 2{,}598{,}960$

18. $\binom{8}{0} = \frac{8!}{0!8!} = \frac{1}{1} = 1$ 19. $\binom{11}{11} = 1$ 20. $\frac{25\cdot24\cdot23}{1\cdot2\cdot3} = 2300$

21. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ formula IIIB. Let $r = n-r$ and substitute for r. $\binom{n}{n-r} = \frac{n!}{(n-r)!(r)!} = \binom{n}{r}$

22. $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0!(n!)} = \frac{n!}{1\cdot n!} = 1$

23. $\binom{8}{2} = \frac{8\cdot 7}{1\cdot 2} = 28$

24. (a) $\binom{9}{2}$ = number of segments but 9 of these are sides.
Hence number of diagonals $= \binom{9}{2} - 9$

(b) $\binom{n}{2} - n = \frac{n(n-1)}{2} - n = \frac{n(n-1)-2n}{2} = \frac{n(n-3)}{2}$

25. (a) $\binom{5}{2} + \binom{5}{3} = \frac{5\cdot 4}{1\cdot 2} + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} = \frac{2(5\cdot 4\cdot 3)}{1\cdot 2\cdot 3} = \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3} = \binom{6}{3}$

(b) $\binom{n}{r} + \binom{n}{r+1} = \frac{n\cdot(n-1)\cdot\ldots\cdot(n-r+1)}{r!} + \frac{n\cdot(n-1)\cdot\ldots\cdot(n-r)}{(r+1)!}$

$= \frac{n(n-1)(n-2)\ldots(n-r+1)(r+1)+n(n-1)\ldots(n-r)}{(r+1)!}$

$= \frac{[n(n-1)\ldots(n-r+1)](r+1+n-r)}{(r+1)!}$

$= \frac{n(n-1)\ldots(n-r+1)(n+1)}{(r+1)!} = \binom{n+1}{r+1}$

26. (a) $\binom{n}{r} = \binom{n}{n-r} \therefore \binom{n}{12} = \binom{n}{n-12} = \binom{n}{10}$ $\quad n-12 = 10$ or $n = 22$

27. $\binom{20}{r} = \binom{20}{r+4} \therefore r+r+4 = 20,\ 2r = 16,\ r = 8$

28. (a) $S = \{\text{top up, top down, side}\} = \{U,D,L\}$
(b) $\mathcal{P}(S) = \{\emptyset,\{U\},\{D\},\{L\},\{U,D\},\{U,L\},\{D,L\},S\}$
(d)

$\emptyset$	$\{U\}$	$\{D\}$	$\{L\}$	$\{U,D\}$	$\{U,L\}$	$\{D,L\}$	S
0	0.35	0.20	0.40	0.55	0.80	0.65	1

(c) $P(U) = 0.35,\ P(D) = 0.20,\ P(L) = 0.45$

29. (a) See Figure 6.5 (b) 2^{36}

(c)	{0,	1,	2,	3,	4,	5}
(d)	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

30. (a) $\binom{9}{3} = 84$ (b) $\binom{5}{3} = 10$ (c) $P(3B) = \frac{10}{84}$ or $\frac{5}{42}$

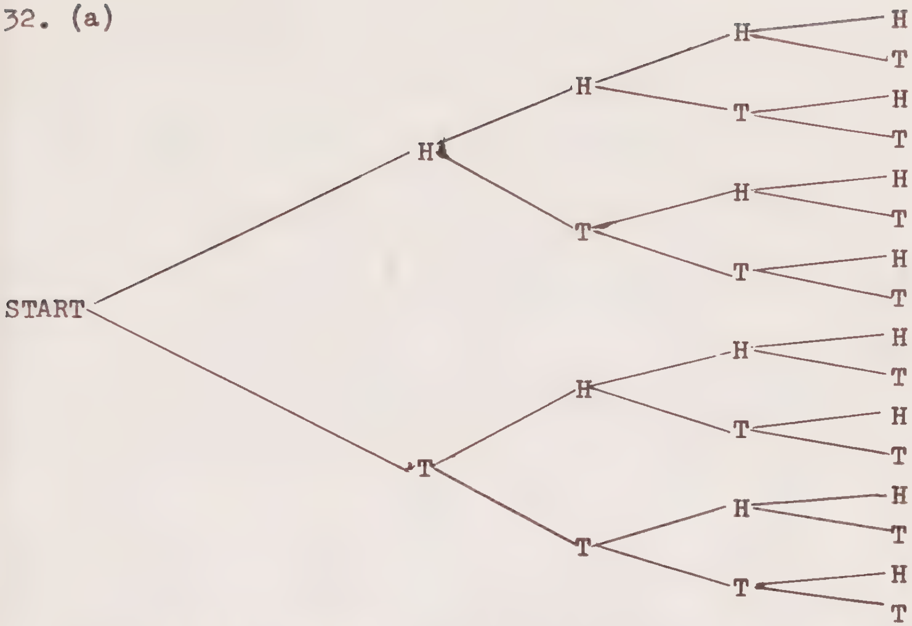
(d) $\binom{4}{3} = 4$ $\quad P(3G) = \frac{4}{84}$ or $\frac{1}{21}$

(e) $P(B \text{ and } G) = 1 - (\frac{5}{42} + \frac{2}{42}) = 1 - \frac{1}{6} = \frac{5}{6}$

(f) There will be 84 elements--10 with 3 boys; 40 with 2 boys, 1 girl, $\binom{5}{2}\cdot\binom{4}{1}$; 30 with 1 boy, 2 girls, $\binom{5}{1}\cdot\binom{4}{2}$; and 4 with 3 girls. Use the elements $\{B_1,B_2,B_3,B_4,B_5,G_1,G_2,G_3\}$ to write out all elements. Each element has a probability of $\frac{1}{84}$.

31. Four kings can be selected in only one way. For the other 48 cards, a hand of nine cards can be selected in $\binom{48}{9}$ ways. This is 1,677,106,640.

32. (a)



(b) {(H,H,H,H),(H,H,H,T),(H,H,T,H),(H,T,H,H),(T,H,H,H),
(H,H,T,T),(H,T,T,H),(H,T,H,T),(T,H,H,T),(T,H,T,H),
(T,T,H,H),(H,T,T,T),(T,H,T,T),(T,T,H,T),(T,T,T,H),
(T,T,T,T)}

(c) Each outcome has probability of $\frac{1}{16}$

(d) {(0H),(1H),(2H),(3H),(4H)}

$P = \frac{1}{16} \quad \frac{4}{16} \quad \frac{6}{16} \quad \frac{4}{16} \quad \frac{1}{16}$

(e) $P = \frac{6+4+1}{16} = \frac{11}{16}$

33. $P(A\cup\emptyset) = P(A)+P(\emptyset)$, $A\cup\emptyset = A \therefore P(A) = P(A)+P(\emptyset)$ or $P(\emptyset)=0$

34. $A\cap\overline{A} = \emptyset \quad A\cup\overline{A} = S \quad P(A\cup\overline{A}) = P(A) + P(\overline{A})$
but $A\cup\overline{A} = S, P(S) = 1 \therefore 1 \;,\; P(A)+P(\overline{A})$ or $P(A)=1-P(\overline{A})$

35. (a) $A\backslash B$ and B are disjoint
(b) Hence $P(A\backslash B) = P(A) - P(A\cap B)$
(c) But $A\cup B = (A\backslash B)\cup B$, and thus $P(A\cup B)=P(A\backslash B)+P(B)$
Substituting (b) in (c), we obtain
$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

Section 6.3 (pp.328-331)

1. (a) A = {(4,1),(4,2),(4,3),(4,4),(4,5),(1,4),(2,4),
(3,4),(5,4),(6,4)}
B = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),
(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),
(3,4),(4,1),(4,2),(4,3),(5,1),(5,2),(6,1)}
C = {(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),
(5,3),(5,5)}

$A \cap B = \{(1,4),(2,4),(3,4),(4,1),(4,2),(4,3)\}$
$A \cap C = \{\ \}$

(b) $n(A) = 11$; $n(B) = 21$; $n(C) = 9$; $n(A \cap B) = 6$; $n(A \cap C) = 0$

(c) $P(A) = \frac{11}{36}$; $P(B) = \frac{21}{36}$; $P(C) = \frac{9}{36}$; $P(A \cap B) = \frac{6}{36}$;

$P(A \cap C) = 0$

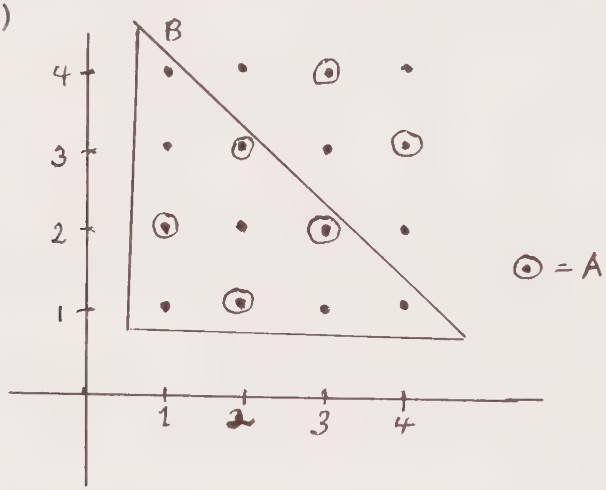
(d) $P(A|B) = \frac{6}{21}$; $P(B|A) = \frac{6}{11}$; $P(A|C) = 0$; $P(C|A) = 0$

2. (a) $P(X \cup Y) = \frac{1}{2}+\frac{1}{4}-\frac{1}{10}$ or $\frac{13}{20}$ (b) $P(X|Y) = \frac{2}{5}$

(c) $P(Y|X) = \frac{1}{5}$ 3. $P(3|\text{odd}) = \frac{1}{4}$

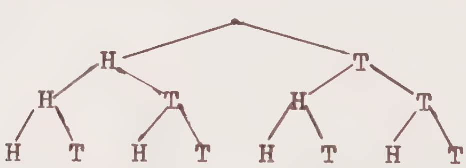
4. $P(A) = \frac{3}{8}$; $P(B|A) = \frac{2}{7}$; $P(A \cap B) = \frac{6}{56} = \frac{3}{28}$

5. (a)



(b) $P(B) = \frac{10}{16}$; $P(A \cap B) = \frac{4}{16}$; $P(A|B) = \frac{2}{5}$

6. (a)



$S = \{(H,H,H),(H,H,T),(H,T,H),(T,H,H),(H,T,T),$
$(T,H,T),(T,T,H),(T,T,T)\}$

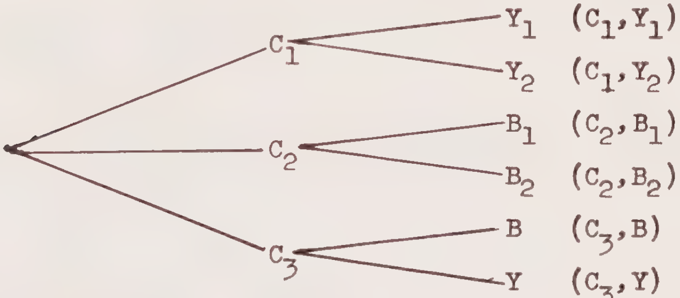
(b) $P(\geq 1 \text{ head}) = \frac{7}{8}$ (c) $P(2 \text{ heads}) = \frac{3}{8}$

7.

	Luminous	Not Luminous	
Blue	8	6	14
Yellow	4	2	6
	12	8	20

$P(\text{Luminous}|\text{Blue}) = \frac{8}{14}$ $P(\text{Yellow}|\text{Not Lunimous}) = \frac{2}{8}$

8.



A = other side blue = $\{(C_2,B_1),(C_2,B_2),(C_3,Y)\}$

B = shows blue = $\{(C_2,B_1),(C_2,B_2),(C_3B)\}$

C = other side yellow = $\{(C_1,B_1),(C_1,B_2),(C_3,B)\}$

$A\cap B = \{(C_2,B_1),(C_2,B_2)\}$

$C\cap B = \{(C_3,B)\}$

$P(A) = \frac{1}{2}$; $P(B) = \frac{1}{2}$; $P(C) = \frac{1}{2}$; $P(A\cap B) = \frac{1}{3}$; $P(C\cap B) = \frac{1}{6}$;

$\therefore P(A|B) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$ $P(C|B) = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$

$P(A|B)\cup P(C|B) = 1$

9. $P(A) = .92$; $P(B) = .80$; $P(C) = .60$
$P(C\cap B) = P(3C)$; $P(C\cap A) = P(3C)$
$P(C|B) = \frac{.25}{.80} = \frac{5}{16}$ $P(C|A) = \frac{.25}{.92}$

10. A = Event first ball is red $P(A) = \frac{5}{12}$
B = Event red on second ball $P(B|A) = \frac{4}{11}$
$P(A\cap B) = \frac{5}{12}\cdot\frac{4}{11} = \frac{5}{33}$
Similarly with red replaced by white
$P(A) = \frac{4}{12}$, $P(B|A) = \frac{3}{11} \therefore P(A\cap B) = \frac{1}{11}$
Similarly with red replaced by blue
$P(A) = \frac{3}{12}$, $P(B|A) = \frac{2}{11} \therefore P(A\cap B) = \frac{1}{22}$
$P(\text{Both of same color}) = \frac{10}{66} + \frac{6}{66} + \frac{3}{66} = \frac{19}{66}$
$P(\text{Different colors}) = 1 - \frac{19}{66} = \frac{47}{66}$

11. $P(A|(A\cup B)) = \frac{P(A\cap(A\cup B))}{P(A\cup B)}$; $A\cap(A\cup B) = A$,
$P(A\cup B) = P(A) + P(B)$; Substituting $P(A|A\cup B) = \frac{P(A)}{P(A)+P(B)}$
where $A\cap B = \emptyset$

12. Let A_1 be event first bulb is good; A_2 the event second bulb is good.

Then $P(A_1) = \frac{6}{9}$; $P(A_2|A_1) = \frac{5}{8}$. Thus

$P(A_1\cap A_2) = \frac{6}{9}\cdot\frac{5}{8} = \frac{30}{72}$ or $\frac{5}{12}$

13. $P(E) = \frac{60}{100} = .6$ (b) $P(U|E) = \frac{P(U\cap E)}{P(E)} = \frac{\frac{25}{100}}{\frac{60}{100}} = \frac{25}{60} = \frac{5}{12}$

14. Let G_1 = First object is good, G_2 = second object is good, D_1 = first object is bad, D_2 = Second object is bad.

$P(G_1) = \frac{2}{3}$ $P(G_2|G_1) = \frac{19}{29}$ $\therefore P(G_1\cap G_2) = \frac{2}{3}\cdot\frac{19}{29} = \frac{38}{87}$

$P(D_1) = \frac{1}{3}$ $P(D_2|D_1) = \frac{9}{29}$ $\therefore P(D_1\cap D_2) = \frac{9}{3}\cdot\frac{9}{29} = \frac{9}{87}$

$P(G_2|D_1) = \frac{20}{29}$ $\therefore P(G_2\cap D_1) = \frac{1}{3}\cdot\frac{20}{29} = \frac{20}{87}$

$P(D_2|G_1) = \frac{10}{29}$ $\therefore P(G_1\cap D_2) = \frac{2}{3}\cdot\frac{10}{29} = \frac{20}{87}$ (Check $\Sigma = \frac{87}{87} = 1$)

15. (a) $\binom{6}{3} = 20$ (b) $\binom{5}{2} = 10$

(c) $P(\overline{B}_3) = 1 - \frac{1}{2} = \frac{1}{2}$ $P(B_3) = \frac{1}{2}$

(d) $P(B_4|B_3) = \frac{P(B_4\cap B_3)}{P(B_3)} = \frac{\frac{4}{20}}{\frac{1}{2}} = \frac{8}{20}$ or $\frac{2}{5}$

16. $P(\text{Dif}) = \frac{30}{36}$ (Dif $\cap$ a three) = 11 $\therefore P(D\cap \text{Three}) = \frac{11}{36}$

$P(3|\text{Dif}) = \frac{\frac{11}{36}}{\frac{30}{36}} = \frac{11}{30}$

17. $P(A\cap B\cap C) = P(C\cap(B\cap A)) = P(B\cap A)\cdot P(C|A\cap B)$
$= P(A)\cdot P(B|A)\cdot P(C|A\cap B)$

18. Use an addition table for 0 to 9 as a sample space
A = The sum of two digits is even
B = Both digits are odd
C = One digit is 0

(a) $n(A) = 50$, $P(A) = \frac{1}{2}$, $n(B\cap A) = 25$

$n(B) = 25$, $P(B) = \frac{1}{4}$, $P(B\cap A) = \frac{1}{4}$

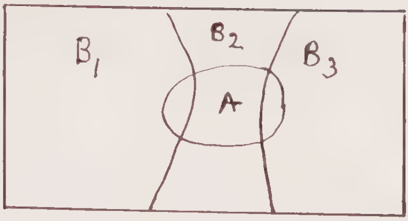
$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

(b) $n(c) = 19$, $n(C\cap A) = 9$, $P(A\cap C) = \frac{9}{100}$

$P(C) = \frac{19}{100}$, $P(C|A) = \frac{\frac{9}{100}}{\frac{50}{100}} = \frac{9}{50}$

Section 6.4 (PP.338-339)

1.



$P(A) = P(B_1)\cdot P(A|B_1)+P(B_2)\cdot P(A|B_2)+P(B_3)\cdot P(A|B_3)$

2. (a) Let B_1 = a white ball is drawn first

$P(B_1) = \frac{5}{8}$ $P(A_1|B_1) = \frac{4}{9}$

B_2 = a black ball is drawn first

$P(B_2) = \frac{3}{8}$ $P(A_1|B_2) = \frac{3}{9}$

A_1 = a white ball is drawn second

We seek to find the P(A)

$P(A_1) = \frac{5}{8}\cdot\frac{4}{9} + \frac{3}{8}\cdot\frac{3}{9} = \frac{29}{72}$

(b) Let A_2 = a black ball is drawn second

$P(A_2|B_1) = \frac{5}{9}$, $P(A_2|B_1) = \frac{6}{9}$

$P(A_2) = \frac{5}{8}\cdot\frac{5}{9} + \frac{3}{8}\cdot\frac{6}{9} = \frac{43}{72}$

Also, $P(A_2) = 1 - P(A_1) = 1 - \frac{29}{72} = \frac{43}{72}$

3. (a) A = First ball is red $P(A) = \frac{36}{60}$; $P(B|A) = \frac{35}{59}$

B = Second ball is red $P(A\cap B) = \frac{36}{60}\cdot\frac{35}{59} = \frac{21}{59}$ or $\frac{105}{295}$

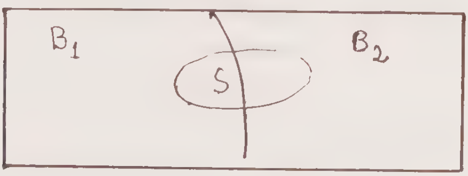
(b) C = First ball is blue $P(C) = \frac{24}{60}$; $P(D|C) = \frac{23}{59}$

D = Second ball is blue $P(C\cap D) = \frac{24}{60}\cdot\frac{23}{59} = \frac{46}{295}$

(c) $\left.\begin{aligned} P(A\cap D) &= P(A)\cdot P(D|A) = \frac{3}{5}\cdot\frac{24}{59} = \frac{72}{295} \\ P(C\cap B) &= P(C)\cdot P(B|C) = \frac{2}{5}\cdot\frac{36}{59} = \frac{72}{295} \end{aligned}\right\} \frac{144}{295}$

or $1 - \left(\frac{105}{295} + \frac{46}{295}\right) = \frac{144}{295}$

4.

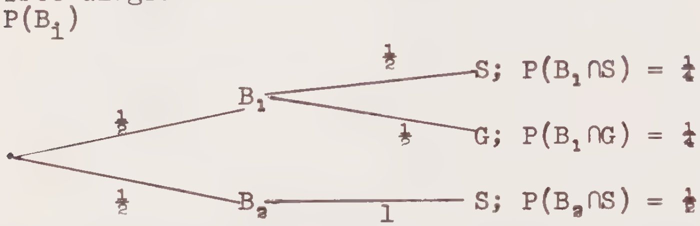


By total probability $P(S) = P(B_1)\cdot P(S|B_1)+P(B_2)\cdot P(S|B_2)$

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 1 = \frac{3}{4}$$

$$P(B_2|S) = \frac{P(B_2 \cap S)}{P(S)} = \frac{P(B_2)\cdot P(S|B_2)}{P(S)} = \frac{\frac{1}{2}\cdot 1}{\frac{3}{4}} = \frac{2}{3}$$

Tree diagram



5. B_1 = White ball first $P(B_1) = \frac{a}{a+b}$; $P(A|B_1) = \frac{c+1}{c+d+1}$

B_2 = Black ball first $P(B_2) = \frac{b}{a+b}$; $P(A|B_2) = \frac{c}{c+d+1}$

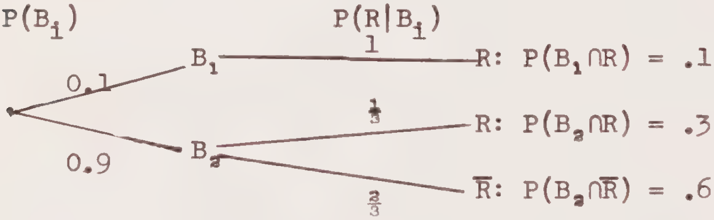
A = White ball second $P(A) = P(B_1)\cdot P(A|B_1)+P(B_2)\cdot P(A|B_2)$

$$= \left(\frac{a}{a+b}\right)\left(\frac{c+1}{c+d+1}\right) + \left(\frac{b}{a+b}\right)\left(\frac{c}{c+d+1}\right) = \frac{a + ac + bc}{(a+b)(c+d+1)}$$

6. B_1 = People who know; B_2 = People who guess;
A = Select Right Shell (R). To find $P(B_1|A)$

$P(B_1|A) = \frac{P(B_1 \cap A)}{P(A)}$; $P(B_1 \cap A) = P(B_1)\cdot P(A|B_1)$

$P(A) = P(B_1)\cdot P(A|B_1) + P(B_2)\cdot P(A|B_2)$



$$\therefore P(B_1|A) = \frac{(0.1)(1)1}{(0.1)(1) + (0.9)(0.3)} = \frac{.1}{.37} = \frac{10}{37}$$

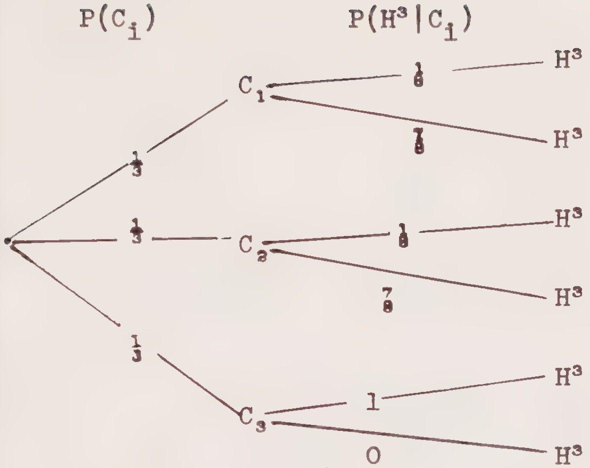
7. Let C_i = Partitioning

Let H^3 = event 3 tosses al heads

We are asked to find $P(C_3|H^3)$

$$P(C_3|H^3) = \frac{P(C_3)\cdot P(H^3|C_3)}{P(H^3)}$$

$P(H^3) = P(C_1)\cdot P(H^3|C_1)+P(C_2)\cdot P(H^3|C_2)+P(C_3)\cdot P(H^3|C_3)$

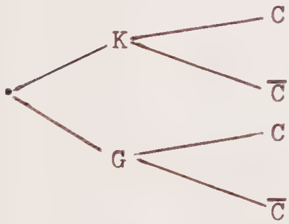


$$\therefore P(C_3|H^3) = \frac{\frac{1}{3}\cdot 1}{\frac{1}{3}\cdot\frac{1}{8} + \frac{1}{3}\cdot\frac{1}{8} + \frac{1}{3}\cdot 1} = \frac{\frac{1}{3}}{\frac{10}{24}} = \frac{4}{5}$$

8. Let K = student knows, G = student guesses,
 C = response is correct.
 We are asked to find $P(G|C)$

$$P(G|C) = \frac{P(G)\cdot P(C|G)}{P(C)}$$

$$P(C) = P(K)\cdot P(C|K) + P(G)\cdot P(C|G)$$



$$\therefore P(G|C) = \frac{(0.2)(\frac{1}{5})}{(0.8)(1)+(0.2)(\frac{1}{5})}$$

$$= \frac{.04}{.84} = \frac{1}{21}$$

9. D = Item is defective, A = Made by A Machine,
 B = Made by B Machine, C = Made by C Machine.
 Find $P(A|D)$, $P(B|D)$, $P(C|D)$

$$P(A|D) = \frac{P(A)\cdot P(D|A)}{P(A)\cdot P(D|A) + P(B)\cdot P(D|B) + P(C)\cdot P(D|C)}$$

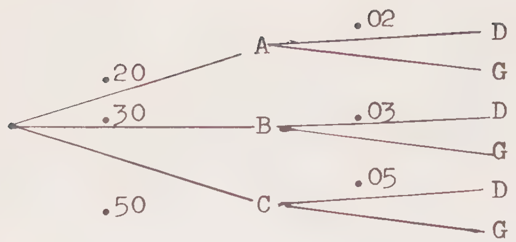
$$= \frac{(.20)(.02)}{(.20)(.02) + (.30)(.03) + (.50)(.05)}$$

$$= \frac{.004}{.004 + .009 + .025} = \frac{.004}{.038} \approx .105$$

$$P(B|D) = \frac{(.30)(.03)}{.038} \approx .240$$

$$P(C|D) = \frac{.025}{.038} \approx .660$$

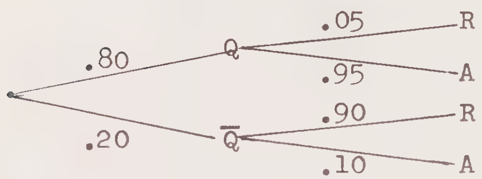
$\Sigma P \approx 1.00$ If it is defective, it came from A, B, or C
$\therefore$ Probability is 1



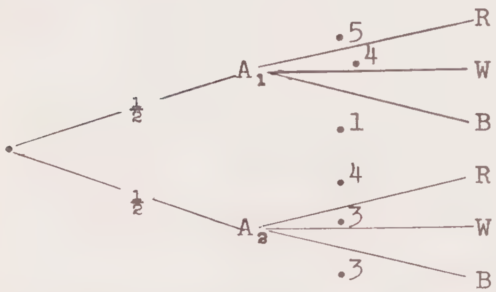
10. Q = person is qualified, $\overline{Q}$ = person is unqualified
R = person is rejected, A = person is accepted.
Find $P(Q|A)$

$$P(Q|A) = \frac{P(Q)\cdot P(A|Q)}{P(Q)\cdot P(A|Q) + P(\overline{Q})\cdot P(A|\overline{Q})}$$

$$= \frac{(.80)(.95)}{(.80)(.95) + (.20)(.10)} = \frac{.760}{.780} \approx .974$$



11.



(a) $P(A) = P(A_1)\cdot P(R|A_1) + P(A_2)\cdot P(A_2)\cdot P(R|A_2)$

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{2}{5} = \frac{9}{20}$$

(b) $P(B) = \frac{1}{2}\left(\frac{1}{10}\right) + \frac{1}{2}\cdot\frac{3}{10} = \frac{4}{20}$

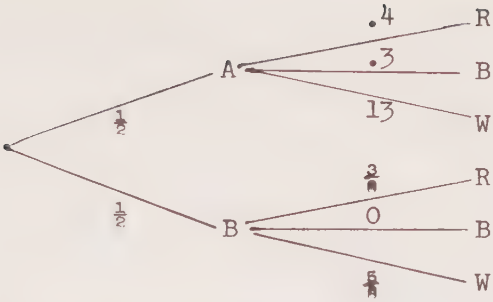
Since there is replacement the two events of drawing a blue are independent

$$\therefore P(B \cup B) = \frac{1}{5}\cdot\frac{1}{5} = \frac{1}{25}$$

12. (a) $P(W|A) = .3$

(b) $$P(A|W) = \frac{P(A)\cdot P(W|A)}{P(A)\cdot P(W|A) + P(B)\cdot P(W|B)}$$

$$= \frac{\frac{1}{2}\cdot\frac{3}{10}}{\frac{1}{2}\cdot\frac{3}{10}+\frac{1}{2}\cdot\frac{5}{8}} = \frac{\frac{3}{20}}{\frac{37}{80}} = \frac{12}{37} \approx .324$$



<u>Section 6.5 (pp.345-348)</u>

1. $P(A) = \frac{1}{4}$; $P(B) = \frac{1}{13}$; $P(C) = \frac{5}{13}$; $P(A \cap C) = \frac{5}{13}$;
   $P(A \cap B) = \frac{1}{52}$; $P(B \cap C) = \frac{1}{13}$; $P(A) \cdot P(B) = \frac{1}{52}$;
   $P(A) \cdot P(C) = \frac{5}{52}$; $P(B) \cdot P(C) = \frac{5}{169}$
   $\therefore$ (A and B) and (A and C) are independent

2. $P(A) = \frac{1}{3}$; $P(B) = \frac{1}{2}$; $P(C) = \frac{2}{3}$; $A \cap B = \{3\}$; $P(A \cap B) = \frac{1}{6}$;
   $A \cap C = \{3\}$; $P(A \cap C) = \frac{1}{6}$; $B \cap C = \{3,4\}$; $P(B \cap C) = \frac{1}{3}$
   $P(A) \cdot P(B) = \frac{1}{6}$; A and B are independent
   $P(A) \cdot P(C) = \frac{2}{9}$; A and C are not independent
   $P(B) \cdot P(C) = \frac{1}{3}$; B and C are independent

3. $P(A) \neq 0$, $P(S) = 1$, $A \cap S = A$ $\therefore$ $P(A) \cdot P(S) = P(A) \cdot P(A \cap S) = P(A)$
   $\therefore$ A and S are independent

4. (a) $P(\overline{A}) = P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B})$ or $P(\overline{A} \cap B) = P(\overline{A}) - P(\overline{A} \cap \overline{B})$
   But $\overline{A}$ and $\overline{B}$ are independent.
   Hence $P(\overline{A} \cap B) = P(\overline{A}) - P(\overline{A})P(\overline{B}) = P(\overline{A})(1 - P(\overline{B}))$
   $= P(\overline{A}) \cdot P(B)$

   (b) $P(\overline{A}) = P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B})$ or $P(\overline{A} \cap \overline{B}) = P(\overline{A}) - P(\overline{A} \cap B)$
   or $P(\overline{A} \cap \overline{B}) = P(\overline{A}) - P(\overline{A}) \cdot P(B) = P(\overline{A})(1 - P(B))$
   $= P(\overline{A}) \cdot P(\overline{B})$

5. $P(A) = \frac{225}{600}$; $P(B) = \frac{400}{600}$; $P(A \cap B) = \frac{150}{600}$
   $\therefore$ $P(A) \cdot P(B) = P(A \cap B) = \frac{1}{4}$.
   Hence by Exercise 4, A and $\overline{B}$, $\overline{A}$ and B, and $\overline{A}$ and $\overline{B}$ are independent.

6. (a) $P(A) = \frac{225}{600}$; $P(B) = \frac{400}{600}$; $P(A \cap B) = \frac{225}{600}$
   Hence $P(A) \cdot P(B) = \frac{1}{4} \neq \frac{3}{8}$

(b) $P(A\cap\overline{B}) = 0$, other probabilities are not 0. Hence $P(A)\cdot P(\overline{B}) \neq P(A\cap\overline{B})$

7. (a)

	Elec.	Non-Elec.	
New	54	36	90
Used	66	44	110
	120	80	200

(b) Use $P(E\cap N) = P(E)\cdot P(N)$

$$P(E\cap U) = \frac{3}{5}\cdot\frac{9}{20} = \frac{27}{100} \qquad \frac{27}{100}\cdot 200 = 54$$

(c) By subtraction from subtotals
(d) This follows from Exercise 4.

8. Use $P(A)\cdot P(B) = P(A\cap B) = .12$
$A\cap B = 48$, $A\cap\overline{B} = 192$, $\overline{A}\cap B = 32$, $\overline{A}\cap\overline{B} = 128$

	A	$\overline{A}$	
B	48	32	80
$\overline{B}$	192	128	320
	240	160	400

9. After 7 games at the most, 2 more games must be played. The probability that A wins given 4 games won is $\frac{3}{4}$, the probability that B wins is $\frac{1}{4}$. $\therefore$ the odds in favor of A winning are 3 to 1. A gets $\frac{3}{4}$ of 4200 or \$3150. B gets \$1050.

10. $P(\overline{M}) = .20$ $P(\overline{E}) = .24$ $P(\overline{M}\cap\overline{E}) = .04$
Since $P(\overline{M})\cdot P(\overline{E}) \neq P(\overline{M}\cap\overline{E})$, the events are not independent.

11. The two relays act independently. Hence
$P(R_1\cap R_2) = P(R_1)\cdot P(R_2) = P^2$

(b) $P(R_1\cap R_2) = .95 \times .95 = .9025$

12. $P(A) = 0.5$ $P(B) = p$ $P(A\cap B) = 0.8$
(a) $P(A\cup B) = P(A)+P(B)$ or $0.8 = 05+p \therefore p = 0.3$
(b) $P(A\cap B) = P(A)+P(B)-P(A\cup B)$ for all events
$P(A\cap B) = P(A)\cdot P(B)$ for independence
$\therefore P(A)\cdot P(B) = P(A)+P(B)-P(A\cup B)$
$(0.5)(p) = 0.5+p-0.8$ or $.5p = .3$, $p = .6$
(c) $p \neq 0.6$

13. $P(D_1) = .1$ $P(D_2) = 0.2$ $P(\overline{D}_1) = .9$ $P(\overline{D}_2) = .95$

By independence (D_1,D_2), $(D_1,\overline{D}_2)$, $(\overline{D}_1,D_2)$ and $(\overline{D}_1,\overline{D}_2)$ are all independent.
$P(D_1)\cdot P(D_2) = .005$ (Both defects exist)

(a) $P(D_1 \cup D_2) = P(D_1) + P(D_2) - P(D_1 \cap D_2) = .1 + .05 - .005 = .145$

(b) $P(\overline{D}_1) \cdot P(\overline{D}_2) = .855$ (No defects)

(c) $P(D_1) \cdot P(\overline{D}_2) = .095$ (Defect of 1st type only)
$P(\overline{D}_1) \cdot P(D_2) = .045$ (Defect of 2nd type only)
one or other not both 0.140

14. The reduced sample space is $\{(H,T), (T,H), (T,T)\}$.
Each has a $P = \frac{1}{3}$

$$\therefore P(H,H) = \frac{n(H,H)}{n(S)} = \frac{1}{3}$$

Review Exercises (pp.348-351)

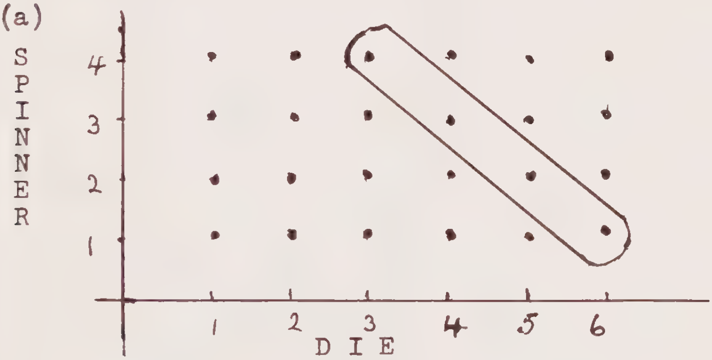
1. (a) n(books) x n(seats) = $3 \cdot 4 = 12$

(b) $P = \frac{2}{12} = \frac{1}{6}$ (c) $7 = (7)_1$

2. (a) $(6)_3 = 6 \cdot 5 \cdot 4 = 120$ (b) $(3)_3 = 6$ (c) $P = \frac{6}{120} = \frac{1}{20}$

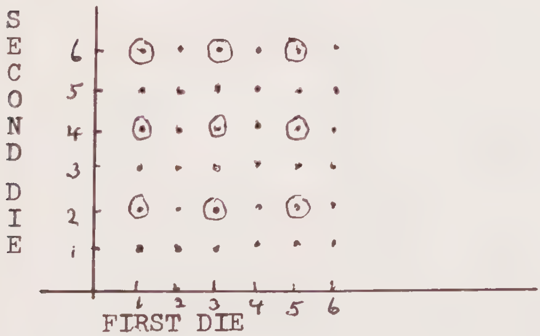
3. (a) $\binom{10}{3} = 120$ (b) $\binom{6}{2} \cdot \binom{4}{1} = 60$ (c) $P = \frac{\binom{4}{3}}{120} = \frac{1}{30}$

4. (a)



(b) $P(\text{sum } 7) = \frac{4}{24} = \frac{1}{6}$

5.

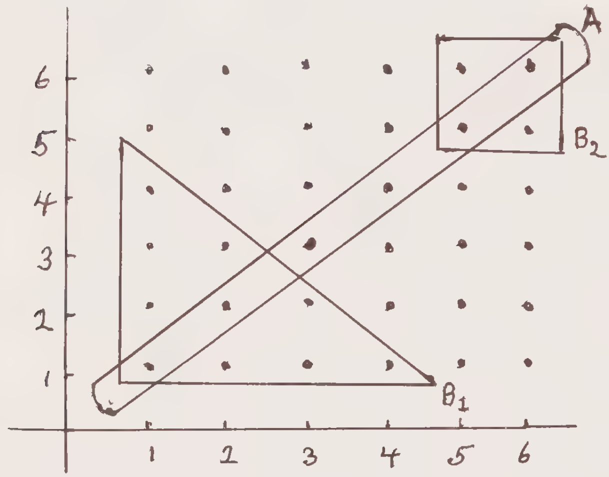


$P(\text{odd}, \text{even}) = \frac{9}{36} = \frac{1}{4}$

6. $P(X\cup Y) = P(X)+P(Y)-P(X\cap Y)$; $P(X\cap Y)=P(X)\cdot P(Y)$;
$P(X) = .05$; $P(X\cup Y) = 0.8$; $P(X\cap Y) = 0.5\cdot P(Y)$. Hence
$0.8 = 0.5+P(Y)-0.5P(Y)$ or $0.3 = 0.5P(Y) \therefore P(Y) = 0.60$

7. See Text, Section 6.2

8.

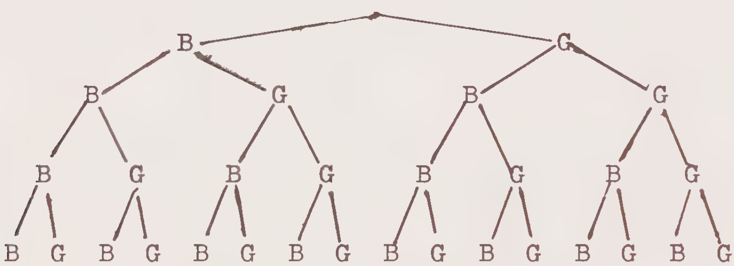


$P(A\cap(B_1\cup B_2)) = \frac{4}{36}$

$P(B_1\cup B_2) = \frac{14}{16}$

$P(A|B_1\cup B_2) = \frac{2}{7}$

9. (a)



(b) $P(B^3,G) = \frac{4}{16} = \frac{1}{4}$ (c) $P[(B^3G)\cup B^4] = \frac{5}{16}$

(c) $P(B^2G^2|B) = \frac{3}{16}$

10. $P(D_1) = 0.12$, $P(D_2) = 0.06$, $P(\overline{D}_1) = 0.88$, $P(\overline{D}_2) = 0.94$

(a) $P(\overline{D}_1\cap\overline{D}_2) = (.88)(.94) = .8272$

(b) $P(D_1\cap\overline{D}_2) = (.12)(.94) = .1128$

$P(\overline{D}_1\cap D) = (.88)(.06) = 0.528$

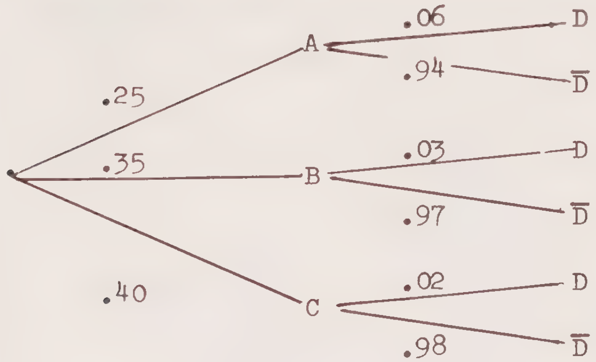
$P(D\cap D) = (.12)(.06) = \frac{.0072}{.1728} = P(\text{Defection})$

(c) $P[(\overline{D}_1\cap D)\cup(D_1\cap\overline{D}_2)|D_1] = \frac{.1656}{.1728} \approx .958$

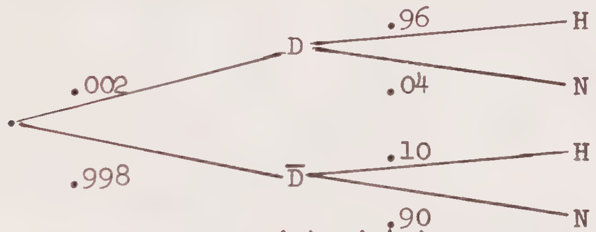
11. (a) $P(A|D) = \dfrac{P(A)\cdot P(D|A)}{P(A)\cdot P(D|A)+P(B)\cdot P(D|B)+P(C)\cdot P(D|C)}$

$= \dfrac{(.06)(.25)}{(.06)(.25)+(.03)(.35)+(.02)(.40)} = \dfrac{150}{335}$

(b) ~ by $P(C|D) = \frac{80}{335}$ and $P(B|D) = \frac{105}{335}$



12. Let D = has disease, $\overline{D}$ = does not have disease, H = test positive, N = test negative



(a) $P(D|H) = \dfrac{P(D)\cdot P(H|D)}{P(D)\cdot P(H|D)+P(\overline{D})\cdot P(H|\overline{D})}$

$$= \frac{(.002)(0.96)}{(.002)(0.96)+(.998)(.10)}$$

$$= \frac{.00192}{.00192+.09980} = \frac{192}{10172} \approx .189$$

$$P(D|N) = \frac{(.002)(.041)}{(.002)(.041)+(.998)(.90)}$$

$$= \frac{.00008}{.00008+.89820} = \frac{8}{89828} \approx .0001$$

13. $P(A) = \frac{3}{10}$, $P(A\cup B) = \frac{6}{10}$, $P(B) = p$

(a) $P(A\cup B) = P(A)+P(B) \therefore \frac{6}{10} = \frac{3}{10}+p \therefore p = \frac{3}{10}$

(b) $P(A)\cdot P(B) = P(A\cap B)$ $P(A\cap B) = P(A)+P(B)-P(A\cup B)$

(c) $(0.3)p = .3+p-.6$ or $.7p = .3$ and $p = \frac{3}{7}$

14. $P(A) = \frac{1}{6}$, $P(B) = \frac{1}{6}$, $P(C) = \frac{1}{6}$, $P(A\cap B) = \frac{1}{36}$, $P(A\cap C) = \frac{1}{36}$

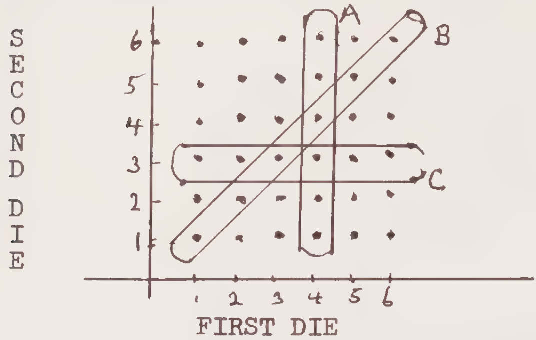
$P(B\cap C) = \frac{1}{36}$

$P(A)\cdot P(B) = P(A\cap B) \therefore (A,B)$ are independent

$P(A)\cdot P(C) = \frac{1}{36} = P(A\cap C) \therefore (A,C)$ are independent

$P(B)\cdot P(C) = \frac{1}{36} = P(B\cap C) \therefore (B,C)$ are independent

$P(A\cap B\cap C) = 0 \therefore$ not all three are independent



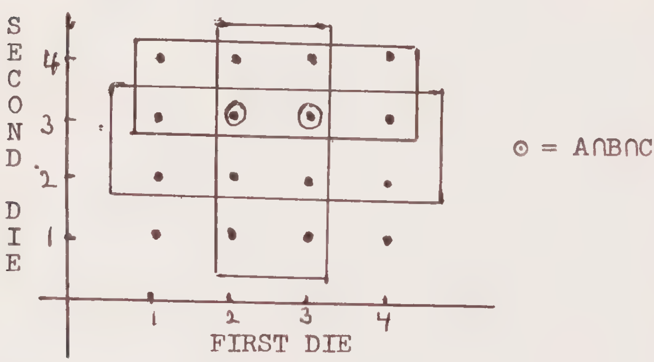
15. $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{2}$

$P(A\cap B) = \frac{1}{4}$, $P(A\cap C) = \frac{1}{4}$, $P(B\cap C) = \frac{1}{4}$, $P(A\cap B\cap C) = \frac{1}{8}$

$P(A)\cdot P(B)\cdot P(C) = \frac{1}{8} = P(A\cap B\cap C)$

$P(A)\cdot P(B) = \frac{1}{4} = P(A\cap B)$, $P(A)\cdot P(C) = \frac{1}{4} = P(A\cap C)$

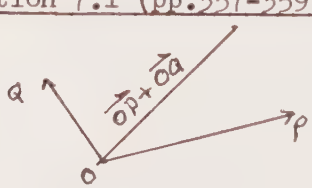
$P(B)\cdot P(A) = \frac{1}{4} = P(B\cap C) \therefore A, B, C$ are independent



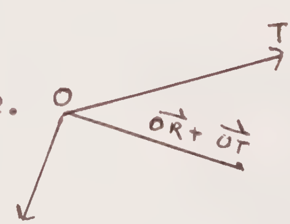
CHAPTER 7

Section 7.1 (pp.357-359)

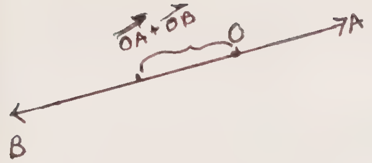
1.



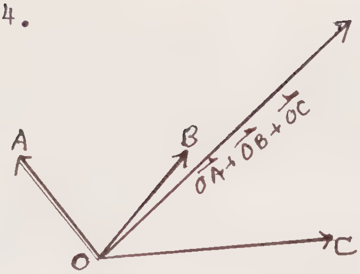
2.



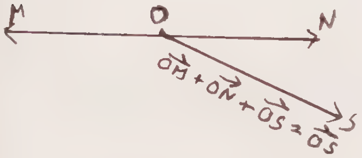
3.



4.



5.



6. $2\sqrt{29}$; 22° 7. 13; 23°
8. $5\sqrt{3}$; 30° 9. $\frac{14\sqrt{3}}{3}$; 90°
10. 19.4; 6°
11. Braking force = $3000\sin 25^\circ$
≈ 1269 lbs.
12. $4500(\sin 20^\circ) \approx 1539$ lbs.

Section 7.2 (pp.363-365)

1. (3,7) 2. (10,-1) 3. (2,0) 4. (0,-4)
5. (a) $(x,y) \to (x+2,y-1)$ (b) $(x,y) \to (x+4,y)$
(c) $(x,y) \to (x+2,y+1)$
6. (a) $(x,y) \to (x-4,y-3)$ (b) $(x,y) \to (x+3,y+3)$
(c) $(x,y) \to (x+7,y+6)$
7. (a) $(x,y) \to (x-2,y+2)$ (b) $(x,y) \to (x+2,y-2)$
(c) $(x,y) \to (x+4,y-4)$
8. (a) $(x,y) \to (x+x_1,y+y_1)$ (b) $(x,y) \to (x+x_2,y+y_2)$
(c) $(x,y) \to (x+x_2-x_1,y+y_2-y_1)$
9. Let $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ be any elements in B. Label $P = (x_1,y_1)$, $Q = (x_2,y_2)$, $R = (x_3,y_3)$
   (a) Closure: $\overrightarrow{OP} + \overrightarrow{OQ}$ is the arrow having terminal point $T = (x_1+x_2,y_1+y_2)$
   (b) Associativity: The terminal point of $(\overrightarrow{OP}+\overrightarrow{OQ})+\overrightarrow{OR}$ has coordinates $(x_1+x_2)+x_3$. The terminal point of $\overrightarrow{OP}+(\overrightarrow{OQ}+\overrightarrow{OR})$ has coordinates $x_1+(x_2+x_3)$. Because real number addition is associative
   $$(x_1+x_2)+x_3 = x_1+(x_2+x_3)$$
   and so $$(\overrightarrow{OP}+\overrightarrow{OQ})+\overrightarrow{OR} = \overrightarrow{OP}+(\overrightarrow{OQ}+\overrightarrow{OR})$$
   (c) $\overrightarrow{OO}$ is in B and is the identity for arrow addition.
   (d) Inverses: For each arrow $\overrightarrow{OP}$ $(P = (x_1,y_1))$, $\overrightarrow{OP'} \in B$ where $P' = (-x_1,-y_1)$. $\overrightarrow{OP}+\overrightarrow{OP'} = \overrightarrow{OO}$
   (e) Commutativity: Let $\overrightarrow{OP}+\overrightarrow{OQ} = \overrightarrow{OT}$ $(T = (x_1+x_2,y_1+y_2))$ and let $\overrightarrow{OQ}+\overrightarrow{OP} = \overrightarrow{OS}$ $(S = (x_2+x_1,y_2+y_1))$.
   But T = S, so $\overrightarrow{OP}+\overrightarrow{OQ} = \overrightarrow{OQ}+\overrightarrow{OP}$

10. The proofs for (a) - (e) are similar, so only part (c) is shown.
Prove: $(r+s)\overrightarrow{OP} = r\overrightarrow{OP} + s\overrightarrow{OP}$
The strategy is to show that the terminal points of $(r+s)\overline{OP}$ and $r\overline{OP} + s\overline{OP}$ are the same point and hence the equality is true.

Let $P = (x_1,y_1)$. Then the terminal point of arrow $(r+s)\overrightarrow{OP}$ is $((r+s)x_1,(r+s)y_1)$
The terminal points of $r\overrightarrow{OP}$ and $s\overrightarrow{OP}$ are (rx_1,ry_1) and (sx_1,sy_1), respectively. Hence arrow $r\overrightarrow{OP} + s\overrightarrow{OP}$ has terminal point

$$(rx_1+sx_1,ry_1+sy_1) = ((r+s)x_1,(r+s)y_1)$$

$$\therefore (r+s)\overrightarrow{OP} = r\overrightarrow{OP} + s\overrightarrow{OP}$$

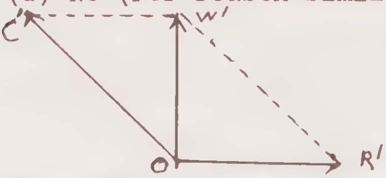
11. $(P_1,+)$ is a commutative group - addition of polynomials is associative and commutative, $0\cdot x+0$ is the identity and for each $(ax+b)\in P_1$, $(-ax-b)\in P_1$ with $(ax+b)+-ax-b) = 0\cdot x+0$
The scalar properties follow easily but require space to demonstrate. The proof of (b) is representative:
Prove: $r((ax+b)+(cx+d)) = r(ax+b)+r(cx+d)$ for all $r \in R$, $(ax+b)$, $(cx+d) \in P_1$

$$\begin{aligned} r((ax+b)+(cx+d)) &= r((a+c)x + (b+d)) \\ &= r(a+c)x+r(b+d) \\ &= rax + rcx + rb + rd \\ &= rax + rb + rcx + rd \\ &= r(ax+b) + r(cx+d) \end{aligned}$$

The preceding steps all follow from properties of scalar multiplication and addition of polynomials.

12. $(I,+)$ is a vector space over Q but NOT over R. $(I,+)$ is a commutative group and each scalar property is satisfied.

13. (a) Yes (b) No (for all $r\in R$, $q\in Q$, rq is not necessarily in Q). (c) No (Z is not a field)
(d) No (For reason similar to the one given in b)

14. 

$|OR'| = 2000$lbs, the thrust in the rod
$|OC'| = 2000\sqrt{2}$ lbs ≈ 2828lbs, the tension in the cable.

Suggestion: Have students discuss this possible unexpected result (the tension is greater than the weight of the anchor). Relate to fishing poles and lines, etc. - the need for using a line that tests at a much greater weight than expected weight for fish.

15. 

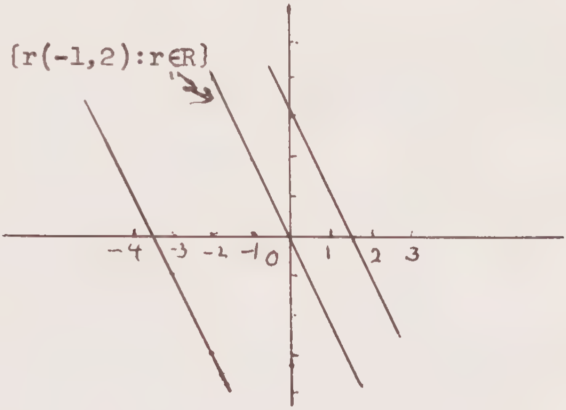
$x = \sqrt{116} \approx 13.4$

$\tan\alpha = .4$ so $\alpha \approx 22^\circ$

Captain should head 22° west of north at a speed of about 13 knots.

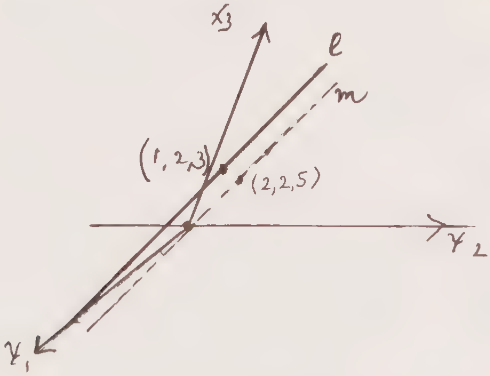
Section 7.3 (pp.373-375)

1. $(-4,9,-2)$ 2. $(2,\frac{4}{3},0)$ 3. Impossible   4. $(-1,-7,11)$
5. $(-12,13,1)$ 6. $(-1,1,5,-6)$ 7. Answers will vary
8.



9. (a) We know that the graph of $m = \{s(2,2,5):s\in R\}$ is a straight line. The graph of $\ell = \{(1,2,3) + s(2,2,5) : s\in R\}$ is the image of m under the translation $T_{(1,2,3)}: (x,y,z) \to (x+1,y+2,z+3)$. Because the image of a line is a line under a translation, the graph of ℓ is a line. Ask students how we know that $m \| \ell$

(b)



10. (a) $0 < t < 1$. $m = (2,3)+ \frac{1}{2}(3,1) = (\frac{7}{2},\frac{7}{2})$
(b) $t \geq 1$; $t \leq 0$
11. A ray with endpoint (0,0) passing through (-2,5)
12. An open ray $\overset{\circ\to}{AC}$, where A = (1,-2,-4), that passes through (2,-4,-8)
13. The point (3,9)
14. The segment $\overline{AB}$ where A = (4,5,6) and B = (5,7,9)

15. The line containing $(0,6)$ and $(3,0)$

16. The parabola that opens down, has vertex at $(0,6)$ with x-intercepts $(\pm\sqrt{3},0)$

17. (a) Vector Form: $\{(-3,5) + r(5,-6): r \in R\}$
Vector equation: $(x,y) = (-3,5) + r(5,-6),\ r \in R$
Parametric equations: $x = -3+5r$
$y = 5-6r$

(b) Vector equation: $(x,y,z) = (1,-2,5) + s(-5,6,-1)$
Parametric equations: $x = 1-5s$
$y = -2+6s$
$z = 5-s$

(c) $(x,y,z) = (5,-6,-4) + s(-1,5,5)$ gives
$x = 5 - s$
$y = -6+5s$
$z = -4+5s$

18. Let $A = \{r(4,-3,3)+(-4,1,2)\}$ and $B = \{r(4,-3,3)+(0,-2,5)\}$ where in each case $r \in R$.
$P \in A$ if there is an r such that
$(-4,1,2) = r(r,-3,3)+(-4,1,2)$; let $r = 0$.
Is $P \in B$? Is there an r satisfying
$(-4,1,2) = r(4,-3,3)+(0,-2,5)$?
If so, $-4 = 4r$ and $1 = -3r-2$ and $2 = 3r+5$
Since $r = -1$ satisfies each equation, $P \in B$.
Similarly $Q \in A$ (let $r = 1$) and $Q \in B$ (let $r = 0$).
Since there is one and only one line through P, Q and since P and Q are both in A and B (whose graphs are lines), $A = B$.

19. Many possible answers. The two most likely are:

(1) $\overleftrightarrow{AB} = \{(4,6,-3) + r(-5,-7,8) : r \in R\}$ and
(2) $\overleftrightarrow{AB} = \{(-1,-1,5) + s(-5,-7,8) : s \in R\}$

A and B are elements of both sets - if you let $r = 0$ and then $r = 1$, you get A and B in set (1); If you let $s = -1$ and then $s = 0$, you get A and B in set (2).

*20. (a) Proof: $(\ell,+)$ is a commutative group.
Let $A = t(a_1,a_2,\ldots,a_n)$, $B = u(a_1,a_2,\ldots,a_n)$, and $C = v(a_1,a_2,\ldots,a_n)$ be any elements of ℓ.

Closure: $A + B = t(a_1,a_2,\ldots,a_n)+u(a_1,a_2,\ldots,a_n)$
$= (ta_1,ta_2,\ldots,ta_n)+(ua_1,ua_2,\ldots,ua_n)$
$= (ta_1+ua_1,ta_2+ua_2,\ldots,ta_n+ua_n)$
$= ((t+u)a_1,(t+u)a_2,\ldots,(t+u)a_n)$
$= (t+u)(a_1,a_2,\ldots,a_n) \in \ell$

Associativity and Commutativity:
$(\ell,+)$ inherits these properties from $(R^n,+)$

Identity: $(0,0,\ldots,0) \in \ell$; let $t = 0$

Inverses: For each element $t(a_1,a_2,\ldots,a_n) \in \ell$, $-t(a_1,a_2,\ldots,a_n) \in \ell$ and their sum is $(0,0,0,\ldots0)$

(b) Scalar properties--the only property that has to be checked is that $r\cdot(t(a_1,a_2,\ldots,a_n)) \in \ell$ because the others are inherited from $(R^n,+)$ over R.

$$r\cdot(t(a_1,a_2,\ldots,a_n)) = r(ta_1,ta_2,\ldots,ta_n)$$
$$= (rta_1,rta_2,\ldots,rta_n)$$
$$= rt(a_1,a_2,\ldots,a_n) \in \ell$$

Make sure students understand why some properties are inherited but others are not.

21. True   22. False (One reason - no identity)

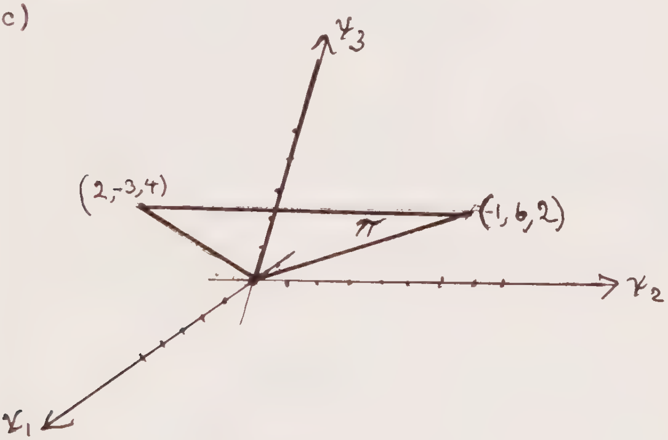
23. True   24. False (One reason - no identity)

Section 7.4 (pp.380-382)

1. (a) $(2,-3,4) \notin \{r(-1,6,2) : r \in R\}$ because there is no r such that $(2,-3,4) = r(-1,6,2)$; that is, there is no r that simultaneously satisfies $2 = -r$, $-3 = 6r$, $4 = 2r$

(b) Many possible answers

(c)



(d) $(x,y,z) = m(2,-3,4) + n(-1,6,2)$;
$x = 2n - n$, $y = -3m + 6n$, $z = 4m + 2n$

2. $(x,y,z) = (1,6,-7) + m(2,-3,4) + n(-1,6,2)$;
$x = 1 + 2m - n$, $y = 6 - 3m + 6n$, $z = -7 + 4m + 2n$

3. (a) $(x_1,x_2) = r(2,3) + (-4,1)$
(b) $(x_1,x_2,x_3) = r(-1,5,-1) + (0,4,-7)$
(c) $(x_1,x_2,x_3) = m(5,1,1) + n(-2,-1,5) + (4,3,-5)$
(d) $(x_1,x_2,x_3,x_4) = r(4,-2,3,3) + s(-1,5,7,-6)$

4. $m = \frac{-3}{2}$ 5. $r = \frac{-2}{3}$, $s = -2$

6. $(B,+)$ is a commutative group
Closure: Let $r(-2,1,3) + s(1,1,4)$ and $r'(-2,1,3) + s'(1,1,4)$ be any 2 elements in B. Then

$r(-2,1,3)+s(1,1,4)+r'(-2,1,3)+s'(1,1,4) =$
$(r+r')(-2,1,3)+(s+s')(1,1,4)$ which is in B.

Identity: $(0,0,0) \in B$ and is the identity.

Inverses: For each $r(-2,1,3)+s(1,1,4)$, $-r(-2,1,3)+ -s(1,1,4)$ is in B with their sum $(0,0,0)$.

Associativity and Commutativity: Inherited

Scalar Properties--the only one that has to be verified is that for all $r' \in R$, $x \in B$, $r' \cdot x \in B$:
$r' \cdot (r(-2,1,3)+s(1,1,4)) = r'r(-2,1,3)+r's(1,1,4)$ which is in B.

*7. Let $S = \{rA + sB : r,s \in R;\ A,B \in R^n\}$
$(S,+)$ is a commutative group. As in (6), closure, existence of an identity, and inverses for each element have to be demonstrated. Closure is the most involved:
Let $x,y \in S$. Then $x = rA + sB$ and $y = r'A + s'B$ for some $r, s, r', s' \in R$

(i) $x + y = rA + sB + r'A + s'B$
(ii) $\qquad = rA + r'A + sB + s'B$
(iii) $\qquad = (r + r')A + (s + s')B$ and so $(x+y) \in S$.

Note: Some or most students will probably want to let $A = (a_1,a_2,\ldots,a_n)$ and $B = (b_1,b_2,\ldots,b_n)$ and work the proof using n-tuples; this is alright, but can be avoided (as in (i)-(iii) above) by recognizing that $(S,+)$ over R inherits properties from $(R^n,+)$ over R. In particular, in (ii) we use the fact that + in $(S,+)$ must be commutative and in (iii) that vectors in S distribute over addition of scalars.

To complete the proof, the scalar properties have to be verified and closure is the only one that needs proof (the others are inherited).

Let $x \in S$, $r' \in R$. Then $x = rA + sB$
$r'x = r'(rA + sB) = (r'rA + r'sB) \in S$. Done!

8. A vector line in R^3

9. A vector plane: If $A \notin \{rB : r \in R\}$ and A and B are both non-zero vectors.
A vector line: If $A \in \{rB\}$ or if exactly one of A or B is the zero vector.
A single vector: If A and B are both the zero vector.

10. (a) Always true (b) sometimes true (what happens if $B = (0,0,0)$?)
(c) Sometimes true (d) Sometimes true
(e) Sometimes true (f) Never true

11. A plane containing $(2,6,1)$ and parallel to the vector plane $\{m(-1,2,7) + n(4,4,-8) : m,n \in R\}$
To see if $(3,2,-16) \in S$, solve
(1) $3 = -m + 4n + 2$
(2) $2 = 2m + 4n + 1$
(3) $-16 = 7m - 8n + 1$

working with (1) and (2) gives $m = 0$, $n = \frac{1}{4}$, which do NOT satisfy (3), hence $(3,2,-16) \notin S$.

12. $(x,y,z) = (1,2,3) + r(2,1,-2) + t(3,2,0)$, $r,t \in R$

13. Many answers possible; here is one:
$R - A = (1,-3,3)$, $C - A = (-1,-6,6)$, so equation takes the form
$(x,y,z) = A + r(R-A) + s(C-A)$
$= (1,3,-2) + r(1,-3,3) + s(-1,-6,6)$, $r,s \in R$

14. The plane is a vector plane (it will pass through the origin) and must contain $(-3,4,1)$ and $(1,8,-2)$.
Equation:
$(x,y,z) = r(-3,4,1) + s(1,8,-2)$, $r,s \in R$

15. A vector plane that will contain $(1,2,3)$ and $(-4,5,1)$
$(x,y,z) = r(1,2,3) + s(-4,5,1)$, $r,s \in R$

16. Note that $\ell_1 \| \ell_2$ and neither are vector lines, so the strategy is to pick 3 non-collinear points and proceed as in Exercise 13.
Let $A = (1,1,1) \in \ell_1$, $B = (1,3,5) \in \ell_1$ and
$C = (-3,5,1) \in \ell_2$. Then $B - A = (0,2,4)$,
$C - A = (-4,4,0)$ and equation is:
$(x,y,z) = (1,1,1) + r(0,2,4) + s(-4,4,0)$, $r,s \in R$

17. $(x,y,z) = r(0,1,0) + s(4,1,-3)$, $r,s \in R$

18. $(x,y,z) = r(1,0,0) + s(0,1,0)$, $r,s \in R$

19. Let the vector representations be I, II, and III. The chart gives for each representation the values of r and s that will produce either A, B, or C

		A	B	C
I	r	0	1	0
	s	0	0	1
II	r	-1	0	-1
	s	0	0	1
III	r	0	1	0
	s	-1	-1	0

Most entries can be found by inspection.

Section 7.5 (pp.388-390)

1. (a) No (b) A vector plane containing $(1,0,-2)$ and $(-1,5,0)$
(c) $10x_1 + 2x_2 + 5x_3 = 0$

2. $T = \{(x_1,x_2,x_3) : x_2 = 5$ and $2x_1 + x_3 = -2\}$

3. $11x_1 + 5x_2 - 9x_3 = 0$

4. A vector equation is $(x,y,z) = r(1,2,3) + s(-4,1,3)$ with non-parametric form: $x - 5y + 3z = 0$

5. $\{(x,y,z) : y - z = 7\}$

6. (a) $x_1 = -x_2-x_3$ (reduced form)

$$\left.\begin{array}{l} x_1 = -r-s \\ x_2 = r \\ x_3 = s \end{array}\right. \text{ gives } (x_1,x_2,x_3) = \begin{array}{l}(-r-s,r,s) \\ = (-r,r,0)+(-s,0,s) \\ = (r(-1,1,0)+s(-1,0,1)\end{array}$$

a vector plane

(b) $x_3 = 3x_1 + 2x_2 - 6$ (reduced form)

$$\begin{array}{l} x_1 = r \\ x_2 = s \\ x_3 = 3r + 2s - 6 \end{array} \quad \text{gives } (x_1,x_2,x_3) = r(1,0,3)+s(0,1,2)+(0,0,-6)$$

a plane

(c) $x_3 = 2x_2 - 5$ (reduced form)

$$\begin{array}{l} x_1 = s \\ x_2 = r \\ x_3 = 2r-5 \end{array} \quad \text{gives } (x_1,x_2,x_3) = r(0,1,2)+s(1,0,0)+(0,0,-5)$$

a plane

(d) $x_1 = -x_2-x_3-x_4$ (reduced form)

$$\begin{array}{l} x_1 = -r-s-t \\ x_2 = r \\ x_3 = s \\ x_4 = t \end{array} \quad \text{gives } (x_1,x_2,x_3,x_4) = r(-1,1,0,0)+s(-1,0,1,0)+t(-1,0,0,1)$$

a vector 3-space in R^4

7. $\left.\begin{array}{l} x_1 = 2x_2 + 4 \\ x_3 = -\frac{1}{4}x_2 + \frac{1}{4} \end{array}\right\}$ (reduced form) Let $x_2 = r$ to obtain

$$\left.\begin{array}{l} x_1 = 2r + 4 \\ x_2 = r \\ x_3 = -\frac{1}{4}r + \frac{1}{4} \end{array}\right\} \text{ gives } (x_1,x_2,x_3) = r(2,1,-\tfrac{1}{4})+(4,0,\tfrac{1}{4})$$

A line in R^3. You might want to talk about the geometric interpretations here, but that is the substance of the next section.

8. No work

9. Let $A = \{(x_1,x_2,x_3): 2x_1+3x_2+x_3 = 0\}$. Since A is a vector plane, select any 2 non-collinear points on A, say $(-1,0,2)$ and $(0,-1,3)$. Then $A = \{r(-1,0,2) + s(0,-1,3) : r,s \in R\}$ Answers will vary depending on selected points.

10. $3x_1-2x_2+x_3 = 4$ represents a plane (not vector) so we need only 3 non-collinear points. If $(0,0,4)$, $(0,-2,0)$, and $(1,1,3)$ are used, one representation is $(x_1,x_2,x_3) = (0,0,4) + r(0,-2,-4) + s(1,1,-1)$, $r,s \in R$

11. As in 10, lead students to observe that the intercepts are usually good choices: x_1-intercept $(-4,0,0)$; x_2-intercept $(0,8,0)$; x_3-intercept $(0,0,2)$. $(x_1,x_2,x_3) = (-4,0,0) + r(4,8,0) + s(4,0,2)$

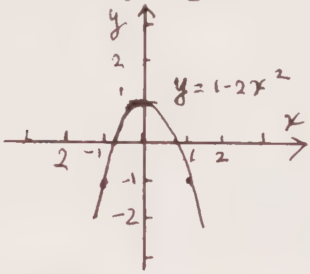
12. Here the choices are harder (one of the reasons that introducing parameters is, in general, the most

efficient method of going from non-parametric to parametric form). Both equations are satisfied by (4,1,0) and (3,0,1) so

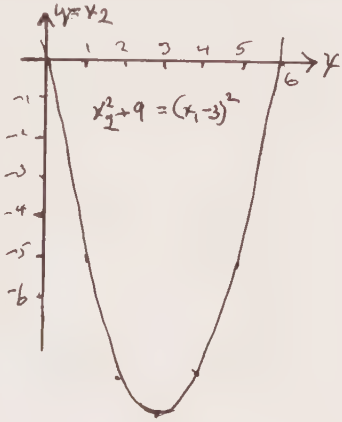
$(x_1,x_2,x_3) = (4,1,0) + r(-1,-1,1)$

Note: we needed to know that the non-parametric system represents a line; however, one cannot always tell this by sight.

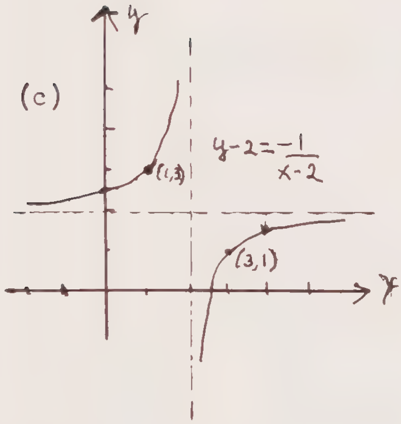
13. (a)



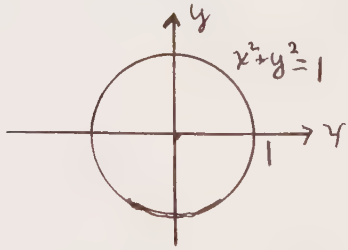
(b)



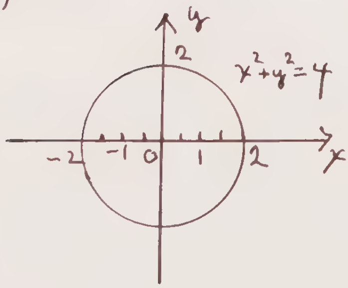
(c)



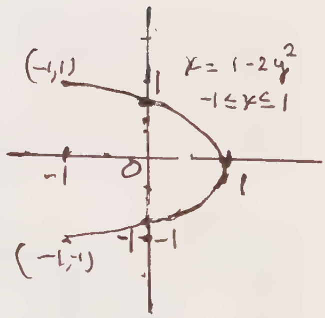
(d)



(e)



(f)



14. (a) $x_1 = ra_1 + sb_1 + c_1$
$x_2 = ra_2 + sb_2 + c_2$
$x_3 = ra_3 + sb_3 + c_3$

(b) Solving $x_1 = ra_1 + sb_1 + c_1$ for r, you get
$r = \frac{1}{a_1}(x_1-sb_1-c_1)$

Assumption: Since $(a_1,a_2,a_3) \neq (0,0,0)$, at least one of the a_i's is not 0. So let it be a_1.
The resulting system is

$$x_1 = x_1$$
$$x_2 = \frac{a_2}{a_1}(x_1 - sb_1 - c_1) + sb_2 + c_2$$
$$x_3 = \frac{a_3}{a_1}(x_1 - sb_1 - c_1) + sb_3 + c_3$$

which, when simplified, is:

$$x_1 = x_1$$
$$x_2 = \frac{a_2}{a_1}x_1 + s\left(b_2 - \frac{a_2 b_1}{a_1}\right) + c_2 - \frac{c_1 a_2}{a_1}$$
$$x_3 = \frac{a_3}{a_1}x_1 + s\left(b_3 - \frac{b_1 a_3}{a_1}\right) + c_3 - \frac{c_1 a_3}{a_1}$$

(c) Solve equation (2) for s noting appropriate valid assumption. Substitution will give

$$x_1 = x_1$$
$$x_2 = x_2$$
$$x_3 = a'x_1 + b'x_2 + c'$$

where a', b', and c' are each expressions in the a_i's, b_i's, and c_i's.

Section 7.6 (pp.394-395)

1. No solutions

2. From
$$\begin{aligned} x + 3y - 2z &= 1 \\ 2x - y + z &= 1 \\ x + 10y - 7z &= 2 \end{aligned} \Rightarrow \begin{aligned} x + 3y + 2z &= 1 \\ -7y + 5z &= -1 \\ 7y - 5z &= 1 \end{aligned} \Rightarrow$$
$$\begin{aligned} x + 3y - 2z &= 1 \\ -7y + 5z &= -1 \end{aligned} \Rightarrow \begin{aligned} 7x + z &= 4 \\ -7y + 5z &= -1 \end{aligned}$$
we can go no further, and so, because x and y are both in terms of z, the solution set is
$$x = \frac{-1}{7}r + \frac{4}{7} \quad y = \frac{5}{7}r + \frac{1}{7} \quad z = r$$
In coordinate form:
$$(x,y,z) = r\left(\frac{-1}{7},\frac{5}{7},1\right) + \left(\frac{4}{7},\frac{1}{7},0\right)$$
Interpretation: The system represents 3 planes that intersect in the line that is represented above.

3. No solutions

4. $(m,n,t) = (6,-2,-1)$ Interpretation: 4 planes that intersect in 1 point--draw a pyramid with a square base as a model; the four planes containing the faces meet in the vertex of the pyramid.

5. From $\ell = \{r,(3,2,1)\}$, $x = 3r$, $y = 2r$, and $z = r$.
Since $z = r$, substitute to get
$x = 3z$ and $y = 2z$; i.e.
$\ell = \{(x,y,z) : x = 3z \text{ and } y = 2z,\ z \in R\}$
$\pi r \ell$ corresponds to the solutions of
$x - 2y + 4z = 6 \quad x - 3z = 0 \quad y - 2z = 0$
The solution is $(x,y,z) = (6,4,2)$

6. $(0,0,0)$

7. Eliminating parameters gives the following system:
$x + y = 5 \quad x - z = 3 \quad 2x+y = 7 \quad 3x+z = 7$
which has no solutions. The lines are skew.

8. (a) $T = P \cap Q \cap R = (P \cap Q) \cap R$
By Theorem 2, $P \cap Q$ is a subspace of R^3, and by Theorem 2 again $(P \cap Q) \cap R$ must be a subspace of R^3

(b) $$\left.\begin{array}{r} x-4y+3z = 0 \\ 4x+2y+z = 0 \\ 2x+10y-5z=0 \end{array}\right\} \Rightarrow \left.\begin{array}{r} x-4y+3z = 0 \\ 18y-11z = 0 \\ 18y-11z = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{r} 9x+5z = 0 \\ 18y-11z=0 \end{array}\right\} \Rightarrow \begin{array}{l} x = \frac{-5}{9}r \\ y = \frac{11}{18}r \\ z = r \end{array}$$
That is, $T = \{r,(\frac{-5}{9},\frac{11}{18},1):r \in R\}$ which shows that

T is a subspace of R^3--it is a <u>vector line</u>.

9. (a) A is represented by $x = -2y - 3z - 4t$.
Letting $y = r$, $z = s$, and $t = v$, in parametric form A is $x = -2r-3s-4v \quad y = r \quad z = s \quad t = v$ or
$A = \{r(-2,1,0,0)+s(-3,0,1,0)+v(-4,0,0,1),\ r,s,v \in R\}$
which is in the form for a subspace of R^4. Similarly for B. Point out that this is a generalization from students' experiences with <u>generating</u> vector lines and vector planes in R^2 and R^3.

(b) Because W is the intersection of 2 subspaces of R^4.

(c) $W = \{(x,y,z,t): x+2y+3z+4t = 0$
and
$4x+3y+2z+t = 0$

Putting the system in reduced form gives:
$$\left.\begin{array}{r} x+2y+3z+4t = 0 \\ 4x+3y+2z+t = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x - z - 2t = 0 \\ y + 2z + 3t = 0 \end{array}$$
Letting $z = r$ and $t = s$,
$x = r+2s \quad y = -2r-3s \quad z = r \quad t = s$
or $W = \{r(1,-2,1,0) + (2,-3,0,1) : s,r \in R\}$
Letting $r = 0$ and $s = 0$, $(0,0,0) \in W$
$r = 0$ and $s = 1$, $(2,-3,0,1) \in W$
$r = 1$ and $s = 0$, $(1,-2,1,0) \in W$
and so on.

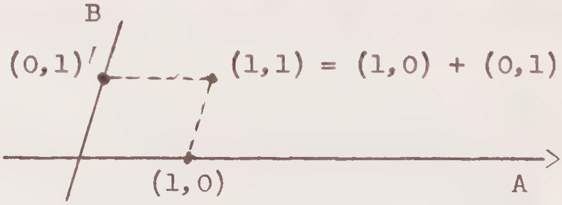
*10. Proof by mathematical induction:
(a) W_1 is a subspace of V by hypothethis
(b) Suppose $W_1 \cap W_2 \cap \ldots \cap W_n$ is a subspace of V.
What about $W_1 \cap W_2 \cap \ldots \cap W_n \cap W_{n+1}$?
$W_1 \cap W_2 \cap \ldots \cap W_n \cap W_{n+1} =$

$(W_1 \cap W_2 \cap \ldots \cap W_n) \cap W_{n+1} = A \cap W_{n+1}$ where A is a subspace of V (from b.) But $A \cap W_{n+1}$ is also a subspace of V because it is the intersection of 2 subspaces (Theorem 2), hence
$$\bigcap_{i=1}^{n+1} W_i \text{ is a subspace of V}$$
Done!

*11. S and W are subspaces of V
$S \oplus W = \{s+w : s \in S, w \in W\}$
Prove that $S \oplus W$ is a subspace of V.
(a) $(S \oplus W, +)$ is a commutative group
<u>Closure</u>: Let (s_1+w_1) and (s_2+w_2) be in $S \oplus W$.
Then
$$\begin{aligned}(s_1+w_1)+(s_2+w_2) &= s_1+w_1+s_2+w_2 \\ &= s_1+s_2+w_1+w_2 \\ &= (s_1+s_2)+(w_1+w_2)\end{aligned}$$
<u>Because</u> S is a <u>subspace</u> of V, $(s_1+s_2) \in S$
<u>and similarly</u> $(w_1+w_2) \in W$.
$\therefore (s_1+w_1)+(s_2+w_2)$ is of the form $s'+w'$ for $s' \in S$, $w' \in W$ and is thus an element in $S \oplus W$.
<u>Identity</u>: The zero vector $\overline{0}$ is in $S \oplus W$ because $\overline{0} \in S$ and $\overline{0} \in W$.
<u>Inverses</u>: Let $(s+w) \in S \oplus W$. $(-s+-w)$ is also in $S \oplus W$ because $-s \in S$ and $-w \in W$. (They are both subspaces.) Since $(s+w)+(-s+-w) = \overline{0}$, each element in $S \oplus W$ will have an inverse in $S \oplus W$.
<u>Associativity and Commutativity</u>: These properties are inherited from V.

(b) Scalar properties (Only the closure under scalar multiplication need be tested.)
Let $(s+w) \in S \oplus W$ and $r \in F$, the scalar field.
Consider $r\cdot(s+w)$: $r\cdot(s+w) = rs + rw$ (because <u>V</u> is a vector space over F) But $rs \in S$ and $rw \in W$ because each are subspaces of V.
$\therefore r\cdot(s+w) = s'+w'$ $(rs = s' \in S,\ rw = w' \in W)$ and so $r\cdot(s+w) \in S \oplus W$.

12. $A \cup B$ is not necessarily a subspace of V whenever A and B are. Consider:
(i) $(R^2,+)$ over R -- the vector space
(ii) $A = \{r(1,0) : r \in R\}$, $B = \{s(0,1) : s \in R\}$ -- the subspaces which happen to be the coordinate axes



(iii) $A \cup B = \{r(1,0),\ s(0,1) : r,s \in R\}$
Now the <u>only</u> elements in $A \cup B$ are those with either first or second component 0. $(0,1) \in A \cup B$ and so is $(1,0) \in A \cup B$. But $(1,0) + (0,1) = (1,1) \notin A \cup B$; i.e., $A \cup B$ is <u>not</u> closed with respect to + and so cannot be a subspace of $(R^2,+)$ over R.

Section 7.7 (pp.400-401)

1. (a) $OA = |A| = \sqrt{14}$; $OB = |B| = \sqrt{14}$; $AB = |A-B| = \sqrt{14}$
   (b) 60° -- triangle is equilateral. Note that
   $$\cos(\angle AOB = \frac{A\cdot B}{|A|\cdot|B|} = \frac{3\cdot 1 + -1\cdot 2 + -2\cdot -3}{\sqrt{14}\cdot\sqrt{14}} = \frac{7}{14} = \frac{1}{2}$$
   $\therefore \angle AOB = 60^\circ$; $\angle OAB = 60^\circ$
2. Only D
3. $|A| = \sqrt{21}$; $B\ (\frac{-2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}})$, $|B| = 1$

   If C is any non-zero vector, the vector $\frac{1}{|C|}C$ will be a unit vector.
4. No, $V = (1,1,2)$ is not a unit vector. A unit vector in the same direction as V is:
   $$U = \frac{1}{|V|}V = \frac{1}{\sqrt{6}}(1,1,2) = (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$$
5. Yes. A unit vector in opposite direction is $(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 0)$
6. (a) A, B, and C generate the x_1,x_2, and x_3-axes, respectively.
   (b) $A\cdot B = 0$, $A\cdot C = 0$, and $B\cdot C = 0$.
   Interpretation: The lines generated by A, B, and C are mutually perpendicular (perpendicular in pairs)
7. $R\cdot Q = (1)(1) + (3)(-1) + (-2)(-1) = 0$
8. $\cos\alpha = \frac{3}{3\sqrt{7}}$
9. $\cos\alpha = 1$ ($\therefore \alpha = 0^\circ$; vectors are collinear)
10. $\cos\alpha = -1$ ($\therefore \alpha = 180^\circ$; vectors are collinear, on opposite sides of (0,0,0)
11. $\cos\alpha = 0$ ($\therefore \alpha = 90^\circ$; lines generated by the two vectors are perpendicular)
12. (a) $\sin\alpha = \frac{\sqrt{57}}{3\sqrt{7}}$ (b) $\sin\alpha = 0$ (c) $\sin\alpha = 0$
    (d) $\sin\alpha = 1$
13. Let $A = (1,1,0)$, $B = (1,1,1)$, $C = (1,0,1)$ and consider ΔABC. There are two ways to proceed:
    Method 1: $AB = 1$, $AC = \sqrt{2}$, $BC = 1$, and since $AB^2+BC^2 = AC^2$ with $AB = BC = 1$, you're done.
    Method 2: $AB = BC = 1$, so ΔABC is at least isosceles. Now $(A-B) = (0,0,-1)$, $(C-B) = (0,-1,0)$ with $(A-B)\cdot(C-B) = 0$ (i.e., the lines generated by $(A-B)$ and $(C-B)$ are perpendicular $\Rightarrow \overleftrightarrow{AB} \perp \overleftrightarrow{BC}$ and we have a right triangle.

    Discuss both methods. In method 2, a diagram would be helpful.
14. Let $A = (a_1,a_2,a_3)$, $B = (b_1,b_2,b_3)$ and $C = (c_1,c_2,c_3)$
    (a) $A\cdot B = a_1b_1+a_2b_2+a_3b_3 = b_1a_1+b_2a_2+b_3a_3 = B\cdot A$
    (b) $A\cdot(B+C) = (a_1,a_2,a_3)\cdot(b_1+c_1,b_2+c_2,b_3+c_3)$
    $= a_1(b_1+c_1)+a_2(b_2+c_2)+a_3(b_3+c_3)$

$$= a_1b_1+a_1c_1+a_2b_2+a_2c_2+a_3b_3+a_3c_3$$
$$= a_1b_1+a_2b_2+a_3b_3+a_1c_1+a_2c_2+a_3c_3$$
$$= A\cdot B + A\cdot C$$

(c) $r(A\cdot B) = r(a_1b_1+a_2b_2+a_3b_3)$
$= ra_1\cdot b_1+ra_2\cdot b_2+ra_3\cdot b_3 = (rA)\cdot B$

(d) $r(A\cdot B) = r(a_1b_1+a_2b_2+a_3b_3)$
$= ra_1b_1+ra_2b_2+ra_3b_3$
$= a_1\cdot rb_1+a_2\cdot rb_2+a_3\cdot rb_3 = A\cdot(rB)$

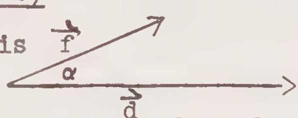
Be sure that students recognize the two different interpretations for the raised dot $(\cdot)$ in the proofs of (a) - (d). It means either multiplication of vectors or multiplication of real numbers.

15. Consider the symbol $(A\cdot B)\cdot C$ where in each instance, $(\cdot)$ means multiplication of vectors (inner product). Since $A\cdot B$ is a scalar, $(A\cdot B)\cdot C$ is not defined. Hence, no matter how you associate, $A\cdot B\cdot C$ makes no sense.

16. Let $A = (-2,4,1)$, $B = (1,5,1)$, and $C=(2,-1,2)$.
Compare $A\cdot(B+C)$ with $A\cdot C + A\cdot C$.
(i) $A\cdot(B+C) = (-2,4,1)\cdot(3,4,3) = -6+16+3 = 13$
(ii) $A\cdot B+A\cdot C = (-2,4,1)\cdot(1,5,1)+(-2,4,1)\cdot(2,-1,2)$
$= 19+(-6) = 13$
$A\cdot(B+C) = A\cdot B + A\cdot C$

Section 7.8 (pp.406-410)

1. The force diagram is



where α takes on the values 0°, 30°, 60°
For $\alpha = 0^\circ$, work is $(20)(20) = 400$ft-lbs
For $\alpha = 30^\circ$, work is $(20)(20)(\cos 30^\circ) \approx 346$ft-lbs
For $\alpha = 60^\circ$, work is $(20)(20)(\cos 60^\circ) = 200$ft-lbs.

2. (a) In $[0,2]$, work = 40 joules; in $[2,6]$ work = 160 joules; in $[6,8]$ work = 0 joules; in $[8,14]$ work = 60 joules.
(b) 260 joules

3. Work $= (50)(3\pi)$ft-lbs ≈ 471ft-lbs

4. $\vec{f}\cdot\vec{d} = 18$ 5. $\vec{f}\cdot\vec{d} = 36+49 = 85$ 6. $\vec{f}\cdot\vec{d} = -2+2+28 = 28$

7. $\vec{f}\cdot\vec{d} = 0$

8. (a) $(3,-2,4)$ (b) $\left(\frac{3}{\sqrt{29}},\frac{-2}{\sqrt{29}},\frac{4}{\sqrt{29}}\right)$ (c) $\{r(3,-2,4): r \in R\}$
Many different answers are possible; the preceeding are the most likely.

9. $x+y+z = -2$ 10. $-3x+2y+4z = 29$

11. (a) $\perp$ x-axis : $x = 0$ (b) $\perp$ y-axis : $y = 2$
(c) $\perp$ z-axis : $z = 4$

12. Let $A = (0,1,3)$ and $B = (-2,3,1)$, then $B-A = (-2,2,-2)$ is a vector normal to the plane whose equation we want.

Equation: $-2x+2y-2z = 18-2\sqrt{2}$ or equivalently,
$x - y + z = \sqrt{2} - 9$

13. Since $(x,y,z)\cdot(-3,4,5) = 0$, $-3x+4y+5z = 0$
There are an infinite number of vectors that satisfy equation; one is $(3,1,1)$ and so $(3,1,1)$ is normal to $\{r(-3,4,5) : r \in R\}$

14. <u>Method 1</u> (Use Vectors)
<u>Let ℓ be</u> the tangent line and ℓ' the line through the origin that is parallel to ℓ
$\ell' = \{(x,y):(x,y)\cdot(-3,4) = 0\}$
$= \{(x,y): -3x+4y = 0\}$
Note that the equation for ℓ must take the form $-3x+4y = d$ (Have students compare with the relationship between $ax+by+cz = 0$ and $ax+by+cz = d$, $d \neq 0$).
To find d, $-3(-3)+4(4) = d$ $\quad 25 = d \quad$ so equation is
$-3x+4y = 25$

<u>Method 2</u> (Use Slope)
<u>Here is</u> a good opportunity to review concept of slope and the use of slope to determine equations <u>in</u> <u>R^2</u>
Slope of line through $(0,0)$ and $(-3,4)$ is $-\frac{4}{3}$;
slope of tangent $= \frac{3}{4}$
Equation:
$$\frac{y-4}{x+3} = \frac{3}{4} \Rightarrow -3x+4y = 25$$

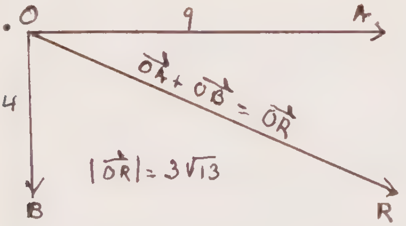
15. $3x+4y+2z = 29$ $\quad$ 16. $\{(x,y,z):x = 1\}$ $\quad$ 17. $\frac{7}{\sqrt{29}}$

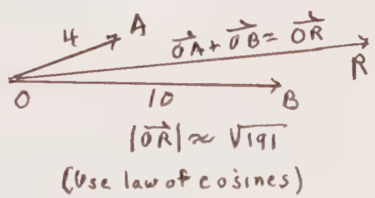
18. $2\sqrt{6}$

19. One way to verify their parallelism is to solve
$$\begin{aligned} 3x - 4y + 12z &= 4 \\ 3x - 4y + 12z &= +2 \end{aligned}$$
The solution set is empty, so planes are parallel. The distance between $(0,0,0)$ and $3x-4y+12z = 4$ is $\frac{4}{13}$; and $3x-4y+12z = 2$ is $\frac{2}{13}$, so the distance between planes is $\frac{4}{13} - \frac{2}{13} = \frac{2}{13}$.

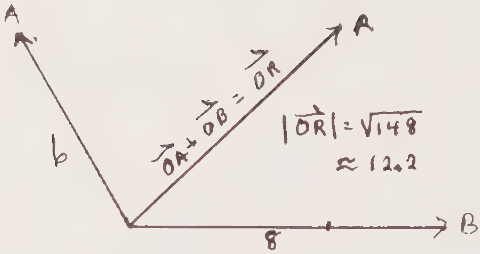
20. $\frac{4}{\sqrt{14}}$

<u>Review Exercises (pp.411-413)</u>

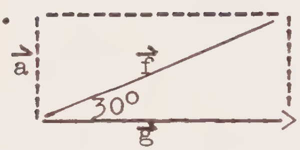
1. 

2. 

3.



4.



$|\vec{a}| = 150$, $|\vec{g}| = 150\sqrt{3}$ so that airplane is gaining altitude at a rate of 150mi/hr while its ground speed is approximately 260mi/hr.

5. (a) $(3,-1,5)$ (b) $(3,3,11)$ (c) -4 (d) -12
(e) 95 (f) not possible

6. $(P_2,+)$ over R is a vector space

7. (a) $\{r(-3,2,1) : r \in R\}$ $x+3z = 0$ and $y-2z = 0$
(b) $\{r(1,3,2)+s(0,1,1) : r,s \in R\}$; $y-z = 0$
(c) $\{(-1,2,4)+r(4,3,-2) : r \in R\}$; $-3x+4y = 11$ and $x+2z = 7$
(d) $\{(1,2,3)+r(-2,-4,-6)+s(3,-6,-1) : r,s \in R\}$;
$8x+5y-6z = 0$

8. $(1,3,4)$ 9. No solutions 10. $\left(\frac{1}{18}, \frac{-3}{2}, \frac{11}{18}\right)$

11. Solution set = $\{(x_1,x_2,x_3) : x_1 = \frac{-1}{5}r + \frac{1}{5}$

$x_2 = \frac{2}{5}r - \frac{7}{5}$

$x_3 = r$

For $r = 1$, $(x_1,x_2,x_3) = (0,-1,1)$ and
for $r = 6$, $(x_1,x_2,x_3) = (-1,1,6)$ are solutions.

12. The equation takes the form $3x-y+z = c$, where c has to be determined.
$3(2) - (-1) + 4 = c$ $11 = c$
Equation: $3x - y + z = 11$

13. (a) $\frac{1}{\sqrt{26}}(4,1,-3) = \left(\frac{4}{\sqrt{26}},\frac{1}{\sqrt{26}},\frac{-3}{\sqrt{26}}\right)$ or its opposite,

$$\left(\frac{-4}{\sqrt{26}},\frac{-1}{\sqrt{26}},\frac{3}{\sqrt{26}}\right)$$

(b) $(4,1,-3)$, $(-4,-1,3)$. In fact, any vector of the form $r(4,1,-3)$, $r \in R$.

(c) $\{(1,1,1)+r(4,1,-3) : r \in R\}$

14. Given line: $\{r(3,-4,1) : r \in R\}$. Note that this line is normal to $3x-4y+z = 0$ and hence is perpendicular to every ~~vector~~ line in that plane. Since $(1,0,-3)$ satisfies $3x-4y+z = 0$, $\{s(1,0,-3) : s \in R\}$ is perpendicular to $\{r(3,-4,1) : r \in R\}$, and hence

$$\{(4,-3,3)+s(1,0,-3) : s \in R\}$$

is a line through $(4,-3,3)$ that is perpendicular to $\{r(3,-4,1) : r \in R\}$. Note that the lines *may* be skew.

15. Work $= \vec{f}\cdot\vec{d} = |\vec{f}|\cdot|\vec{d}|\cos 20^\circ \approx (10)(18)(.9397) \approx 169$

MAINTAINING SKILLS AND UNDERSTANDING - II

Exponential and Logarithmic Functions (pp.415-418)

1. $5^4 = 5^3\cdot 5 = 5^2\cdot 5\cdot 5 = 5^1\cdot 5\cdot 5\cdot 5 = 5\cdot 5\cdot 5\cdot 5$

2. (a) $x > 1$; $x-1 > 0$; $x^2-2x+1 > 0$; $x^2 > x+(x-1)$ or $x^2 > x$

(b) By definition $x^n = x^{n-1}\cdot x$; $x^{n-1}\cdot x > x^{n-1}\cdot 1$ (Since $x > 1$) $\therefore x^n > x^{n-1}$

3. If $0 < x < 1$ then $\frac{1}{x} > 1$; $(\frac{1}{x}-1) > 0$ $\frac{1}{x^2} - \frac{2}{x} + 1 > 0$; $\frac{1}{x^2} > \frac{1}{x} + (\frac{1}{x}-1)$ or $\frac{1}{x^2} > \frac{1}{x} \rightarrow x^2 < x$

(b) $0 < x < 1 \therefore x$ is positive. $x < 1 \therefore x^{n-1}\cdot x < x^{n-1}\cdot 1$ or $x^n < x^{n-1}$

4., 5., 6., See textbook 7. a^7 8. n^9 9. $(a-b)^7$

10. $(-2)^7 = \frac{1}{128}$ 11. $(a\cdot b)^x\cdot b^2$ 12. a^{2n} 13. h^{5+x+y}

14. 1 15. -1 16. C^5 17. 1 18. x^6 19. 2^{12}

20. $(-2)^{15}$ 21. $a^0 = 1$; $a^{-1} = \frac{1}{a}$; $a^{-n} = \frac{1}{a^n}$; rule in text.

22. $\frac{1}{8}$ 23. $\frac{1}{16}$ 24. $\frac{1}{h^3}$ 25. rs 26. $\frac{1}{c}$ 27. $\frac{a^2b^2+1}{a^3b^4}$

28. $\frac{ab}{b-a}$ 29. 4.65×10^{11} 30. 7.3×10^{-7}

31. 1.65076373×10^6 32. 4.72×10^{-14} 33. 20,000,000

34. 0.0000000321 35. .000(26 zeros)000162 36. 50

37. 5×10^{21} 38. 7×10^{13} 39. $25{,}000(1.02)^{12} \approx 31{,}706.00$

40. $x^{\frac{p}{q}} = \sqrt[q]{x^p} = (\sqrt[q]{x})^p$

41. $x > 0$; if n is odd $y = \sqrt[n]{x}$, if n is even $y = \sqrt[n]{x}$ or $-\sqrt[n]{x}$

42. $\sqrt[4]{80} = \sqrt[4]{16\cdot 5} = \sqrt[4]{16}\sqrt[4]{5} = 2\sqrt[4]{5}$ 43. $\sqrt{(-5)^6} = \sqrt{5^6} = 5^3=125$

44. 2 45. 7 46. 3 47. $4\sqrt{2} + \sqrt{3}$ 48. $6\sqrt{3}$

49. $\frac{4\sqrt{12}}{3}$ 50. $\frac{\sqrt[4]{8x^3y^2}}{2xy}$ 51. See text 52. 243 53. 32

54. 9 55. $\frac{1}{216}$ 56. $\frac{1}{b}$ 57. $3\sqrt{3}\sqrt[3]{4}$ or $3\sqrt[6]{432}$ 58. $128g^2x^6$

59.

x	0	1	.5	-.5	-1	2	-2	3	-3	$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{5}{2}$	$-\frac{5}{2}$
2^x	1	2	1.414	.707	.500	4	.250	8	.125	2.82	.353	5.656	.177

60. Figure 4.1, Mapping is one-to-one $|R \to R_+$

61. Consider sequence 1, 1.4, 1.41, 1.414, ...$\to \sqrt{2}$.
Limit 2^x as $x \to \sqrt{2}$ is $2^{\sqrt{2}}$
Consider (2^x) as $2^1, 2^{1.4}, 2^{1.41}, 2^{1.414}, \ldots \to 2^{\sqrt{2}}$
$2^{\frac{5}{4}} = 2.38$; $2^{\frac{3}{2}} = 2.82$; $2^{\sqrt{2}} \approx 2.665$

62.

x	-3	$-2\frac{1}{2}$	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
3^x	.037	.064	.111	.192	.333	.577	1	1.732	3	5.19	9.0	15.6	27
3^{-x}	27	15.6	9.0	5.19	3	1.73	1	.517	.333	1 92	.111	.064	.037

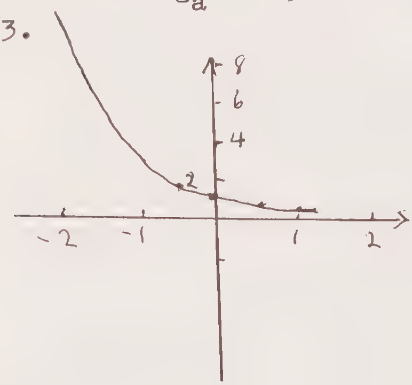
(*)
67. (a) $A = 1000e^{.06} = 1061.80$ (b) $A = 1000(1.015)^4 = 1061.36$

68. $A = 100 \cdot e^{10(-.038)} = 68.3861$

69. (a) See Ex. 60 (b) Mapping is one-to one
(c) Inverse is reflection about $y = x$
$(x,y) \to (y,x)$ yields inverse $x = 2^y$; $y = \log_2 x$

70. $a^x = N \to \log_a N = 4$, hence $a^{\log_a N} = N$

(*)63.



60.

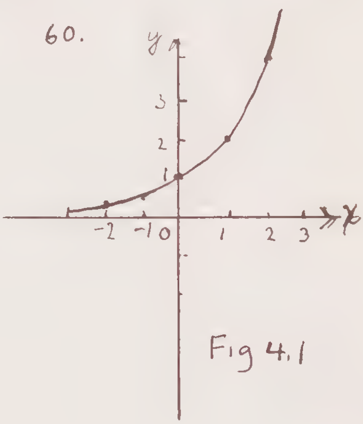
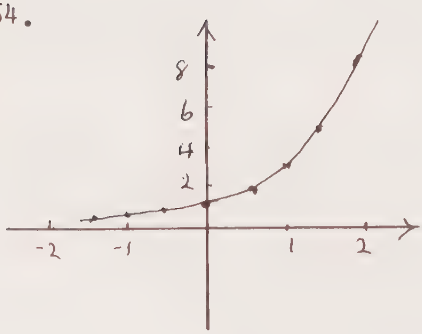


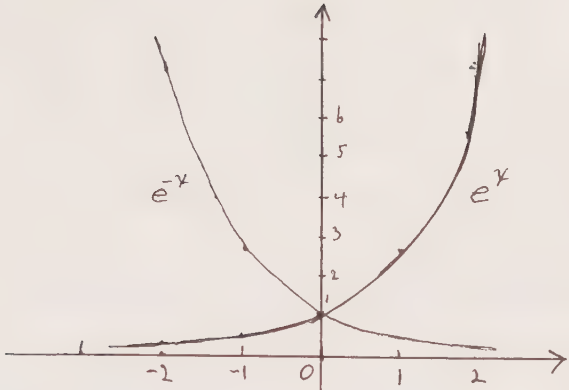
Fig 4.1

64.



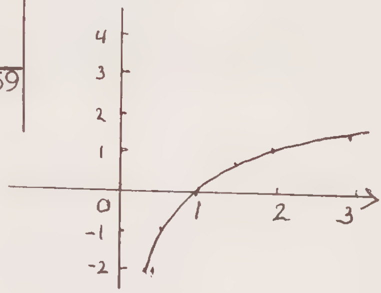
65. $R \to R^+$

66.



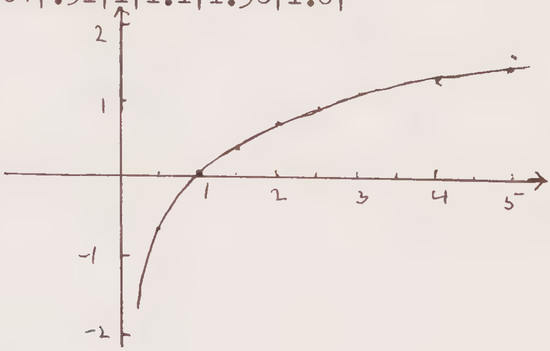
71. (a)

x	.5	1	1.5	2	2.5	3
y	-1	0	.587	1	1.33	1.59



(b)

x	.5	1	1.5	2	2.5	e	3	4	5
y	-.69	0	.40	.64	.91	1	1.1	1.38	1.6



72. Make a table

x	2^x
2	4
2.25	4.756
?	5.000
2.50	5.657

$\frac{C}{.25}$ $\frac{244}{901}$

C = .067

X = 2.25+.067 = 2.317

73. $10^{\frac{1}{2}} = x$, $x = 3.162$ 74. $5^{\frac{3}{2}} = x$, $x = 11.18$

75. $x^2 = 16$, $x = 4$ 76. $7^x = 7^{-2}$; $x = -2$

77. $4^{\frac{3}{2}} = x$; $x = 8$ 78. $2^x = \frac{1}{16} = 2^{-4}$; $x = -4$ 79. 1

80. 2   81. 0   82. -1 or 9-10   83. -3 or 7-10

84. 1.86332; 2.97635; 0.61805; 9.80754-10; 7.26482-10

85. $\overline{3}.9956$ 86. $\overline{1}.3319$ 87. log x=(log8.41+log69.7)-$\frac{1}{2}$log4.58

88. 65,335   89. 544.3   90. 1.465   91. 19.04

92. $\log\frac{a}{b}=(\log a)-(\log b)$; $\frac{\log a}{\log b} = (\log a)\div(\log b)$

93. $x \log 34.56 = \log 2.45 \quad x = \frac{.38917}{1.53857} \approx .252$

94. See slide rule manual

<u>Circular Functions: Periodic Models (pp.420-423)</u>

1. $v = at$; $v = r\omega$; $at = r\omega$, $\frac{\omega}{t} = \alpha \therefore a = r\alpha$

2. $r = 3\text{ft}$; $\omega = 1200\pi\frac{\text{rad.}}{\text{min.}}$; $v = 3\cdot 1200\pi\frac{\text{ft.}}{\text{min.}} = \frac{3600\pi\text{ft.}}{\text{min.}}$

3. $x\frac{\text{mi.}}{\text{hr.}}\cdot 5280\frac{\text{ft.}}{\text{mi.}} \times \frac{1\text{hr.}}{3600\text{sec.}} = \frac{22x\text{ft.}}{15\text{sec.}}$ To change a number of miles per hour to a number of feet per second multiply by $\frac{22}{15}$.

4. $r = \frac{13}{12}\text{ft.}$; $v = 52\cdot\frac{22\text{ft.}}{15\text{sec.}}$; $\omega = (52 \times \frac{22}{15})\div\frac{13\text{rad}}{12\text{sec}} \approx 11.2\text{RPS}$

5. Larger, $\omega = \frac{48\text{rad}}{1\text{sec}}$; Smaller, $\omega = \frac{48}{\frac{2}{3}} = 72\frac{\text{rad}}{\text{sec}}$

6. $\frac{v}{\omega} = r \quad \frac{a}{\alpha} = r \therefore \frac{v}{\omega} = \frac{a}{\alpha}$

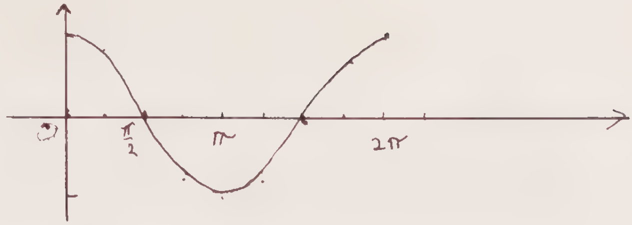
7. $\omega = 72\text{RPS} = 72.2\pi\frac{\text{rad}}{\text{sec}}$; $v_1 = 6\times 144\pi\frac{\text{ft.}}{\text{sec.}} = 864\pi\frac{\text{ft.}}{\text{sec.}}$
$v_2 = 4\cdot 144\pi\frac{\text{ft.}}{\text{sec.}} = 576\pi\frac{\text{ft.}}{\text{sec.}}$

	Degrees	Radians	Rev.
8.	160	$\frac{8}{9}\pi$	$\frac{4}{9}$
9.	114.59	2	$\frac{1}{\pi}$
10.	2700	15π	$7\frac{1}{2}$
11.	30	$\frac{\pi}{6}$	$\frac{1}{6}$
12.	1152	$\frac{32\pi}{5}$	$3\frac{1}{5}$
13.	1280	$\frac{64\pi}{9}$	$3\frac{5}{9}$
14.	236.16	4.12	$\frac{2.06}{\pi}$
15.	22.5	$\frac{\pi}{8}$	$\frac{1}{16}$

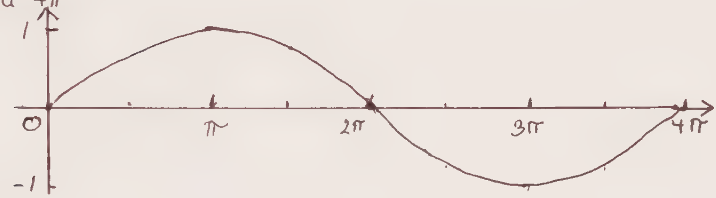
16. $\frac{200 \times 12}{5\pi} \approx 153$ coils

17. (a) 210°, 330° or $\frac{7}{6}$rad, $\frac{11}{6}$rad (b) 3.6646, 5.7596

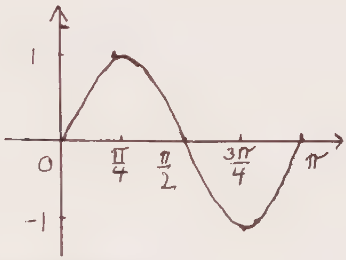
18. Period 2π



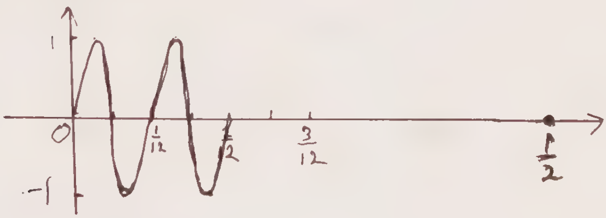
19. Period 4π



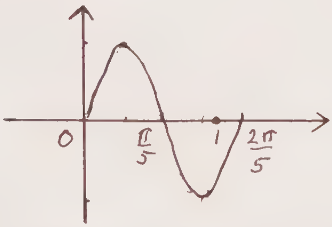
20. Period π



21. Period $\frac{1}{12}$, frequency 12.



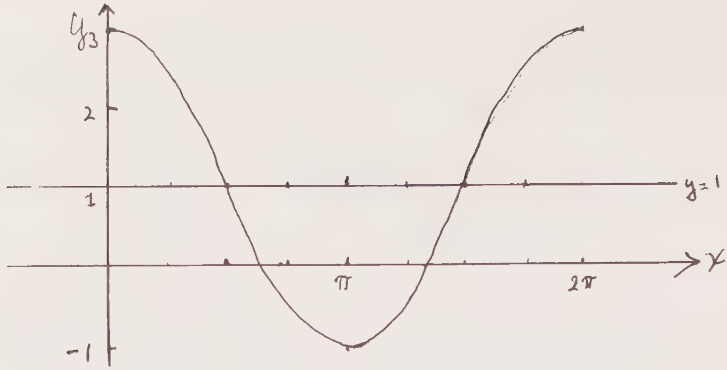
22. Period $\frac{2}{5}\pi$



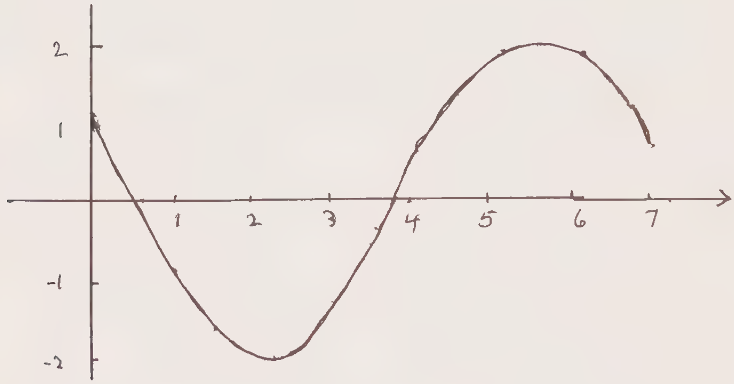
23. See Text   24. Use $\sin(x-y) = \sin(x+(-y))$.

25. $\cos 4x = \cos(2x+2x)$; $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

26.



27.



28. Amplitude 6; period $\frac{1}{60}$; frequency 60

29. $\sin 22\frac{1}{2}^{\circ} = \sqrt{\dfrac{1-\frac{\sqrt{2}}{2}}{2}} = \frac{1}{2}\sqrt{2-\sqrt{2}} \approx .382$

30. $\frac{32}{\sqrt{2}}(\cos 45^{\circ}+i\ \sin 45^{\circ})$ 31. $16(\cos 300^{\circ}+i\ \sin 300^{\circ})$

32. $(a^2+b^2)[\cos(\tan^{-1}\frac{b}{a}) + i\ \sin(\tan^{-1}\frac{b}{a})]$

33. $6\sqrt{2} + 6\sqrt{2}i$ 34. $2.588 + 9.659i$ 35. $a\cos\theta + ia\sin\theta$

36. $17.83 + 16.06i$ 37. $r = 27$, $\sqrt[3]{r} = 3$, $\frac{1}{3}\theta = 72^{\circ}$;
roots are: $3(\cos 72^{\circ}+i\ \sin 72^{\circ})$, $3(\cos 192^{\circ}+i\ \sin 192^{\circ})$, $3(\cos 312^{\circ}+i\ \sin 312^{\circ})$

38. $(\cos 30^{\circ}+i\ \sin 30^{\circ})$; $(\cos 150^{\circ}+i\ \sin 150^{\circ})$; $(\cos 270^{\circ}+i\ \sin 270^{\circ})$

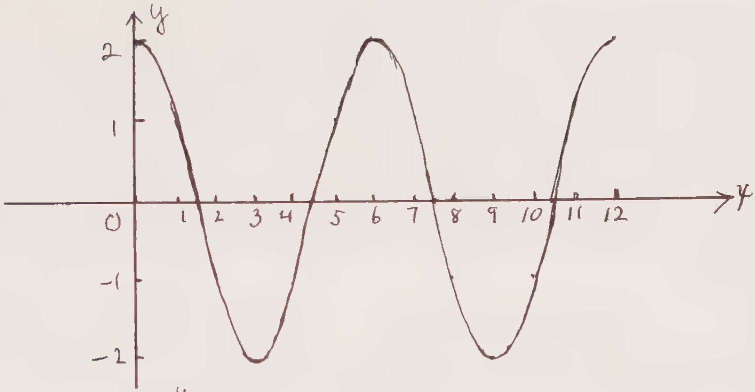
39. See textbook   40. Replace tan x by $\frac{\sin x}{\cos x}$

41. $\sec^2 x+\csc^2 x = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \frac{1}{\cos^2 x\cdot\sin^2 x} = \sec^2 x\cdot\csc^2 x$

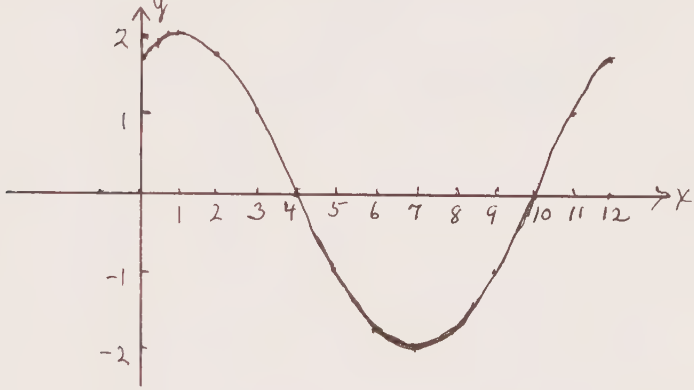
42. Replace cot x by $\frac{\cos x}{\sin x}$ 43. $30^{\circ}, 150^{\circ}, 210^{\circ}, 330^{\circ}$

44. 45° 45. $\emptyset$

46.



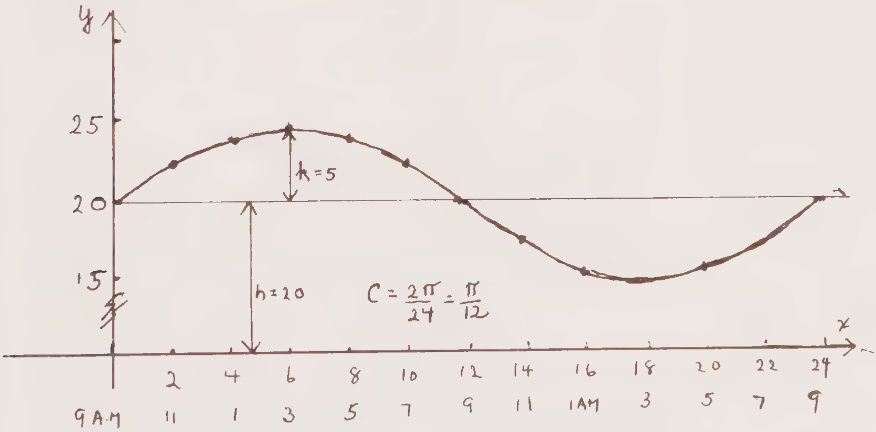
47.



48. $y = k \sin 880\pi t$

49. $h = 20$; $k = 5$; period $= 24$; frequency $= \frac{1}{24}$;

$y = 20 + 5\sin\frac{\pi x}{12}$



Conditional Probability (pp.423-426)

1. The outcome space is that of throwing two dice. Let x be any integer $1 \leq x \leq 6$. Then x occurs 11 ways as one element of the pair, 1 time as a double, and it fails to occur 24 ways. P(neither die shows x) $= \frac{24}{36}$;
   P(one die shows x) $= \frac{11}{36}$; P(both dice show x) $= \frac{1}{36}$
   [$\frac{24}{36}(-1) + \frac{11}{36}(1) + \frac{1}{36}(2) = -\frac{11}{36}$ dollars is the expectation.]

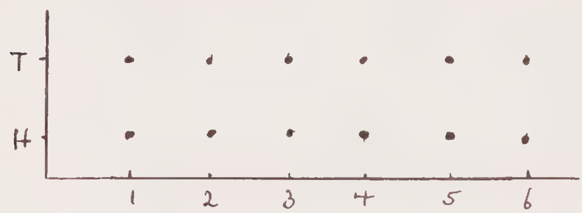
2. 120 3. 720 4. 1 5. 56 6. 126

7. Use $1! = 1$ and $n! = n\cdot(n-1)!$ 8. Use Cartesian Product

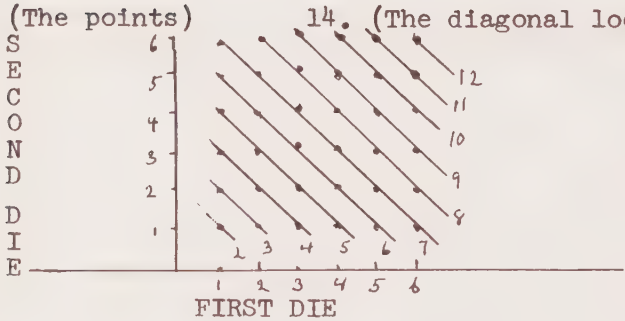
9. See textbook 10. Use $\binom{n}{r}r! = (n)_r$

11. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$; Let r be n-r;
   $\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$

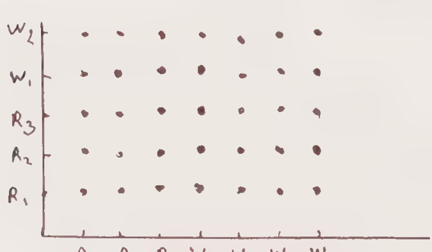
12.



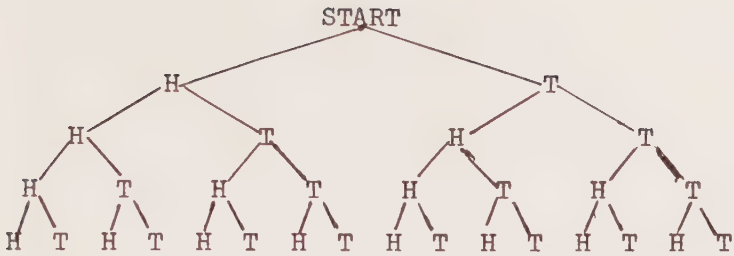
13. (The points) 14. (The diagonal loops)



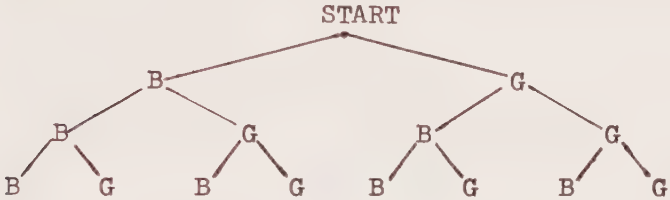
15.



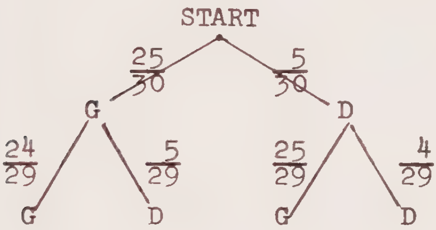
16.



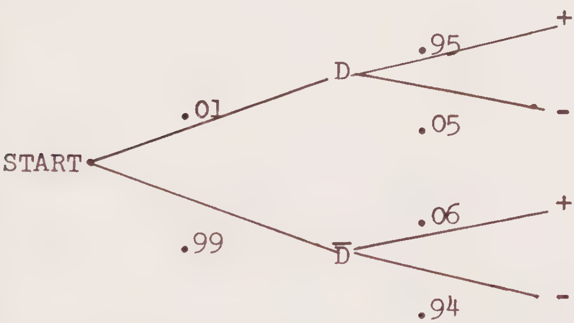
17.



18.



19.



20. $A = (A \setminus B) \cup (A \cap B)$; $A \setminus B$ and $A \cap B$ are disjoint;
$P(A) = P(A \setminus B) + P(A \cap B)$ or $P(A \setminus B) = P(A) - P(A \cap B)$

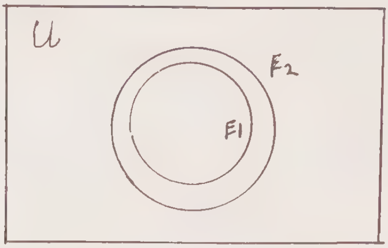
21. $A \cup B = (A \setminus B) \cup B$; $A \setminus B$ and B are disjoint;
$P(A \cup B) = P(A \setminus B) + P(B)$; now substitute from Ex. 20

22. $\mathcal{P}(S)$ is set of all subsets.
$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\{, \{a,b,c\}\}$

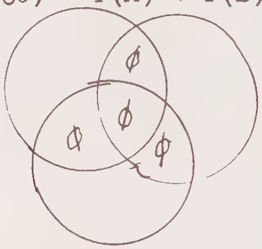
23. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$
Use $(a+b)^n$ with $a = 1$, $b = 1$ to prove this.

24. $(p+q)^6 = \binom{n}{0}p^6 + \binom{n}{1}p^5q + \binom{n}{2}p^4q^2 + \binom{n}{0}p^3q^3 + \binom{n}{4}p^2q^4 + \binom{n}{5}pq^5$
$+ \binom{n}{6}q^6$ with $n = 6$

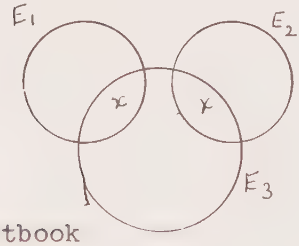
25. $E_1 \subset E_2 \Rightarrow n(E_1) \leq n(E_2)$. $\frac{n(E_1)}{n(U)} \leq \frac{n(E_2)}{n(U)}$; $P(E_1) \leq P(E_2)$



26. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C)$
$+ P(A \cap B \cap C)$.
$A \cap B = A \cap C = B \cap C = \emptyset \Rightarrow A \cap B \cap C = \emptyset$.
Thus $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

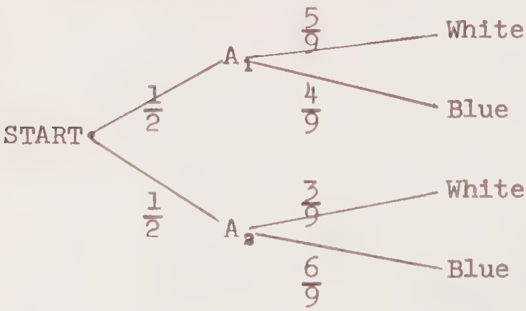


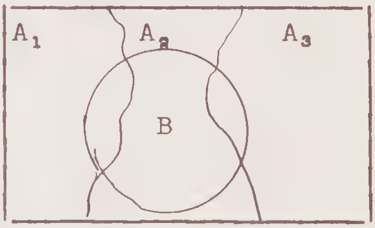
27. $E_1 \cap E_3 \neq \emptyset$, $E_2 \cap E_3 \neq \emptyset$, but $E_1 \cap E_2 \cap E_3 = \emptyset$



28. See Textbook

29. A_1 = Event Jar A is selected
A_2 = Event Jar B is selected
B = Event Ball is Blue
W = Event Ball is White





(6) $P(B) = P(E_1)\cdot P(B|E_1)+P(E_2)\cdot P(B|E_2)$

$$= \frac{1}{2}\cdot\frac{4}{9} + \frac{1}{2}\cdot\frac{6}{9} \text{ or } \frac{5}{9}$$

30. (See Ex. 9) $P(E_2|W) = \dfrac{P(E_2)\cdot P(W|E_2)}{P(E_1)\cdot P(W|E_1)+P(E_2)\cdot P(W|E_2)}$

$$= \frac{\frac{1}{2}\cdot\frac{3}{9}}{\frac{1}{2}\cdot\frac{5}{9} + \frac{1}{2}\cdot\frac{3}{9}} = \frac{\frac{3}{18}}{\frac{8}{18}} \text{ or } \frac{3}{8}$$

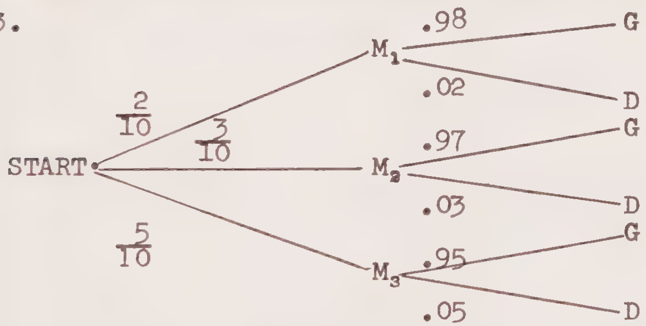
31. See Figure of Exercise 18.
F_1 = Event first tube is good $P(F_1) = \frac{25}{30}$
F_2 = Event second tube is good
B = Event second tube is bad

$P(F_2|F_1) = \frac{24}{29} \therefore P(F_1 \cap F_2) = \frac{5}{6}\cdot\frac{24}{29} = \frac{20}{29}$

$P(B|F_1) = \frac{5}{29} \therefore P(F_1 \cap B) = \frac{5}{6}\cdot\frac{5}{29} = \frac{25}{174}$

32. See Figure of Exercise 29.
$P(B) = P(A_1)\cdot P(B|A_1)+P(A_2)\cdot P(B|A_2)+P(A_3)\cdot P(B|A_3)$

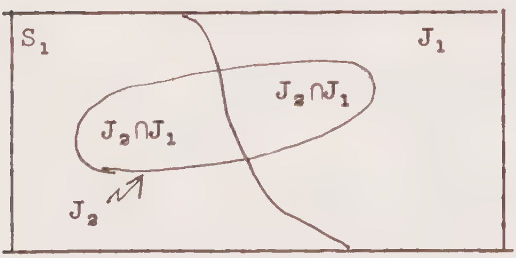
33.



D = event item is defective
M_i = event item is from machine i

$P(D) = P(M_1)\cdot P(D|M_1)+P(M_2)\cdot P(D|M_2)+P(M_3)\cdot P(D|M_3)$
$= (.2)(.02)+(.3)(.03)+(.5)(.05) = .038$

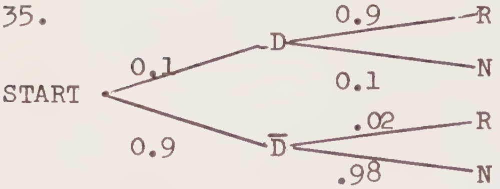
34.



S_1 = Event Sr is selected first
J_1 = Event Jr is selected first
J_2 = Event Jr is selected second

$$P(J_2) = P(S_1)\cdot P(J_2|S_1) + P(J_1)\cdot P(J_2|J_1)$$
$$= \frac{5}{8}\cdot\frac{3}{7} + \frac{3}{8}\cdot\frac{2}{7} = \frac{21}{56}$$

35.

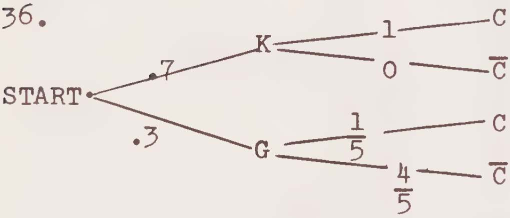


D = Event has the disease $\overline{D}$ = Event does not have disease
R = Event X-ray shows +
N = Event X-ray shows negative

$$P(R) = P(D)\cdot P(R|D)+P(\overline{D})\cdot P(R|\overline{D})$$
$$= .1\cdot .9+.9\cdot .02 = .108$$

$$P(D|R) = \frac{P(D)\cdot P(R|D)}{P(R)} = \frac{(1.)(.9)}{.108} \text{ or } \frac{5}{6}$$

36.



K = Event student knows, C = Event response is correct
G = Event student guesses,
$\overline{C}$ = Event response is incorrect

$$P(C) = P(K)\cdot P(C|K) + P(G)\cdot P(C|G)$$
$$= .7\cdot .1 + .3\cdot\frac{1}{5} = \frac{38}{50}$$

$$P(G|C) = \frac{P(G)\cdot P(C|G)}{P(C)} = \frac{.3\cdot\frac{1}{5}}{\frac{38}{50}} = \frac{3}{38}$$

37. See textbook 38. See textbook

39. (a) $P(S) = \frac{2}{3}$, $P(C) = \frac{1}{3}$, $P(S\cap C) = \frac{90}{300}$

$\therefore P(S)\cdot P(C) \neq P(S\cap C)$; not independent

(b) $P(C|\overline{S}) = \dfrac{P(C\cap\overline{S})}{P(S)} = \dfrac{\frac{10}{300}}{\frac{100}{300}} = \dfrac{1}{10}$

(c) $P(C|S) = \dfrac{P(C\cap S)}{P(S)} = \dfrac{\frac{90}{300}}{\frac{100}{300}} = \dfrac{9}{10}$

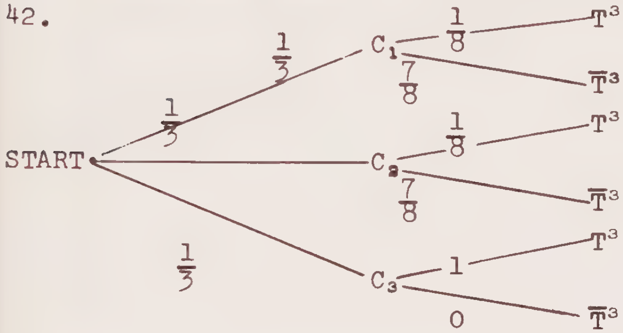
40.

	G	$\overline{G}$	Totals
M	105	45	150
$\overline{M}$	70	30	100
	175	75	250

41. $P(A) = \frac{1}{4}$; $P(B) = \frac{1}{13}$; $P(C) = \frac{2}{52}$

$P(A \cap C) = \frac{2}{52}$; $P(B \cap C) = \frac{4}{52}$; A and B, and A and C are independent; B and C are not independent.

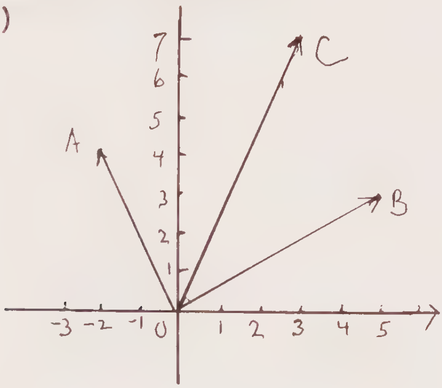
42.



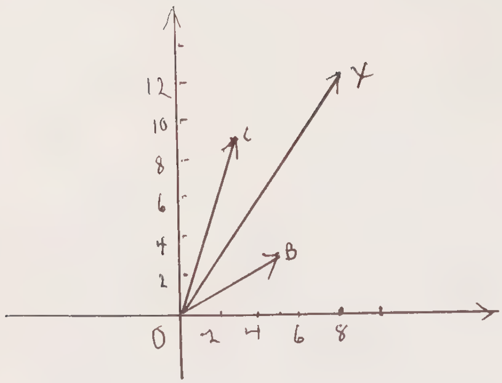
Find $P(C_3 | T^3) = \dfrac{\frac{1}{3}\cdot 1}{\frac{1}{3}\cdot\frac{1}{8} + \frac{1}{3}\cdot\frac{1}{8} + \frac{1}{3}\cdot 1}$ or $\frac{4}{5}$

Vector Spaces (pp.427-431)

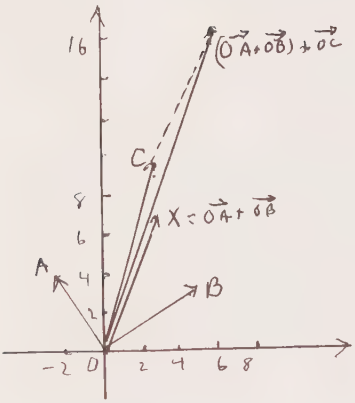
1. (3,7)



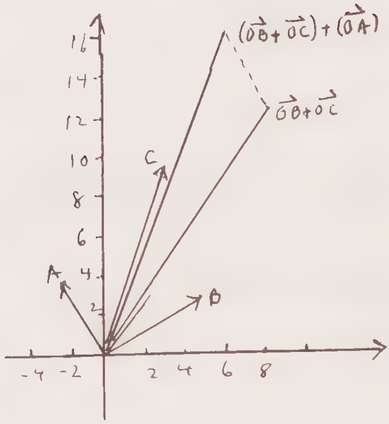
2. (8,12)



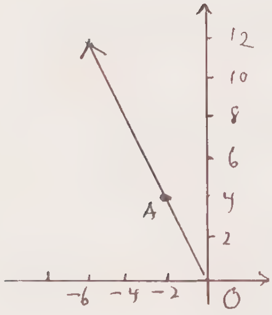
3. (6,16)



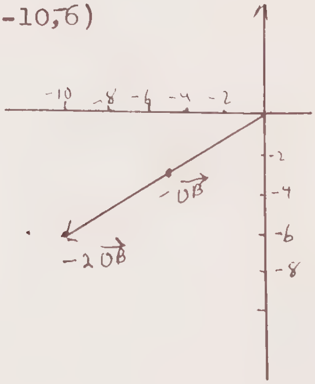
4. (6,16)



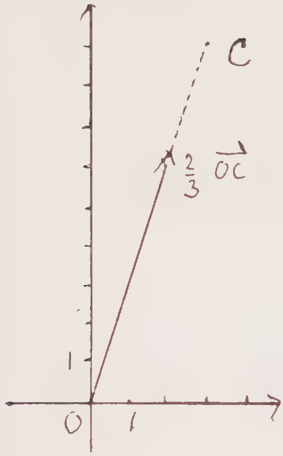
5. (-6,12)



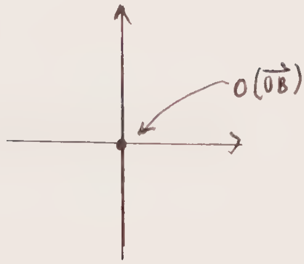
6. (-10,-6)



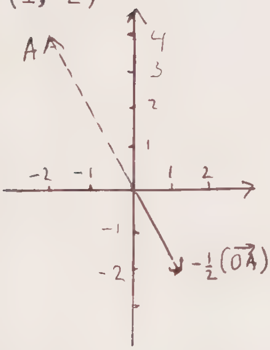
7. (2,6)



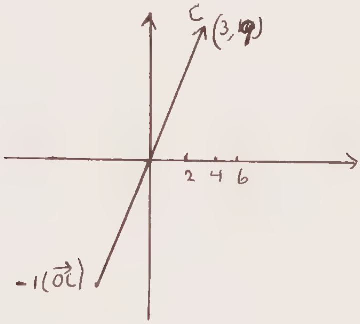
8. (0,0)



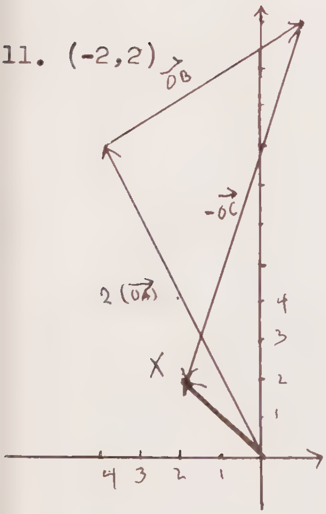
9. (1,-2)



10. (-3,-9)



11. (-2,2)



12. $(-1,\ 5,\ 7)$

13. $(-1,5,7)+(3,-1,2)=(2,4,9)$

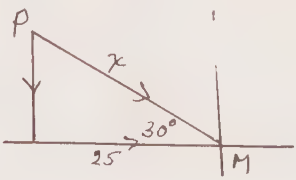
14. $(1,2,3)+(1,2,6) = (2,4,9)$

15. $(2,4,6)$

16. $(1,\ \frac{-3}{2},-\ 2)$

17. $(3,-1,2)-(4,6,8)=(7,-7,-6)$

18. $\cos 30^{\circ} = \frac{25}{x}$, $x \approx 28.87$



19. $\sin 20^\circ = \frac{x}{300}$, $x \approx 102.6$

20. $(x,y) \to x+3, y+3$; $(3,3)$

21. $(x,y) \to x-4, y+4$; $(-4,4)$

22. $(x,y) \to x-2, y+10$; $(-2,10)$

23. $(x,y) \to (x+(c-a)), (y+(d-b)$; $(c-a, d-b)$

24. $\overrightarrow{OP}+\overrightarrow{OQ} = (x_1+x_2, y_1+y_2)$; $r(\overrightarrow{OP}+\overrightarrow{OQ}) = (rx_1+rx_2, ry_1+ry_2)$;
$r\overrightarrow{OP} = (rx_1, ry_1)$; $r\overrightarrow{OQ} = (rx_2, ry_2)$
$r\overrightarrow{OP}+r\overrightarrow{OQ} = (rx_1+rx_2, ry_1+ry_2)$

25. 26. 27. Follow same procedure as in Exercise 24.

28. Let ax^2+bx+c; dx^2+ex+f be any two polynomials with real coefficients. $\therefore$ their sum is of the same form. The identity element is the 0 polynomial.

29. Use (a,b,c) and (d,e,f) to represent the two polynomials and proceed as in Exercises 24-27.

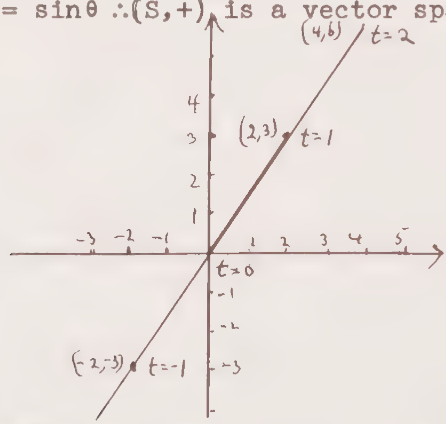
30. See Textbook.

31. See Textbook. (b) $V+r$ has no meaning; $(r+s)+V$ has no meaning.

32. No, it is a real number.

33. $a,s \in R$, $\sin\theta = A$, $\sin\phi = B$, then
(i) $r(\sin\phi+\sin\theta) = r\sin\theta+r\sin\phi$
(ii) $(r+s)\sin\theta = r\sin\theta + s\sin\theta$
(iii) $(r\cdot s)\sin\theta = r\cdot(s\sin\theta)$
(iv) $1\cdot\sin\theta = \sin\theta$ $\therefore (S,+)$ is a vector space.

34. (a)



(b) $t_1(2,3) = (2t_1, 3t_1) = (x_1, x_2)$ $\therefore$ $x_1 = 2t_1$, $x_2 = 3t_2$
$\frac{x_1}{x_2} = \frac{2}{3}$ or $3x_1 = 2x_2$

35. $T_1+T_2 = T_3$; $-T_1+T_1 = 0$, $T_1+T_2 = T_2+T_1$
$T_1+(T_2+T_3) = (T_1+T_2)+T_3$; Abelian Group.

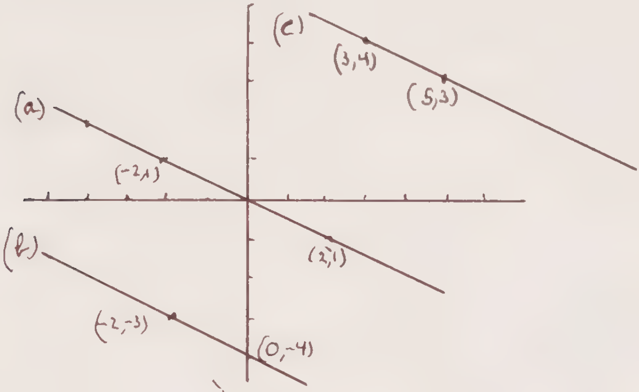
(See Figure of Exercise 34)

36. Q is a line in 3-space determined by origin (0,0,0) and (1,3,2) [r= 1].

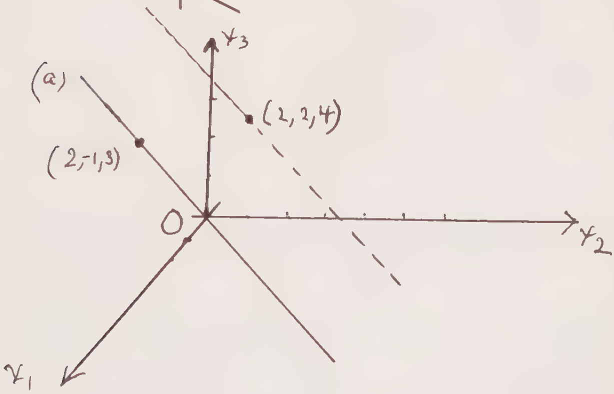
37. Yes (6,2,-4)   38. Yes (14,21)   39. No

40. Yes (16,-10,-2)   41. Yes (32,-18,-10)

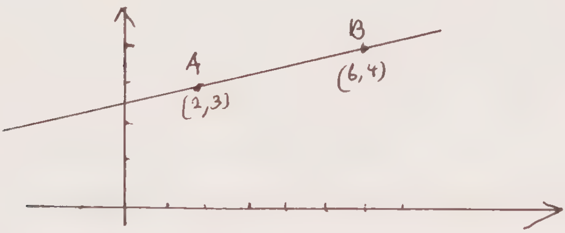
42. Lines are parallel



43.



44. r = 0, (2,3); r = 1, (6,4)



45. $\overline{AB} = \{(x_1,x_2) = r(4,1) + (2,3);\ 0 \leq r \leq 1\}$

$\overline{AB} = \{(x_1,x_2) = r(4,1) + (2,3);\ 0 \leq r\}$

$\overline{BA} = \{(x_1,x_2) = r(4,1) + (2,3);\ 1 \geq r\}$

Mid Point AB = $\left(\frac{2+6}{2}, \frac{3+4}{2}\right) = (4, 3\frac{1}{2})$ or

$\{x_1,x_2\} = \{r(4,1) + (2,3);\}\ r = \frac{1}{2}$

46. $B-A = (+3,+1)\quad \ell = \{(x_1,x_2) = t(3,1) + (2,3)\}$

47. $x_1 = 3t+2 \quad x_2 = t+3$

48. $B-A = (-4,3,-3);\quad \ell = \{(x_1,x_2,x_3)=t_1(-4,3,-3)+(4,-1,-2)\}$

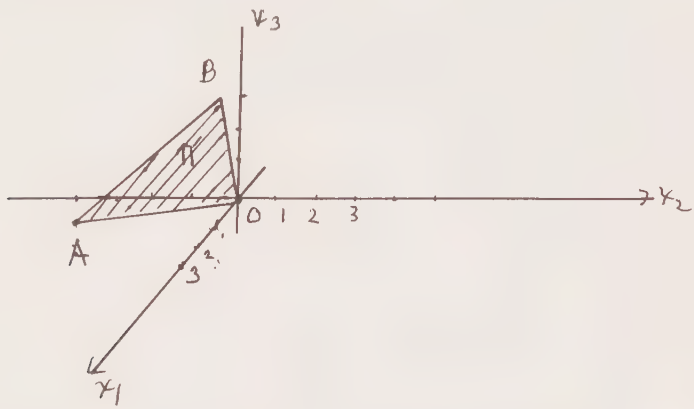
$\ell = \{(x_1,x_2,x_3)=t_2(-4,3,-3)+(0,2,-5)\}$

$t_1 = n \rightarrow t_2 = n-1$

49. Point $(1,2,3)$ is not on $\ell = \{(x_1,x_2,x_3) = r(2,1,3)\}$

50. See Textbook. $\{r(2,-3,1)\}$

51.



$\pi = \{(x,-3) + s(3,1,4)\}$ or $\{(x_1,x_2,x_3) = (2r+3s,-3r+s,$ $r+4s\}$ $x_1 = 2r+3s;\ x_2 = -3r+s;\ x_3 = r+4s.$

52. $(B-A) = (1,-2,1)$; C is not on $\overleftrightarrow{AB}$; $(C-A) = (3,-1,-1)$
Then $\pi = \{r(1,-2,1) + s(3,-1,1) + (0,2,1)\}$. Not a vector or plane.
Note: $(0,2,1)$ may be replaced by $(1,0,2)$ or $(3,1,0)$.

53. (a) $x_1 = 3r+1;\ x_2 = r+3;\ \{(x_1,x_2) = r(3,1) + (1,3)\}$
(b) $\{(x_1,x_2,x_3) = r(1,1,-5) + (7,0,-4)\}$
(c) $\{(x_1,x_2,x_3) = r(3,1,2) + s(-2,1,3) + (1,-3,2)\}$

54. $\pi = \{(x_1,x_2,x_3) = r(-2,1,2) + t(0,2,3) + (3,3,3)\}$

55. $\pi = \{(x_1,x_2,x_3) = r(1,4,-3) + s(-2,8,1)\}$

56. $\pi = \{(x_1,x_2,x_3) = r(3,2,1) + s(1,5,-4)\}$

57. $x_1 = -4r+1,\ x_2 = r+1$; eliminate r; $x_1+4x_2-5 = 0$

58. $x_1 = -s+3t+1;\ x_2 = 2s+t;\ x_3 = s-2;\ x_1-3x_2+7x_3+13 = 0$

59. $x_1-x_2-1 = 0$

60. $\{(x_1,x_2,x_3) = r(3,1,2) + s(1,0,-1)\}$;
$x_1 = 3r+s,\ x_2 = r,\ x_3 = 2r-s;\ x_1-5x_2+x_3 = 0$

61. See Textbook, Section 7.5

62. See Textbook

63. $x_1 = r,\ x_2 = \frac{3}{2}r + \frac{1}{2}$

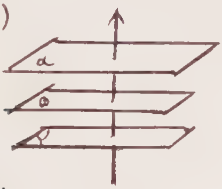
64. $x_1 = 2r-5;\ x_2 = r;\ x_3 = r+1$

65. $x_1 = r;\ x_2 = s;\ x_3 = 3-r-s$

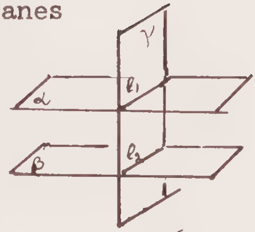
66. $x_1 = 3r+3;\ x_2 = r;\ x_3 = 5r+1$

67. Assume α, β, γ are distinct planes

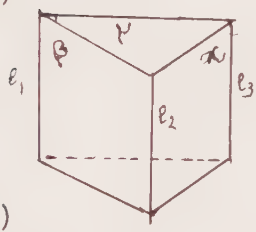
(a)



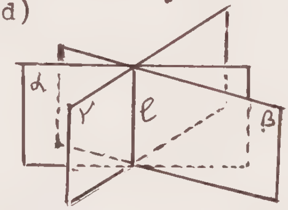
(b)



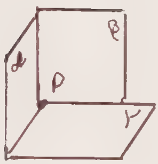
(c)



(d)



(e)



(a) 3 parallel planes
(b) 2 parallel lines of intersection
(c) 3 parallel lines of intersection
(d) 1 line of intersection
(e) 1 point of intersection

68. $x_1 = 2;\ x_2 = 2;\ x_3 = 2$. Intersect in a point.

69. $5x_1-2x_1-3x_3 = 0;\ 6x_1+x_2+3x_3 = 0$
$\{(0,0,0),\ x_1 = k,\ x_2 = 11k,\ x_3 = \frac{17}{3}k\}$
Intersection is a vector line.

70. First 3 equations yield $x = 7$, $y = 0$, $z = -3$. Does not satisfy the fourth equation. Fourth plane intersects at other points $\therefore$ No solution.

71. R^3, R^2, R^1 and the origin R^0. (Three space, planes, lines, points)

72. (a) -11 (b) -11 (c) 32 (d) -14

73. $B-A = (-4,6,2);\ C-B = (8,6,-2);\ (B-A)\cdot(C-B) = 0$
$\therefore AB \perp BC$
$D-C = (4,-6,-2);\ D-A = (8,6,-2);\ CD \parallel AB,\ AD \parallel CB,\ CD \perp DA$
$AB = \sqrt{(-4)^2+6^2+2^2} = \sqrt{56};\ BC = \sqrt{8^2+6^2+(-2)^2} = \sqrt{104}$
$K = \sqrt{56}\cdot\sqrt{104} = 8\sqrt{91} \approx 76.312$

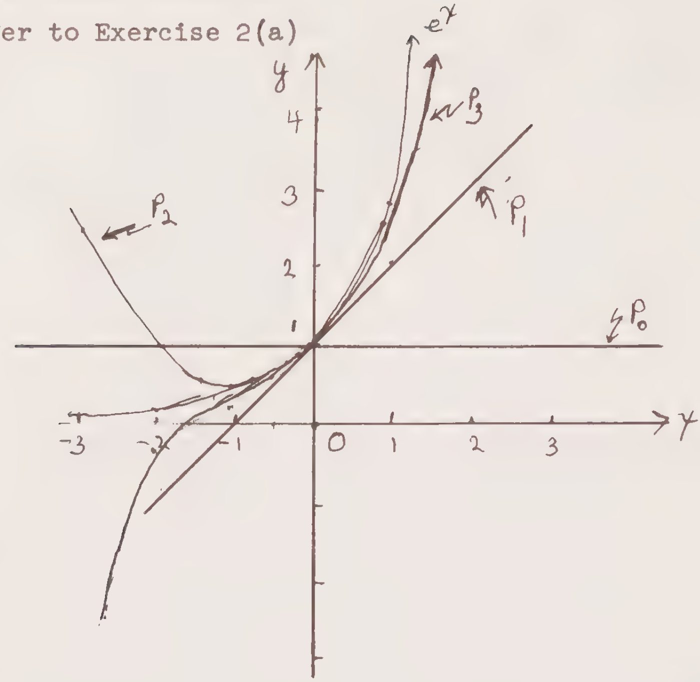
74. From Ex. 18 $\vec{f} = \frac{50\sqrt{3}}{3} = 28.9$ lb. $W = \vec{f}\cdot d = 28.9 \times 50$ ft.lb. ≈ 1445 ft.lb.

EPILOGUE

Section E.1 (pp.435-436)

1. Refer to Exercise 2(a)

2. (a)



(b)

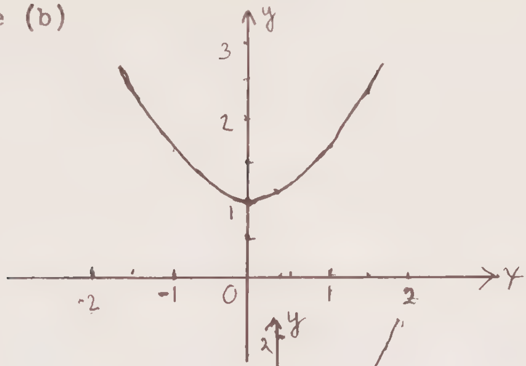
x	$P_0(x)$	$P_1(x)$	$P_2(x)$	$P_3(x)$	e^x
-2	1	-1	1	-.3333	0.1353
-1	1	0	.5	.3333	0.3679
-.5	1	.5	.625	.6042	0.6065
0	1	1	1	1	1
.5	1	1.5	1.625	1.6458	1.6487
1	1	2	2.5	2.6667	2.7183
2	1	3	5	6.3333	7.3891

(c) The table suggests that as n gets large, $P_n(x)$ approximates e^x quite well. If your students have access to a computer terminal, they can investigate $|P_n(x)-e^x|$ for various values of x and increasing n. Geometrically, for n "large," the graph of P_n "hugs" the graph of $f:x \to e^x$. An optional activity is to have students graph $f:x \to e^x$, P_0, P_1, P_2, P_3, ... on the same set of axes.

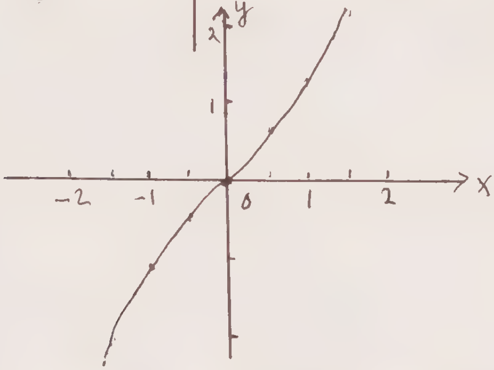
3. Let $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$; $P_3(3.14) \approx 14.230$

From the table, $22.198 < e^{3.14} < 24.533$ so $P_3(x)$ is NOT a good approximation to e^x for $x = 3.14$ (More terms are required--have students compare graph of $y = e^x$ with that of P_4 from Exercise 2.)

4. (a) See (b)
   (b)



5.



6. For a -- f use $\sin h(x) = \dfrac{e^{x}-e^{-x}}{2}$ and $\cos h(x) = \dfrac{e^{-x}+e^{-x}}{2}$

(a) $\cos h(-x) = \dfrac{e^{-x}+e^{-(-x)}}{2} = \dfrac{e^{-x}+e^{x}}{2} = \cos h(x)$

This compares with $\cos(-x) = \cos(x)$

(b) $\sin h(-x) = \dfrac{e^{-x}-e^{-(-x)}}{2} = \dfrac{e^{-x}-e^{x}}{2} = -\dfrac{e^{x}-e^{-x}}{2}$

$= -\sin h(x)$

This compares with $\sin(-x) = -\sin x$

(c) $\cos h^2(x) - \sin h^2(x) = \left(\dfrac{e^{x}+e^{-x}}{2}\right)^2 - \left(\dfrac{e^{x}-e^{-x}}{2}\right)^2$

$= \dfrac{e^{2x}+2+e^{-2x}}{4} - \dfrac{e^{2x}-2+e^{-2x}}{4} = 1$

This compares with $\cos^2 x - \sin^2 x = \cos 2x$ <u>or</u>
$\cos^2 x + \sin^2 x = 1$

(d) $\sin h(2x) = \dfrac{e^{2x}-e^{-2x}}{2} = 2\left(\dfrac{e^{x}-e^{-x}}{2}\right)\left(\dfrac{e^{x}+e^{-x}}{2}\right)$

$= 2\sin h(x)\cdot\cos h(x)$

This compares with $\sin 2x = 2\sin x\cdot\cos x$

(c) $\cos h^2(x) = \dfrac{e^{2x}+2+e^{-2x}}{4}$

$\sin h^2(x) = \dfrac{e^{2x}-2+e^{-2x}}{4}$

$\cos h^2(x) + \sin h^2(x) = \dfrac{2e^{2x}+2e^{-2x}}{4}$

$= \dfrac{e^{2x}+e^{-2x}}{2} = \cos h(2x)$

(f) $\sin h(x)\cos h(y) + \sin h(y)\cos h(x) =$

$$= \left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{y}+e^{-y}}{2}\right)+\left(\frac{e^{y}-e^{-y}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$= \frac{e^{x+y}-e^{-x-y}}{4} + \frac{e^{x+y}-e^{-x-y}}{4} = \frac{e^{x+y}-e^{-(x+y)}}{2}$$

$= \sin h(x+y)$

This compares with $\sin(x+y) = \sin x\cos y + \cos x\sin y$

Section E.2 (pp.441-443)

1. $\sqrt{2}e^{\frac{i\pi}{4}}$ 2. $e^{i\pi}$ 3. $e^{\frac{i\pi}{2}}$ 4. $2e^{\frac{i\pi}{3}}$

2.

Rectangular	Polar	Trigonometric	Exponential
$1 - i$	$(\sqrt{2},\frac{7\pi}{4})$	$\sqrt{2}(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$	$\sqrt{2}e^{\frac{7i\pi}{4}}$
$\frac{3}{2} + \frac{3i\sqrt{3}}{2}$	$(3, \frac{\pi}{3})$	$3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$	$3e^{\frac{i\pi}{3}}$
$2i$	$(2, \frac{\pi}{2})$	$2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$	$2e^{\frac{i\pi}{2}}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$(1,\frac{2\pi}{3})$	$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$	$e^{\frac{2i\pi}{3}}$
$-4i$	$(4,\frac{3\pi}{2})$	$4(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$	$4e^{\frac{3i\pi}{2}}$
$-1 + i$	$(\sqrt{2},\frac{3\pi}{4})$	$\sqrt{2}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$	$\sqrt{2}e^{\frac{3i\pi}{4}}$
$-e$	(e,π)	$e(\cos\pi + i\sin\pi)$	$e^{1+i\pi}$
$-2\sqrt{3} + 2i$	$(4,\frac{2\pi}{3})$	$4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$	$4e^{\frac{2i\pi}{3}}$
$-5i$	$(5,\frac{3\pi}{2})$	$5(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$	$5e^{\frac{3i\pi}{2}}$
1	$(1,2\pi)$	$(\cos 2\pi + i\sin 2\pi$	$e^{2i\pi}$

6. The terminal sides of the angles of measure π and 3π are the same (the negatively directed x-axis), hence the vectors represented by $e^{i\pi}$ and $e^{3i\pi}$ have the same direction. Since $|e^{i\pi}| = |e^{3i\pi}| = 1$,

$$e^{i\pi} = e^{3i\pi} = -1$$

7. As in Exercise 6, $\frac{\pi}{3}$ and $\frac{7\pi}{3}(\frac{\pi}{3} + 2\pi)$ have the same terminal side so the vectors $e^{\frac{i\pi}{3}}$ and $e^{\frac{7i\pi}{3}}$ have the same direction; the magnitudes

are 1, so $e^{\frac{i\pi}{3}} = e^{\frac{7i\pi}{3}}$

In rectangular form,

$$e^{\frac{i\pi}{3}} = \cos\frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

8. $r_1 = r_2$ and $\theta_1 = \theta_2 + 2n\pi$, $n \in$ Integers
9. (a) $2\sqrt{2}(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4})$; $2\sqrt{2}(\cos\frac{9\pi}{4} + i \sin \frac{9\pi}{4})$

   (b) $2\sqrt{2}e^{\frac{i\pi}{4}}$; $2\sqrt{2}e^{\frac{9i\pi}{4}}$

*10. (a) $\frac{1+i\sqrt{3}}{2}$ (b) $.95+.33i$ (c) $7.39i$

   (d) $e^{2+\frac{i}{2}} = e^2 \cdot e^{\frac{i}{2}} = e^2(\cos\frac{1}{2} + i \sin\frac{1}{2})$

   $\approx e^2(.88 + .48i) \approx 7.39(.88 + .48i)$

   $\approx 6.50 + 3.55i$

11. $e^{i\theta} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \ldots$
12. $e^{-ix} = \cos x - i \sin x$
13. From $e^{ix} = \cos x + i \sin x$ and $e^{ix} = \cos x - i \sin x$ add and subtract to derive results.

<u>Section E.3 (pp.446-448)</u>

1. $\frac{-1+i\sqrt{3}}{2} = e^{\frac{2i\pi}{3}}$; $(e^{\frac{2i\pi}{3}})^3 = e^{2i\pi} = 1$
2. $1+i = \sqrt{2}e^{\frac{i\pi}{4}}$; $(\sqrt{2}e^{\frac{i\pi}{4}})^6 = 8e^{\frac{3i\pi}{2}} = -8i$
3. $2\sqrt{3} + 2i = 4e^{\frac{i\pi}{6}}$; $(4e^{\frac{i\pi}{6}})^5 = 1024e^{\frac{5i\pi}{6}} = -512\sqrt{2} + 512i$
4. $(-\sqrt{3}+i)^3 = 2e^{\frac{5i\pi}{6}}$; $(2e^{\frac{5i\pi}{6}})^3 = 8e^{\frac{5i\pi}{2}} = 8i$
5. To solve $z^3 = 27$, let $z = re^{i\theta}$ and then

   $r^3e^{3i\theta} = 27e^{2i\pi} \Rightarrow r^3 = 27$ and $3\theta = 2\pi + 2n\pi$; i.e.,

   $r = 3$ and $\theta = \frac{2\pi}{3} + \frac{2n\pi}{3} = \frac{2\pi(n+1)}{3}$

   The 3 roots are:

   $$e^{\frac{2i\pi}{3}} = \frac{-3+3i\sqrt{3}}{2}$$

   $$3e^{\frac{4i\pi}{3}} = \frac{-3-3i\sqrt{3}}{2}$$

   $$3e^{\frac{6i\pi}{3}} = 3$$

6. The roots are:

$$e^{\frac{i\pi}{6}} = \frac{\sqrt{3}+i}{2} \qquad e^{\frac{5i\pi}{6}} = \frac{-\sqrt{3}+i}{2} \qquad e^{\frac{9i\pi}{6}} = -i$$

7. $\sqrt{8}e^{\frac{\pi i}{6}} = \frac{\sqrt{24}+i\sqrt{8}}{2} = \sqrt{5}+i\sqrt{2} \qquad \sqrt{8}e^{\frac{7\pi i}{6}} = -\sqrt{6}-i\sqrt{2}$

8. $e^{\frac{2i\pi}{3}} = \frac{-1+i\sqrt{3}}{2} \qquad e^{\frac{4i\pi}{3}} = \frac{-1-i\sqrt{3}}{2} \qquad e^{\frac{6i\pi}{3}} = 1$

9. $e^{\frac{2i\pi}{4}} = i$; $e^{\frac{4i\pi}{4}} = -1$; $e^{\frac{6i\pi}{4}} = -i$; $e^{\frac{8i\pi}{4}} = 1$

10. $1 = e^{2i\pi}$, so if $z = re^{i\theta}$ is a solution of $z^n = 1$,
$r^n e^{ni\theta} = 1e^{2i\pi}$, $r^n = 1$ and $n\theta = 2\pi + 2k\pi$, $k \in$ Int.
So $r = 1$ and $\theta = \frac{2\pi}{n} + \frac{2\pi}{n}\cdot k$, $k \in$ Int.
The n, nth roots are found by letting $k = 0,1,2,\ldots,$ $n-1$. For $k = 0$, we find that one root is
$r = e^{\frac{2\pi i}{n}}$. The other roots are:
For $k = 1$, $e^{\frac{4\pi i}{n}} = r^2$, For $k = 2$, $e^{\frac{6\pi i}{n}} = r^3$,

For $k = n-1$, $e^{2\pi i} = r^n = 1$.
Hence the n nth roots are of the form
$1, r, r^2, \ldots, r^{n-1}$ where $r = e^{\frac{2\pi i}{n}}$.

11. (a)

$\cdot$	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
1	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$	1
$e^{\frac{4\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$	1	$e^{\frac{2\pi i}{3}}$

(b) Let $S = \{1,r,r^2\}$ where $r = e^{\frac{2\pi i}{3}}$.
(S,$\cdot$) is a commutative group:
(i) Multiplication of complex numbers is associative, so multiplication on S is associative.
(ii) The identity is 1
(iii) Each element has an inverse (See Table)
(iv) The operation is commutative as can be seen by the symmetry in the table across the main

diagonal (upper left to lower right).

(c) $1 + e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}} = 0$; $1 \cdot e^{\frac{2\pi i}{3}} \cdot e^{\frac{4\pi i}{3}} = 1$

12. In parts (a) and (b) we use DeMoivre's Theorem in its trigonometric form and
$a + bi = c + di$ IFF $a = c$ and $b = d$
(i.e., equate real and imaginary parts).

13. $\cos 3x + i \sin 3x = (\cos x + i \sin x)^3$

$$= \cos^3 x + 3i \cos^2 x \sin x - 3\cos \sin^2 x - i \sin^3 x$$

$$= (\cos^3 x - 3\cos x \sin^2 x) + i(3\cos^2 x \sin x - \sin^3 x)$$

Hence $\cos 3x = \cos^3 x - 3\cos x \sin^2 x$

$\sin 3x = 3\cos^2 x \sin x - \sin^2 x$

14. By equating real and imaginary parts,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

15. Verification

16. From the series $\cos(.5) = .8776$ and $\sin(.5) = .4794$ which agree with the tabled values of .8776 and .4794 (CRC Standard Mathematical Tables, 1959)

*17.

Time	Voltage
0	20
$\frac{1}{120}$	(-)19.7
$\frac{1}{60}$	20.2
1	15.8

18. (a) $e^{ix} \cdot e^{iy} = (\cos x + i\sin x) \cdot (\cos y + i\sin y)$

$= (\cos(x+y) + i\sin(x+y)$ By DeMoivre's Theorem

$= e^{i(x+y)}$

(b) $(e^{ix})^y = (\cos x + i\sin x)^y$

$= \cos xy + i\sin xy$ By DeMoivre's Theorem

$= e^{ixy}$

(c) $\frac{e^{ix}}{e^{iy}} = \frac{\cos x + i\sin x}{\cos y + i\sin y} = \frac{(\cos x + i\sin y)(\cos(-y) + i\sin(-y)}{(\cos y + i\sin y)(\cos(-y) + i\sin(-y)}$

$= \frac{\cos(x-y) + i\sin(x-y)}{\cos 0 + i\sin 0} = \cos(x-y) + i\sin(x-y)$

$= e^{i(x-y)}$

Review Exercises (pp.449-450)

1. From series $e^{.5} = 1.6458$, $e^1 = 2.6667$, $e^2 = 6.3333$. The table values are 1.6487, 2.7183 and 7.3891. The differences occur because the first 4 terms alone do not define a "close" polynomial approximation to e^x -- the more terms, the closer the approximation.

Answers to Exercises 2-10 are given in the following table:

	Rectangular	Polar	Trigonometric	Exponential
2.	$\frac{1+i}{2}$	$(\frac{\sqrt{2}}{2}, \frac{\pi}{4})$	$\frac{\sqrt{2}}{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$	$\frac{\sqrt{2}}{2}e^{\frac{\pi i}{4}}$
3.	$-2\sqrt{3}+2i$	$(4, \frac{2\pi}{3})$	$4(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$	$4e^{\frac{2i\pi}{3}}$
4.	$\frac{-3\sqrt{2}+3i\sqrt{2}}{2}$	$(3, \frac{3\pi}{4})$	$3(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$	$3e^{\frac{3i\pi}{4}}$
5.	$-2i$	$(2, \pi)$	$2(\cos\pi+i\sin\pi)$	$2e^{2\pi i}$
6.	$\frac{-\sqrt{6}}{2}+\frac{i\sqrt{2}}{2}$	$(\sqrt{2}, \frac{5\pi}{6})$	$\sqrt{2}(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$	$\sqrt{2}e^{\frac{5i\pi}{6}}$
7.	i	$(1, \frac{\pi}{2})$	$\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$	$e^{\frac{i\pi}{2}}$
*8.	$.878+.479i$	$(1, \frac{1}{2})$	$\cos\frac{1}{2}+i\sin\frac{1}{2}$	$e^{\frac{i}{2}}$
*9.	$14(.878+.479i)$	$(14, \frac{1}{2})$	$14(\cos\frac{1}{2}+i\sin\frac{1}{2})$	$14e^{\frac{i}{2}}$
*10.	Same as Exercise 8.			

11. $(1-i)^8 = (\sqrt{2}e^{\frac{7i\pi}{4}})^8 = 16e^{14i\pi} = 16$

12. $(\frac{\sqrt{3}}{3}+\frac{i}{3})^5 = (\frac{2}{3}e^{\frac{\pi i}{6}})^5 = \frac{32}{243}e^{\frac{5\pi i}{6}} = \frac{-16\sqrt{3}}{243}+\frac{16i}{243}$

13. The roots are 1, r, r^2, r^3, r^4 where $r = e^{\frac{2\pi i}{5}}$

14. Let $z = re^{i\theta}$ be a solution of $z^6 = -64$. Then
$r^6e^{6i\theta} = 64e^{\pi i}$ and so $r^6 = 64$ and $6\theta = \pi+2k\pi$, $k \in$ Int.
That is, $r = 2$ and $\theta = \frac{\pi}{6}+\frac{k\pi}{3}$, $k \in$ Integers
The 6 6th roots are found by letting $k = 0,1,\ldots,5$
The roots are:

(1) $2e^{\frac{\pi i}{6}} = \sqrt{3}+i$ (2) $2e^{\frac{\pi i}{2}} = 2i$ (3) $2e^{\frac{5\pi i}{6}} = -\sqrt{3}+i$

(4) $2e^{\frac{7\pi i}{6}} = -\sqrt{3}-i$ (5) $2e^{\frac{3\pi i}{2}} = -2i$ (6) $2e^{\frac{11\pi i}{6}} = \sqrt{3}-i$

15. The roots are $2e^{\frac{2\pi i}{3}}$, $2e^{\frac{4\pi i}{3}}$, and $2e^{2\pi i} = 2$.

(a) Product of roots:

$$2e^{\frac{2\pi i}{3}} \cdot 2e^{\frac{4\pi i}{3}} \cdot 2 = 8e^{\frac{6\pi i}{3}} = 8e^{2\pi i} = 8$$

(b) Sum of roots:

$$2e^{\frac{2\pi i}{3}} = -1+i\sqrt{3} \qquad 2e^{\frac{4\pi i}{3}} = -1-i\sqrt{3}$$

$$2 + 2e^{\frac{2\pi i}{3}} + 2e^{\frac{4\pi i}{3}} = 2 - 1 + i\sqrt{3} - 1 - i\sqrt{3} = 0$$

16. $(1+i)^6 = \binom{6}{6}1^6+\binom{6}{5}1^5\cdot i+\binom{6}{4}1^4\cdot i^2+\binom{6}{3}1^3\cdot i^3+\binom{6}{2}1^2\cdot i^4)$

$$+\binom{6}{1}1\cdot i^5+i^6 = 1+6i-15-20i+15+6i-1 = -8i$$

$$(1+i)^6 = (\sqrt{2}e^{\frac{\pi i}{4}})^6 = 8e^{\frac{6\pi i}{4}} = 8e^{\frac{3\pi i}{2}} = -8i$$

APPENDIX (pp.464)

1. $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$

2. $16x^4+96x^3y+216x^2y^2+216xy^3+81y^4$

3. $b^n+ \frac{nb^{n-1}k}{m} + \frac{n(n-1)}{1\cdot 2}\frac{b^{n-2}k^2}{m^2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\frac{b^{n-3}k^3}{m^3} + \ldots$

$+\binom{n}{6}\frac{b^{n-6}k^6}{m^6} +\ldots+ \frac{k^n}{b^m}$

4. $\cos^4x+4i\cos^3x\sin x-6\cos^2x\sin^2x-4i\cos x\sin^3x+\sin^4x$
(this is equal to $\cos 4x+i\ \sin 4x$)

5. $1 - \frac{n}{1}\cdot\frac{x}{n} + \binom{n}{2}(\frac{x}{n})^2 - \binom{n}{3}(\frac{x}{n})^3+\ldots$ or

$1 - x + \frac{1}{2!}(1-\frac{1}{n})x^2 - \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})x^3 + \frac{1}{4!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})x^4-\ldots$

6. $(a+b)^n = \binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{n}b^n$

if $a = 1$, $b = 1$, then $a+b = 2$ and
$2^n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}$

7. See Exercise 6.
$(1+\frac{x}{n})^n = 1 + x + (1-\frac{1}{n})\cdot\frac{x^2}{2!} + (1-\frac{1}{n})(1-\frac{2}{n})\cdot\frac{x^3}{3!}$

$$+(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\frac{x^4}{4!}+ \ldots$$

If n is very large, $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}$ are close to zero, and $1-\frac{1}{n}, 1-\frac{2}{n}, 1-\frac{3}{n}$, are close to 1. Hence for $n \to \infty$ and $x = 1$, the terms are close to $1, 1, \frac{1}{2!}, \frac{1}{3!}, \ldots$

8. $$\binom{4}{2}+\binom{4}{3} = \frac{4\cdot 3}{1\cdot 2} + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3} = \frac{4\cdot 3\cdot 3}{1\cdot 2\cdot 3} + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3} = \frac{4\cdot 3(3+2)}{1\cdot 2\cdot 3}$$
$$= \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} = \binom{5}{3}$$

$$\binom{7}{3}+\binom{7}{4} = \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3} + \frac{7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4} = \frac{7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4} + \frac{7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4}$$
$$= \frac{2(7\cdot 6\cdot 5)\cdot 4}{1\cdot 2\cdot 3\cdot 4} = \frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4} = \binom{8}{4}$$

9. $$\binom{n}{r} + \binom{n}{r+1} = \frac{n(n-1)\ldots(n-r+1)}{1\cdot 2\cdot \ldots \cdot r} + \frac{n(n-1)(n-2)\ldots(n-r)}{1\cdot 2\cdot 3\cdot \ldots \cdot (r+1)}$$
$$= \frac{(r+1)\cdot n(n-1)\ldots(n-r+1)}{(r+1)!} + \frac{n(n-1)(n-2)\ldots(n-r)}{(r+1)!}$$
$$= \frac{n(n-1)(n-2)\ldots(n-r+1)[(r+1)+(n-r)]}{(r+1)!}$$
$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n+1)}{(r+1)!}$$
$$= \binom{n+1}{r+1}$$

10. Think of it as a tree diagram.

SUGGESTED TEACHER'S TESTS WITH ANSWERS

Prologue

In 1-6 find any and all grammatical errors.

1. LET Y = 3*Y - 7X
2. FØR N = 2 TØ -5 STEP -1
3. LET X = ((X+2/X) - 3½
4. IF M = N, THEN GØ TØ 45
5. READ IN X, Y, Z
6. PRINT THE SUM IS X

In 7 and 8 use hand simulation to determine the output of the given programs.

7.
```
05 READ N
15 FØR J = 1 TØ N
25 LET S = ((J-1)*S + J)
35 PRINT S;
45 NEXT J
55 DATA 5
65 END
```

8.
```
10 READ N
20 LET D = 2
30 IF N/D = INT(N/D) THEN 60
40 LET D = D + 1
50 GØ TØ 30
60 PRINT D
70 DATA 77
80 END
```

9. Write a BASIC program that will calculate $1^3 + 2^3 + 3^3 + \ldots + n^3$ for given value of n

10. Write a BASIC program that will calculate the sum and product of any two given numbers in clock arithmetic modulo 11.

Solutions to Sample Chapter Test

1. 7*X   2. O.K. as is

3. Unmatched parentheses and 3½ not acceptable

4. No comma and no GØ TØ in an IF THEN   5. No IN

6. To print the words need quotes around "THE SUM IS"

7. 1 3 9 31 129

8. 7 (finds smallest factor of N. except 1)

9. Many possibilities:

```
10 INPUT N
20 FØR J = 1 TØ N
30 LET S = S + J↑3
40 NEXT J
50 PRINT S
60 END
```

10. Though BASIC has a MOD function, one can write a fairly simple program to do things in mod 11 without it.

```
 10 INPUT A, B
 15 LET C = A + B
 20 IF C < 11 THEN 50
 30 LET C = C - 11
 40 GØ TØ 20
 50 PRINT A; "+"; B; "="; C
 60 LET D = A*B
 70 IF D < 11 THEN 100
 80 LET D = D - 11
 90 GØ TØ 70
100 PRINT A; "*"; B; "="; D
110 END
```

Chapter 1

In Questions 1 through 5, write the first five terms of sequence defined.

1. $a_n = (-2)^n$ 2. $a_n = n(n+1)$

3. $a_1 = 3$, $a_2 = 4$, $a_n = (a_{n-2})(a_{n-1})$, when $n > 2$

4. The geometric sequence in which $a_1 = 8$, $r = -\frac{1}{2}$

5. The arithmetic sequence in which $a_1 = 8$, $d = -\frac{1}{2}$

In Questions 6 through 11, tell whether or not the given sequence converges. If it does, then tell what the limit is.

6. $a_n = n^2$ 7. $a_n = \frac{1}{n^2}$

8. The geometric sequence in which $a_1 = -1$, $r = 2$.

9. The geometric sequence in which $a_1 = -1$, $r = \frac{1}{2}$

10. The arithmetic sequence in which $a_1 = -1$, $d = \frac{1}{2}$

11. $a_n = \frac{n+3}{n}$

12. Find the sum of the first 100 terms of the arithmetic sequence in which $a_1 = 4$ and $d = 5$.

13. Write the numerical expression (you need not simplify) for the twentieth term of the geometric sequence in which $a_1 = 4$ and $r = 2$.

14. Write the first five terms of the series associated with the geometric sequence 4, 2, ...

15. What is the sum of the infinite series in Question 14?

16. Write the nth term of the series associated with the geometric sequence 3, 6, 12, ...

17. Find two numbers whose sum is 15, given that the arithmetic mean of the numbers exceeds their geometric mean by $\frac{3}{2}$.

18. Prove by mathematical induction:
    For every $n \in Z^+$, $1 + r + r^2 + \ldots + r^{n-1} = \frac{1-r^n}{1-r}$

Solutions to Sample Chapter Test

1. $-2, 4, -8, 16, -32$ 2. $2, 6, 12, 20, 30$

3. $3, 4, 12, 48, 576$ 4. $8, -4, 2, -1, \frac{1}{2}$

5. $8, 7\frac{1}{2}, 7, 6\frac{1}{2}, 6$ 6. No 7. Yes; 0

8. No 9. Yes; 0 10. No 11. Yes; 1

12. $50(4 + 499) = 25{,}150$ 13. $a_{20} = 4(2^{19})$

14. $4, 6, 7, 7\frac{1}{2}, 7\frac{3}{4}$ 15. 8 16. $3 + 6 + 12 + \ldots + 3(2^{n-1})$

17. $\frac{15}{2} - \frac{3}{2} = \sqrt{x(15-x)}$

$$6 = \sqrt{x(15-x)}$$

$x^2 - 15x + 36 = 0$ The numbers are 3 and 12

18. When $n = 1$, $\frac{1-r^n}{1-r} = \frac{1-r}{1-r} = 1.$

Assume $1 + r + r^2 + \ldots + r^{k-1} = \frac{1-r^k}{1-r}$

Then $a + r + r^2 + \ldots + r^{k-1} = \frac{1-r^k}{1-r} + r^k$

$$= \frac{1+r^k + r^k - r^{k+1}}{1 - r}$$

$$= \frac{1 - r^{k+1}}{1 - r}$$

Chapter 2

No Chapter test is suggested on Modelling. This chapter is for the purpose of motivation........

Chapter 3

In Questions 1 through 4 solve the quadratic equations in the field $(C,+,\cdot)$

1. $x^2 - 4x + 20 = 0$
2. $2x^2 + \sqrt{2}x - 2 = 0$
3. $2x^2 + 3x + 4 = 0$
4. $x + \frac{9}{x} = 0$
5. Write a quadratic equation (in which the coefficient of x^2 is 1) whose two solutions are $3 + i$ and $3 - i$.

In Questions 6 through 14 simplify each expression by writing it in the form $a + bi$ where $\underline{a}$ and $\underline{b}$ are real numbers.

6. $(\sqrt{2} + 3i) + (4 - 7i)$
7. $(4 - i) - (-6 + 8i)$
8. $(8 - 2i)(3 + i)$
9. $(6 - 3i)(5 - 4i)$
10. $(\sqrt{2} + \frac{1}{2}i)(\sqrt{2} - \frac{1}{2}i)$
11. $(a + bi)(a - bi)$
12. $\frac{2 + i}{2 - i}$
13. $\frac{3 + 5i}{4 - 7i}$
14. $\frac{a + bi}{c + di}$
15. Solve for $\underline{z}$: $(3i)z = 6 + 2i$.
16. Solve for $\underline{z}$: $(2i)z + 5i = 4 - 3z$
17. Find all possible values of $z = a + bi$ if the product of $\underline{z}$ and its conjugate is 13, and $a + b = -1$.
18. On coordinate paper show the unit square with vertices at (0,0), (1,0), (1,1), and (0,1). Then show the image of this square under this spiral similarity:

$$z \to (3 + 4i)z$$

19. Answer the following TRUE or FALSE. Remember that TRUE is taken to mean true <u>without exception</u>.

    (1) If f is a continuous function and $f(a) > 0$ and $f(b) > 0$, then the equation $f(x) = 0$ has no solutions in the interval $a < x < b$.

    (2) The real number system is isomorphic to the complex number system.

    (3) $(z)(\overline{z}) \geq 0$, where "$\overline{z}$" denotes the conjugate of z.

    (4) In a complex plane z and $\overline{z}$ are symmetric with respect to the x-axis.

    (5) If the discriminant of a quadratic equation (with real coefficients) is zero, then the equation has two real roots.

20. Determine all possible values of $\sqrt{3 + 4i}$ if this is defined as a complex number whose square is $3 + 4i$.

Solutions to Sample Chapter Test

1. $2 + 4i$, $2 - 4i$ 2. $-\sqrt{2}$, $\frac{1}{2}\sqrt{2}$ 3. $\frac{3 \pm \sqrt{23}i}{4}$

4. $\pm 3i$ 5. $x^2 - 6x + 10 = 0$ 6. $(4 + \sqrt{2}) + (-4)i$

7. $10 + (-9)i$ 8. $26 + 2i$ 9. $18 - 39i$ 10. $2\frac{1}{4}$

11. $a^2 + b^2$ 12. $\frac{3}{5} + \frac{4}{5}i$ 13. $\frac{-23}{65} + \frac{41}{65}i$

14. $\frac{ac+bd}{a^2+b^2} + \frac{bc-ad}{a^2+b^2}i$ 15. $\frac{2}{3} - 2i$ 16. $\frac{2-23i}{13}$

17. $2 - 3i$, $-3 + 2i$

18. The image is a square with vertices as (0,0), (3,4), (-1,7), and (-4,3).

19. (1) false (2) false) (3) true (4) true)
(5) false

20. $2 + i$, $-2 - i$

Chapter 4

In Questions 1 - 6, tell whether the statement is true or false.

1. The graph of $f:x \to (\frac{1}{3})^x$, $x \in R$ is strictly decreasing

2. $4^{-\frac{3}{2}} = (\frac{1}{4})^{\frac{3}{2}}$ 3. $\log 72 = 1 + \log 7.2$

4. The domain of $y = \log_{10}x$ is $x \geq 0$

5. $4^{\log_4 4} = 4$ 6. $\log_{10}rs = \log_{10}r + \log_{10}s$

In Questions 7 - 12, evaluate each expression.

7. 2^{-2} 8. $4\pi^0$ 9. $(\frac{1}{5})^{-1}$ 10. $\log_2(\frac{1}{8})$ 11. $8^{-\frac{5}{3}}$

12. $\log_2 4 + \log_4 2$

Sketch a graph of each of the following functions. Identify at least three points on each sketch.

13. $f:x \to 4^{x-2}$ 14. $g : x \to \log_4 x$

15. Solve for t : $5^{t+1} = 29$

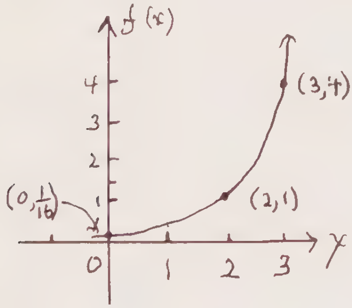
16. The number of bacteria increase according to the formula $N(t) = 3\cdot 10^4 \cdot 2^{4t}$; t in days.
How long will it take for the number of bacteria to double?

Solutions to Sample Chapter Test

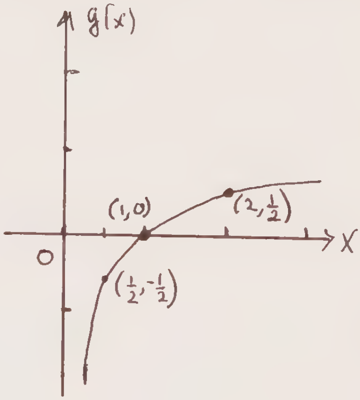
1. T 2. T 3. T 4. F 5. T 6. T 7. $\frac{1}{4}$ 8. 4

9. 5 10. -3 11. $\frac{1}{32}$ 12. $\frac{5}{2}$

13.



14.



15. 1.09 16. t = $\frac{1}{4}$ day

Chapter 5

Without using the text tables, evaluate each of the following (1-3):

1. $\cos(2x + \pi)$ at $x = \frac{\pi}{3}$ 2. $\sin 22\frac{1}{2}^{\circ}$ 3. $\tan\frac{\pi}{3}$

4. Sketch a graph for two periods of $2\cos(4x)$

5. Complete the table below:

Cartesian Coordinates	Polar Coordinates
(5, -5)	- - - -
- - - -	$[6, \frac{5\pi}{3}]$

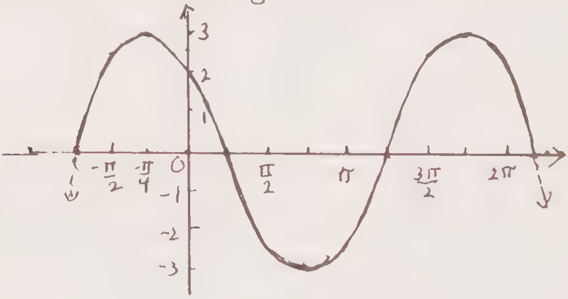
6. Matching:

___ 1. $\cos(x+y)$	a. $((\cos x+1)/2)^{\frac{1}{2}}$
___ 2. $\sin 2x$	b. $\cos^2 x - \sin^2 x$
___ 3. $\sin(x-y)$	c. $\cot x$
___ 4. $\tan(\frac{\pi}{2}-x)$	d. $\cos x \cos y - \sin x \sin y$
___ 5. $\cos\frac{1}{2}x$	e. $-\tan x$
	f. $2 \cos x \sin x$
	g. $\cos y \sin x - \sin y \cos x$

7. Prove that for all x : $1+\tan^2 x = \sec^2 x$

8. Explain 2 different physical situations modelled by periodic functions, describing the interpretation of period and amplitude in each case.

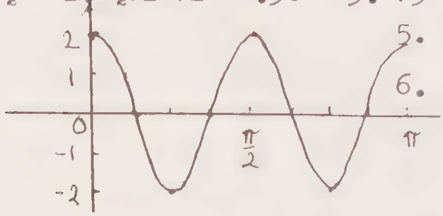
9. Write the rule for a circular function that has a graph like the following:



Solutions to Sample Chapter Test

1. $\frac{1}{2}$ 2. $\frac{1}{2}\sqrt{2-\sqrt{2}} \approx .38$ 3. $\sqrt{3}$

4.



5. $[5\sqrt{2}, \frac{5\pi}{4}]$ $(3, 3\sqrt{3})$

6. d, f, g, c, a

7. $1 + \tan^2 x = 1 + \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

8. many possible answers

9. $3 \cos(x + \frac{\pi}{4})$

Chapter 6

1. A committee of four is to be selected at random from 6 girls and 4 boys.
   (a) How many committees can be formed?
   (b) What is the probability that the committee has 2 girls and 2 boys?
   (c) In part (a) each committee can appoint its members as chairman, vice-chairman, secretary and treasurer of the committee. How many differently ordered committees of four members are thus possible?

2. A box contains 30 red and 20 white balls. At random two balls are selected, one after the other without replacement. Find
   (a) the probability that both balls are white.
   (b) the probability that both balls are red.
   (c) the probability that one ball is white, the other red.

3. In a factory, three workers produce at a rate of 40, 50, and 60 items a day. Their rate of defective items is 2%, 5%, and 4% daily. At the end of the day all produced items are grouped in one pile. If an item is selected at random what is the probability that it is defective? State the events you used and illustrate graphically.

4. In a certain country the population of the age group 30 to 50 years is 60% male. In this group the probability of a male having a heart attack is 0.25 and for a female it is 0.15. A person is reported to have a heart attack. What is the probability it is a woman?

5. Two types of defects occur in manufacturing an automobile axle bearing. The two defects occur independently of each other. From long experience the probabilities of these defects occurring are .04 and .02 respectively. Find for a randomly selected bearing from the finish line, that it has
   (a) both defects
   (b) only one of the defects
   (c) no defects

*6. On a final multiple-choice examination, it is known that students know 70% of the answers and guess on the others. There are four choices on each question, only one of which is correct. Draw a tree diagram of this situation and find the probability that a question answered correctly was guessed.

Solutions to Sample Chapter Test

1. (a) $\binom{10}{4} = 210$ committees (b) $\dfrac{\binom{6}{2}\binom{4}{2}}{210} = \dfrac{90}{210}$ or $\dfrac{3}{7}$

   (c) $(10)_4 = 5040$

2. Let W_1 be event white first, W_2 event white second. Then

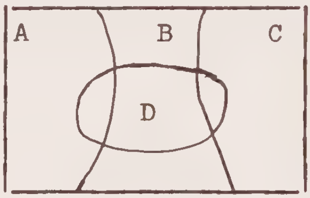
$$P(W_1) = \frac{2}{5},\ P(W_2|W_1) = \frac{19}{49}$$

(a) $P(W_1 \cap W_2) = \frac{2}{5}\cdot\frac{19}{49} = \frac{38}{245}$. Similarly, for R_1, R_2 we have $(R_1) = \frac{3}{5}$, $P(R_2|R_1) = \frac{29}{49}$.

(b) $P(R_1 \cap R_2) = \frac{3}{5}\cdot\frac{29}{49} = \frac{87}{245}$, since $W_1 \cap W_2$ and $R_1 \cap R_2$ are mutually exclusive.

(c) $P(R \cup W) = 1 - \frac{38+87}{245} = \frac{120}{245}$

3. $P(A) = \frac{40}{150}$, $P(B) = \frac{50}{150}$, $P(C) = \frac{60}{150}$



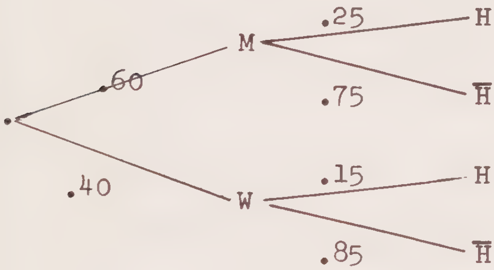
$P(D|A) = \frac{2}{40}$, $P(D|B) = \frac{5}{50}$,

$P(D|C) = \frac{4}{60}$

$$P(D) = \frac{4}{15}\cdot\frac{1}{20} + \frac{1}{5}\cdot\frac{1}{10} + \frac{2}{5}\ \frac{1}{15}$$

$$= \frac{4 + 6 + 8}{300} = \frac{18}{300} = \frac{3}{50}$$

4.



$$P(W|H) = \frac{P(W)\cdot P(H|W)}{P(W)\cdot P(H|W) + P(M)\cdot P(H|M)}$$

$$= \frac{(.40)(.15)}{(.40)(.15) + (.60)(.25)} = \frac{.06}{.2650} = \frac{2}{7}$$

5. Let D_1 be event has first defect, D_2 be the event has second defect, $\overline{D}_1$ be not the first defect, $\overline{D}_2$ be not the second defect.
$P(D_1) = .04$, $P(D_2) = .02$, $P(\overline{D}_1) = .96$, $P(\overline{D}_2) = .98$

Since events are independent $P(E_1)\cdot P(E_2) = P(E_1 \cap E_2)$

(a) $P(D_1 \cap D_2) = (.04)(.02) = .0008$ (it has both defects)

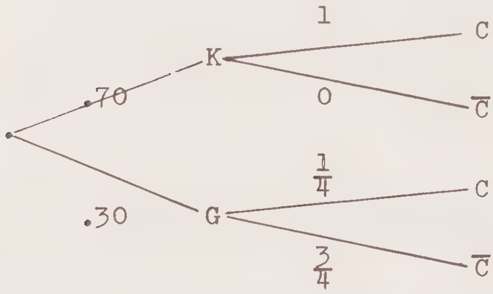
(c) $P(\overline{D}_1 \cap \overline{D}_2) = (.96)(.98) = .9408$ (it has no defects)

(b) $1 - (.9408 + .0008) = .0584$

*6. Let K be event student knows, G be event student guesses, C be correct response, $\overline{C}$ be incorrect.

$$P(G|C) = \frac{P(G)\cdot P(C|G)}{P(G)\cdot P(C|G) + P(K)\cdot P(C|K)}$$

$$= \frac{(.30)(.25)}{(.30)(.25) + (.70)(1)} = \frac{0750}{7750} = \frac{1}{11}$$



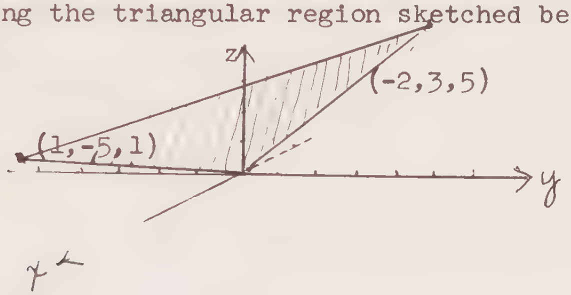
Chapter 7

In Questions 1-5, let R = (1,0,0), S = (0,1,0), T = (-1,4,3), and V = (-2,1,-2). If possible, complete:

1. R + S   2. R•S   3. R + S + T   4. R•S•T   5. T•(T + V)

6. Find a parametric representation for the vector line through (-2,1,3)

7. Find a parametric representation for the vector plane containing the triangular region sketched below



8. Find a standard (non-parametric representation) for the plane containing (0,-1,4), (-2,0,6), and (1,-1,1).

9. For this system of equations, find the solution set, describe the solution set geometrically, and if the set contains more than 1 element, find z.

   $x + y + 2z = -2 \quad 2x - y + z = 4 \quad 4x + y + 5z = 0$

10. Let A = (1,-1,1), B = (-2,-1,1) and C = (0,-3,-2). Determine:
    (a) the length of $\overline{AC}$
    (b) the cosine of angle ABC

Solutions to Sample Chapter Test

1. (1,1,0)   2. 0   3. (0,5,3)   4. Not possible

5. 26   6. $(x,y,z) = r(-2,1,3);\ r \in R$

7. $(x,y,z) = r(1,-5,1) + s(-2,3,5);\ r,s \in R$

8. $3x - 2y + z = 6$ 9. Solution set: $x = -r - \frac{14}{3}$

$$y = -r + \frac{8}{3}$$

$$z = r$$

10. (a) $AC = \sqrt{14}$ (b) $\cos\theta = \frac{2}{\sqrt{17}}$

Epilogue

In Questions 1 - 3, write each complex number in rectangular, polar, trigonometric, and exponential form.

1. $-\sqrt{7}$ 2. $(3, \frac{2\pi}{3})$ 3. $(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^3$

4. Change $\sqrt{2} + i\sqrt{2}$ to exponential form to compute $(\sqrt{2} + i\sqrt{2})^8$. Express the result in a + bi form.

5. In exponential and rectantular form, find all the roots of $z^4+z = 0$ (Hint: Factor $z^4 + z$)

6. In exponential form, find all of the roots of $z^6 = 1$
   (a) Plot the roots to show their geometric pattern
   (b) By cubing each of the roots of $z^6 = 1$, identify those that are also solutions of $z^3 = 1$.

Solutions to Sample Chapter Test

1. $-\sqrt{7}$; $(\sqrt{7},\pi)$; $\sqrt{7}(\cos\pi + i\sin\pi)$; $\sqrt{7}e^{i\pi}$

2. $\frac{-3}{2} + \frac{3i\sqrt{3}}{2}$; $(3, \frac{2\pi}{3})$; $3(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$; $3e^{\frac{2i\pi}{3}}$

3. -1; $(1,\pi)$; $(\cos\pi + i\sin\pi)$; $e^{\pi i}$

4. $\sqrt{2} + i\sqrt{2} = 2e^{\frac{\pi i}{4}}$, $(\sqrt{2}+i\sqrt{2})^8 = (2e^{\frac{\pi i}{4}})^8 = \sqrt{12}e^{\frac{8\pi i}{4}}$

$$= 512e^{2\pi i} = 512$$

5. Roots are: $0, -1, \frac{1 \pm i\sqrt{3}}{2}$

6. Roots are: $e^{\frac{2n\pi i}{6}} = e^{\frac{n\pi i}{3}}$; $n = 0,1,2,3,4,5$

n	Root	Root Cubed
0	1	1
1	$e^{\frac{\pi i}{3}}$	$e^{\pi i} = -1$
2	$e^{\frac{2\pi i}{3}}$	$e^{2\pi i} = 1$
3	$e^{\frac{3\pi i}{3}}$	$e^{3\pi i} = -1$
4	$e^{\frac{4\pi i}{3}}$	$e^{4\pi i} = 1$
5	$e^{\frac{5\pi i}{3}}$	$e^{5\pi i} = -1$

Hence the solutions of $z^6 = 1$ that are also solutions of $z^3 = 1$ are:

$$1, e^{\frac{2\pi i}{3}}, \text{ and } e^{\frac{4\pi i}{3}}$$