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ABSTRACT

The reorganization of the traditionally separated branches of mathematics into a unified single study is discussed in this report. First, significant events from 1800 to 1950 in the field of creative mathematics that led to the contemporary view of the nature of mathematics are highlighted. This review is followed by a discussion of significant movements in curricula reform sponsored by international organizations, and persons, which contributed to the unified concept. Then the preparation, production, innovation, and evaluation of a unified secondary mathematics program is described. The report closes by emphasizing the need to extend the program of unified mathematics to the majority of secondary school students. (Author/DT)

FINAL REPORT

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TOWARD
A UNIFIED MATHEMATICS CURRICULUM
for
THE SECONDARY SCHOOL

A REPORT

of

The Origin, Work, and Development

of

UNIFIED MATHEMATICS

SECONDARY SCHOOL MATHEMATICS CURRICULUM

IMPROVEMENT STUDY (SSMCIS)

TEACHERS COLLEGE, COLUMBIA UNIVERSITY

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PREFACE.

NATURE AND PURPOSE OF THE REPORT

From 1950 to 1965 the United States of America was engaged in an active investigation and reform of its school mathematics curricula. Most of this activity was carried on within traditional curriculum framework, that is, teaching arithmetic, algebra, and geometry as self-contained blocks of mathematical knowledge. By initiating an early start (in the 7^{th} or 8^{th} grades) of the traditional sequence of high school courses, the more able students were enabled to study calculus during the last year of secondary school instruction.

By 1960 however it had become evident that the traditional separation of mathematics study into the several isolated branches was no longer indicative of its nature or its uses in contemporary society. Today algebra has become a subject based on structures and their realizations. Geometry, and its newer extensions, has now become a study of various types of spaces and much the teaching of ordinary synthetic Euclidean geometry is of little subsequent use. The newer concepts of sets, relations, functions and operations have become basic to all of the traditional branches. Probability, statistics, numerical computation, computers and mini-calculators, all yield a quite different aspect to the treatment of problem solving, applications, and modelling. Moreover, placed in a contemporary setting, the important content of the traditional branches is interlocked through fundamental structures that have become the backbone of all mathematics. We now recognize the fact that we must teach this contemporary conception of mathematics if our students' knowledge is not to be anachronistic when they enter adult society.

The need to reorganize the mathematics of the traditional separated branches into a unified single study was recognized by most European countries as early as 1960. How to accomplish this reorganization in actual classroom teaching was then unknown and in the first attempts mistakes were made. But by 1965 some definite procedures and desirable goals had sufficiently matured to indicate a type of unified secondary school mathematics that could be both possible and worthy of attainment. This awareness gave rise to a proposed study to create a unified curriculum, educate teachers in its content, pedagogy and goals and to experiment with selected students in New York suburban secondary schools. The proposal was accepted and financed first by the Federal Office of Education from 1965 to 1969, and then by the National Science Foundation (1969-1976).

This report is in a sense a historical one. First there is a highlighting of significant events from 1800 to 1950 in the field of creative mathematics that led to the contemporary view of the nature of mathematics. This review is followed by a discussion of significant movements in curricula reform sponsored by international organizations, and persons, which contributed to the unified concept. Then the preparation, production, innovation and evaluation of a unified mathematics program is described. The report closes by emphasizing the need to extending the program of unified mathematics to the majority of secondary school students.

It is our hope that this report may serve as a partial guide to curriculum improvement in the decades ahead. A continuous search, under controlled experimentation, for ways of adapting mathematics study to the capacities of the human mind and the needs of society, is a necessary activity of educational development. In any society that is seeking the better life for its people, the growth of scientific knowledge, the use of electronic computers and calculators, and the extension of mathematical methods into almost all other disciplines, demand a minimum knowledge of contemporary mathematics for all its members.

Howard F. Fehr

SSMCIS FINAL REPORT TO NSF

TOWARD A UNIFIED MATHEMATICS CURRICULUM

Chapter I

SIGNIFICANT DEVELOPMENTS IN MATHEMATICS LEADING TO A UNIFIED CONCEPT

The growth and development of mathematics during the last two centuries is a tremendous achievement. To depict this growth in detail would require volumes of great length. Here we must be satisfied with a bird's eye view of the significant developments leading to the present day conception of our subject. While all branches interacted in leading to the contemporary structures, for the sake of clarity we consider the several branches independently and then show how they became more or less a unified body of knowledge.

In the literature on mathematical education one frequently finds reference to mathematics called traditional, classical, old, new, modern, fused, integrated, contemporary, and unified. Much confusion and unwarranted distinctions have arisen from these categorizations. Most mathematicians look upon their subject as one of continuous development over a long span of centuries and having a contemporary aspect which changes as time moves on. When in 1930, Van der Waerden labelled his book Modern Algebra, a name now frequently applied to the treatment of algebra in the manner of his book, he did not intend to say that it was new-born. All he meant to convey by the term "modern" was the way the subject was conceived and being taught by his colleagues (at Hamburg) and himself. In the fourth edition of his book, 1956, he dropped the word Modern, using only the word Algebra. Most of the so-called classical or traditional mathematics finds itself, along with newer conceptions and new topics in the body of important present day mathematical knowledge.

I. Algebra.

The algebra taught in secondary schools during the twentieth century, and still dominant in instruction, may be described as classical algebra. It is concerned with operating on expressions, finding solutions to equations, and applying special techniques (factoring, simplifying, reducing, etc.) to algebraic forms. The material is applied to solving "word" problems. This algebra originated over 4,000 years ago and reached a climax by the end of the nineteenth century.

The first viewpoint of algebra began with the Babylonians and Egyptians who created numerical methods to solve various problems encountered in the organized activities of early civilization. Most certainly the demands of a growing commerce, with exchanges of various money currencies, led to the need for characterizing the answer to a special problem as an "unknown". This primitive algebra made no use of symbolization except for a system of numeration. So far as is known, solutions to these problems were not achieved by any form of general reasoning or by proof. I was strictly a numerical and an empirical algebra -- a listing of verbal instructions for obtaining an answer to a particular type of problem. The methods used relied on numerical tables, giving the squares and cubes of whole numbers. Within these limitations, the Babylonians were able to solve certain types of cubic and quadratic equations, systems of two linear equations, and certain quadratic systems.

While we look upon this work as rudimentary and primitive, nevertheless, the Babylonians must be credited with a substantial achievement, and in at least one aspect, the solution of a cubic equation, were not surpassed until the beginning of the 16th century. However, the lack of symbolization was a great deterrent to creating an organized science and for over 2,000 years prevented the generalizations and abstractions so germane to the development of Algebra. For this reason their contribution should be called pre-algebra.

The Egyptians, contemporary with the Babylonians, developed an even weaker algebra, so far as complexity and depth of reasoning were concerned. Although they referred to an "unknown" and had a symbol for it, their method of solution was one of guessing an answer and then making a correction to it. If an answer was not correct, it was usually easy to see in what ratio to the given numbers in the problem, the answer was either in excess or in defect, and it was necessary only to alter the "guessed" answer in accordance with this ratio. This is a forerunner to recently highly developed methods of interpolation and iterative processes in approximating solutions to problems.

One of the significant gaps in the development of mathematics was the failure of the Greeks to formulate a concept of rational and irrational numbers, or real numbers. However, they did develop a highly systematized theory of ratio of whole numbers which contains, in many ways, the modern development of rational numbers. It may have been the failure of the Greeks to develop a concept of irrational numbers that led them to pursue, as their great contribution, the axiomatic study of space--the only axiomatic theory in mathematics until the 19^{th} century.

The proof, given by Euclid, that there is no ratio of two whole numbers to represent the measure of the diagonal of a unit square is well known, Because of this viewpoint, all measure theory involving incommensurable segments was treated geometrically, not algebraically. From Euclid until the time of Newton and Leibnitz, every thing that was not geometry was designated as algebra.

In the later Greek period (100-300 A.D.) two Greek mathematicians, Heron and Diophantus, did make contributions to the development of algebra. Diophantus made the first breakthrough toward symbolization in what is now called syncopated algebra. Here he used letters to represent the unknowns in the equation, and also special symbols for addition, multiplication, and equality. Thus for the first time, algebra went beyond mere verbal instructions for performing certain operations on numbers and unknowns. However his algebra remained essentially numerical and each problem was given its own special mode of solution. For Diophantus, a quadratic equation had two, one or no solutions, only a positive whole number, or ratio of such numbers being accepted as roots. It is significant to note that in the domain of whole numbers and their ratios, he was correct in his interpretation. Finally, it must be noted that his solution of one equation in several unknowns, for example $3x + 4y = 5z$, initiated a field of investigation that incubated much of modern number theory.

During the dark ages, the study of mathematics, as with all other knowledge, declined, and was not revived until the Hindus and Arabs brought new computational techniques into play. The Hindus were the first to introduce a form of proof into algebra. They had a primitive idea of the homogeneity of terms in an equation. Thus $x \cdot x$ has a counterpart the area of a square and $3 \cdot x \cdot x$ is the volume of a rectangular parallelepiped with edges 3, x, and x. Omar Khayyam in his treatise on algebra solved quadratic equations in this manner.

The Arabs were more interested in number than in geometry. They introduced the irrational numbers, but not understanding their nature, referred to them as fictitious. To the Arabs belongs the credit of giving the name algebra to this branch of mathematics. The Arab mathematician Mohammed Ibn Musa Al-Khwarizmi, among several books he wrote, entitled one of them Hisab Aljebr w'al Muqabala or the science of transposition and balancing. The Arab "Aljebr" was latinized into "algebra" as we use it today.

Beginning about 1200 A.D. Europe entered into a period of increased mathematical creativity, especially in the field of algebra. The Italian school starting with Fibonacci, and culminating with Ferrari, Tartaglia, Cardan, Bombelli, focused its attention on discovering a general solution for cubic equations. The Hindus had already given a general solution of the quadratic by completing the square in the form of the universally known formula for the roots of $ax^2 + bx = -c$. (At this time the number 0 was not yet accepted, and one never found an equation written in the form $ax^2 + bx + c = 0$.) With Tartaglia's formula, given by Cardan in his book, the general solution of the cubic was completed and a new search began for the general solution of the fourth degree equation.

By 1600, the work done in solving equations had resulted in a large collection of special devices (algorithms) for finding solutions. The French mathematician, Francois Vieta, examined all these special procedures and succeeded in finding a general algebraic theory for finding positive solutions of equations of the first four degrees. This was made possible by the creation of a system of symbolization that had not previously existed for variables, constants, and operations. Vieta called this logistica speciosa, where he used the vowels a, e, i, o, u, to represent variables and consonants, b, c, d, f, g ..., to represent knowns or constants. Later Rene Descartes introduced the procedure of using the last letters of the alphabet, x, y, z, w, ..., as "unknowns" in the sense of a number whose value is to be found, and the first letters, a, b, c, ..., as arbitrary constants, a system quite commonly used today.

Around 1600, zero became accepted as a number and equations took on the form $ax^2 + bx + c = 0$. With this, algebra became a science of symbolic calculation on letters and numbers as contrasted with arithmetic which always operated on numbers. This algebra is epitomized in content and concept in Euler's Introduction to Algebra (1760) in which algebra is defined as The Theory of Calculation with Quantities. This is the first view of classical algebra. The subject matter is an assortment of topics such as can be found in any of the present-day secondary school textbooks on the subject.

The second view of algebra was initiated in the fifteenth and sixteenth centuries by the Italian school with its attention to equations. Having solved the fourth degree equation in all generality, attention was now given to fifth and higher degree equations. A host of theorems were developed during the next three hundred years, as well as special procedures on isolating roots, relating roots to coefficients, the number of roots, the fundamental theorem of algebra, and the well-known impossibility of a finite general prodecure for finding the roots of an equation of degree greater than four. All of this knowledge found its way into the textbooks. By 1860, with the publication of Serret's Algebra, a hundred years after Euler's publication, the second, and indeed even present day view of classical algebra emerged, namely The Science of the Solution of Equations. In fact, in Serret's text one finds for the first time the highest point in the algebraic theory of equations, namely, Galois theory which is a first milestone in the development of a contemporary algebra.

The Development of the Contemporary Aspect of Algebra.

It is difficult to fix the date of the birth of modern algebra, that is, when it could be recognized as a unique and different study from that of classical algebra. Perhaps the first date is 1910 when Steinitz's The Algebraic Theory of Fields was published. Here there is a systematic treatment of operations upon abstract elements, that is, things that are no longer numbers, variables, or figures of classical arithmetic, algebra, or geometry. The next book to accomplish this same task was Modern Algebra by Van der Waerden, published in 1931. In 1941, the first book of this nature in English appeared as A Survey of Modern Algebra by Birkhoff and MacLane. These dates confirm the proper use of the word "modern."

The creation of modern algebra was not instantaneous; it was preceded by a century of intensive creativity in research and development of new ideas. During this time, there were three great streams of mathematical endeavor, running concurrently, which can be listed as follows:

1. Algebraic, in the sense of solution of equation, given in the studies of Lagrange and Gauss, with Abel and Galois developing the theory of groups of permutations, and the formalizing of this study by Jordan and Serret.

2. Geometric, in that the geometrical explanation of complex numbers by Wessel, Argand and Gauss led to the study of vectors which blossomed into what today is

called linear algebra. From a pure point of view, this geometric development was highly influenced by the study of geometric transformations and the operation of composition of transformations. Furthermore, the invention of non-euclidean geometries led to relations between geometries and groups of transformations, as given by Felix Klein. This was one of the movements that led to the significant feature of contemporary mathematics, called its unity.

3. Arithmetic. During the nineteenth century beginning with Gauss (Disquisitiones Arithmeticae. 1801) the unfolding of the nature of number was accomplished. Gauss gave the first introduction to some of the fundamental ideas of modern algebra, e.g. equivalence relations, finite commutative groups, and extensions. During this period, the principal researchers on number were Dirichlet, Kummer, Kronecker, Weierstrass, Meray, Cantor, Dedekind, and Hilbert. The American mathematicians Benjamin Peirce, his son C.S. Peirce, Gibbs, and Dickson also made important contributions.

It would be out of place to attempt here a complete historical development of these three streams -- algebra, leading to groups and operational systems; geometry, leading to linear algebra; and arithmetic, leading to an understanding of number and some ideas of modern algebra. We give only some significant examples.

Returning to the middle ages, we find the first step toward the "modern" was the discovery and use of complex numbers. Cardan used them but called them fictitious. Bombelli showed that they could be roots of a cubic equation. DeMoivre, in 1725, gave a complete theory of computation for these numbers, using the forms $a + bi$ and $r (\cos \theta + i \sin 0.)$. The geometric representations of these numbers, given around 1800 by Wessel (Denmark), Argand (France) and Gauss (Germany), paved the way for the acceptance of these numbers as bona-fide mathematical entities. After thirty years of work, Gauss in 1831 gave a purely algebraic theory, independent of any geometric interpretation, that is accepted today. A complex number is an ordered pair of real numbers (a,b) with the properties

$(a, b) = (c, d)$ if $a = c$ and $b = d$ (equality)

$(a, b) + (c, d) = (a + c, b + d)$ (addition)

$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$ (multiplication)

Now the set $R = \{(a,0)\}$, a subset of the set of complex numbers, has the same structure as the set of real numbers. Further, since $(0,1) \cdot (0,1) = (-1,0)$ we can say that a square root of $(-1,0)$ is $(0,1)$ which is denoted by the symbol i. Since the set $P = \{(0,b)\}$ can be identified with the set $\{(b,0) \cdot (0,1)\}$,

$$(a,b) = (a,0) + (0,b) = a + bi$$

Returning to the geometric representation, the number i was also interpreted as a rotation operator of $\frac{\pi}{2}$ centered at the origin. Later Cayley gave a matrix representation for complex numbers which for the first time introduced the idea of a structure and its realization.

Gauss' logical exposition of the complex number system was a turning point in the development of mathematics because it marked a first application of the postulational method to algebra and also because it opened the way for an explanation of systems of ordered triples (a,b,c), ordered quadruples (a,b,c,d), and finally ordered n-tuples, $(a_1,a_2,\ldots,a_n)$. In 1834, the British mathematician G. Peacock published an algebra in which the first volume treated the algebra of the domain of whole numbers and introduced the communtative, associative, and distributive laws. So novel was this that it gained no acceptance until 1870. In his second volume Peacock extended the algebra to include rational and real numbers by the so-called Law of Mathematical Permanence. This principal is no longer held, but in essence it decreed that if a number system is extended to contain new elements, then the operations on the elements of the new system must be so defined that the properties of the old system will continue to hold. This idea of extension is the first hint of the manner in which subsequently vectors were extended to n dimensions, finite or infinite.

The second stream leading to modern algebra arose from the continued study of solution of equations. Lagrange sought the reasons for the success in the solution of equations of the first four degrees and the stumbling block to the solution of higher degree equations. For the quadratic equation $x^2 + bx + c = 0$, the roots are related by $x_1 + x_2 = -b$ and $x_1x_2 = c$. Lagrange investigated the relation of the roots of the cubic

$$x^3 + px + q = 0$$

Using Cardan's well known formula, namely,

$$A^3 = \frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^2}{27}} \qquad B^3 = \frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

the roots of the cubic are

$$A + B,\ A\omega + B\omega^2,\ A\omega^2 + B\omega$$

where $\omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega^2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ (the complex roots of unity). Designating these three roots by x_1, x_2, and x_3, respectively Lagrange first noted that there can be six permutations of these roots which can be grouped into two sets.

$$(x_1,x_2,x_3); \quad (x_2,x_3,x_1); \quad (x_3,x_1,x_2) \qquad \text{(I)}$$
$$(x_1,x_3,x_2); \quad (x_3,x_2,x_1); \quad (x_2,x_1,x_3) \qquad \text{(II)}$$

Lagrange proved that the expression

$$(x_i + \omega x_j + \omega^2 x_k)^3$$

(where (i,j,k) is any permutation of $(1,2,3)$)

took on only one value for any permutation in Group I and another value for any permutation of Group II. In fact, a little algebra will show, recalling $1 + \omega + \omega^2 = 0$, that

$$x_1 + \omega x_2 + \omega^2 x_3 = 3B; \quad x_1 + \omega x_3 + \omega^2 x_2 = 3A \qquad \text{(III)}$$

Finally the relation $x_1 + x_2 + x_3 = 0$, added to those in III, reduced the solution of the cubic to that of a system of three equations of the first degree. This is one of the first examples of reducing an algebraic problem to linearity.

Lagrange also investigated the roots of a fourth degree equation and showed that the expression $(r_1 \cdot r_2) + (r_3 \cdot r_4)$ took on only three distinct values when the roots r_1, r_2, r_3, r_4 of the equation were permuted in every possible way (4! or 24 ways). This was sufficient to suggest that the problem of solving equations of the fifth degree or higher was related to certain expressions which were in some way invariant under permutations of the roots.

For over 300 years the problem of finding a finite algorithm for determining the roots of an equation, which involved only the operations addition, subtraction, multiplication, division, and extracting roots, challenged the greatest mathematicians. Lagrange came very close when he sensed that the key to the problem involved studying permutations of the roots of equations. Fifty years after this conjecture, Niels Abel and Evariste Galois solved the problem--there are no formulas for equations of degree five or higher. This ushered in a new era in the development of algebra--the beginning of the theory of groups and the resulting study of structures.

Galois continued the study which led to the creation of group theory by examining the set of permutations of the roots of polynomial equations of given degree n. He thus derived the simpler properties of composition of permutations of roots. A binary operation was performed on other objects than variables and numbers.

The third stream leading to modern algebra concerned itself with numbers. In 1801, Gauss defined congruence of integers -- a brand new idea in mathematics. Two integers a and b are congruent, modulo m, if and only if a and b give the same remainder when divided by m. An important outcome of this definition: $a \equiv b \pmod{m}$, is that the relation "$\equiv$" is an example of an equivalence relation; that is, "congruence, modulo m" is reflexive, symmetric, and transitive. Gauss was the first to show the importance of this relation. He further showed that this equivalence relation partitions the set Z of integers into disjoint subsets whose union is all of Z. E.g. if the relation is congruence modulo 3, the subsets, called equivalence classes, are

$$s_0 = \{0, \pm 3, \pm 6, \pm 9, \pm 12, \ldots\}$$
$$s_1 = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$$
$$s_2 = \{\ldots, -7, -4, -1, 2, 5, 8, \ldots\}$$

Now an algebra of classes can be formed consisting of three elements s_0, s_1, s_2. These classes can be operated on according to the tables shown.

+	s_0	s_1	s_2
s_0	s_0	s_1	s_2
s_1	s_1	s_2	s_0
s_2	s_2	s_0	s_1

$\cdot$	s_0	s_1	s_2
s_0	s_0	s_0	s_0
s_1	s_0	s_1	s_2
s_2	s_0	s_2	s_1

This same idea was applied by Cauchy and others to congruence of polynomials. Cauchy considered congruence modulo $(x^2 + 1)$. The remainder of two given polynomials after division by $x^2 + 1$ are then polynomials of the first degree. Thus for any two such functions one has

$$f(x) = a + bx \pmod{x^2 + 1}; \quad g(x) = c + dx \pmod{x^2 + 1}$$

From this it is easy to show that

$$f(x) + g(x) \equiv (a + c) + (b + d)x \pmod{x^2 + 1} \text{ and}$$

$$f(x) \cdot g(x) \equiv (ac - bd) + (ad + bd)x \pmod{x^2 + 1}.$$

These formulas, along with the geometric interpretation by Gauss and others for complex numbers showed that the algebra of complex numbers is reduced to that of the congruence of polynomials with real coefficients, modulo $(x^2 + 1)$. This was a forerunner of the concept of isomorphism.

These investigations led to the extension of operational systems beyond the complex numbers. In 1843, Hamilton invented quaternions. Early work on complex numbers showed that they could be used to describe rotations and dilations in the plane. After years of unsuccessful attempts, he finally perceived that the obstacle to success in extending complex numbers was the commutative law of multiplication. According to Peacock's law of permanence, this property should be retained. However, in a moment of insight (after 15 years of deliberation), Hamilton banished the law of permanence, developed his quaternions, and freed mathematicians to extend their investigations to a much wider variety of algebraic structures.

In the manner that complex numbers were represented as ordered pairs of real numbers, Hamilton's hypercomplex numbers (or quaternions) were ordered quadruples (r, a, b, c) for which

(1) $(r, a, b, c) = (r', a', b', c')$ if
$r=r'$, $a=a'$, $b=b'$, $c=c'$.

(2) $(r, a, b, c) + (r', a', b', c') = (r + r', a + a', b + b', c + c')$.

Using a suitable definition for multiplication, e.g.

$$(r, a, b, c)(r', a', b', c') =$$
$$(rr' - aa' - bb' - cc',\ ra' + ar' + bc' - cb',\ rb' + br' + ca' - ac',\ rc' + cr' + ab' - ba')$$

and designating (1,0,0,0) by u, (0,1,0,0) by i, (0,0,1,0) by j and (0,0,0,1) by k, the product of any two of these elements may be computed by using the following table. Note: $i^2 = j^2 = k^2 = -u$.

$\cdot$	u	i	j	k
u	u	i	j	k
i	i	-u	k	-j
j	j	-k	-u	i
k	k	j	-i	-u

In 1844, F.G. Grassman published the results of a more general theory of n-tuples. Here he showed the vast richness of structure that was available through the use of only a few postulates. He called his development the Theory of Extensions (Ausdehnungslehre) and although he anticipated many results of later algebraists, his work remained largely unknown until after 1900 when it was applied to quantum theory in physics.

The study of matrices was part of the third big stream leading to modern algebra. They were brought into being by Cayley around 1860 as he was studying linear transformations on equations with two variables. Ignoring the variables, Cayley abstracted the coefficients of the transformation and wrote them as a square array. Thus the transformation

$$x' = ax + by$$

$$y' = cx + dy$$

could be completely described by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Cayley developed an albebra of matrices of order n by using the properties of linear transformations on n variables. He also showed how to use 2 x 2 matrices to represent quaternions and later (1884) how to represent the complex numbers by a certain subset of the set of 2 x 2 matrices.

To summarize, the nineteenth century produced three great streams of algebraic development. One of these was the work in pure arithmetic dealing with the extension of the real numbers to complex numbers and quaternions, leading to ordered pairs, triples, quadruples and n-tuples of real numbers. Another stream was that of classical algebra as exemplified in the attempted solutions of higher degree equations. This stream led to the study of permutation groups and finite groups which was greatly elaborated at the end of the last century. The final stream had a distinctive geometric flavor as seen in Hamilton's quaternions, Grassman's extensions and Cayley's matrices, all of which led to the development of linear algebra and vector spaces. By 1910, all these streams became one great confluence which we now term as modern algebra.

<u>Algebra Today</u>.

The preceding pages have traced classical algebra from its vague numerical beginnings to that of calculations with quantities an[illegible] finally to the study of solution of equations. The merging of the three great streams just described gave need for a new definition of algebra as the study of structures.

This new definition of algebra has slowly evolved over the past two centuries with two intrinsic characteristics of all mathematics. First, little by little, the theory of groups was extended (and narrowed) to include structures such as rings, integral domains, fields, monoids, semigroups, etc. The concept of a group with other structures operating on its elements led to the development of vector spaces, modules, algebras, and so on. For the mathematician, it was the use of, and the recognition of, fundamental strucutres as given above that represented a tremendous breakthrough in mathematical thought. Problems that previously were insoluble by the techniques of classical algebra were now examined from a different point of view and in many cases were solved. In geometry and analysis, algebraic structures have become a unifying thread which has recently extended into all branches of mathematics. The university instructor who gives a course in modern analysis finds that unless his students have a good foundation in contemporary algebra he cannot discuss important concepts, for example,that of an operator. Nor can he discuss the behavior of operators acting on Banach or Hilbert spaces (both arising from the vector space structure), the continuity of operators on these spaces, or the spectral theorem in any of its particular forms. In geometry the concept of a group enabled the mathematician to describe different geometries in terms of groups of transformations. In fact, the recognition of structure in geometry actually contributed to a new definition of the subject as given later. The new concept of algebra became a tremendous tool in the further expansion of all mathematics.

The second characteristic is that the structures of abstract algebra began to find application in the description of physical phenomena. In 1890, the Russian crystalographer E.S. Fedorov showed how group theory could be applied to classify systems of points in space which describe the atomic structure of crystals. This marked the first time that group theory had been applied to solve a previously unresolved problem in science. Later, J.W. Gibbs, an American scientist, used an algebra of ordered triples to help in the theory of refining oil by cracking crude oil. Recent applications of matrices and the techniques of linear algebra (whose foundation is built around the concept of a vector space) have helped to solve technological problems in all sciences -- physical, behavioral, biological, astrophysical. They have also contributed to the basic requirement for acceptance of any new development in mathematics -- that is, it must have application outside of mathematics.

The current view of algebra can be described as an ever generalizing study of structures for which there is

(1) A set of elements (undefined).

(2) A set of statements relating the elements (the structure).

(3) A logic for drawing inferences.

(4) A series of propositions that can be proved in the structure (the theory).

(5) A search and study of significant realizations of the theory (the application).

From all that has been presented above, it is clear that algebra today contains all the essential substance of classical algebra. But it is also a completely differently conceived organization in which numbers and equations are subservient to the structures, the realizations of the structures, and the host of activities and applications that can be derived from both the structure and its realizations. From 1910 to 1955, the development of abstract and linear algebra was conceived as an advanced study far removed from the high school instruction. Today we recognize that this modern algebra must be at the very core of the algebra we teach in secondary school -- of course, presented in a form adapted to high school student maturity. Moreover, the unity of all mathematics demands that this algebra be taught with every possible appropriate intervention into geometry and analysis.

II. Geometry.

The history of the origins of geometry is similar to the development of the concepts of number and algebra. Arising out of practical activity (observation of the sun, stars, and physical pehnomena) and a need to describe his surroundings, man slowly conceptualized concrete geometric forms until they took on a meaning of their own. We know that the earliest definition of geometry was that of a study of "earth measure". From texts such as the Moscow (circa 1850 B.C.) and Rhind (circa 1650 B.C.) papyri, it is possible to examine the type of problems which were being solved more than 3,000 years ago. All 110 problems solved in these two texts are numerical, and twenty-six of these are geometric -- mostly concerned with the problems of calculating areas and volumes. The Moscow papyrus even contains a correct description of the formula for the volume of the frustrum of a square pyramid. Geometry remained a calculating science for over 1,000 years, before it became a deductive subject. This was the contribution of the Greeks.

The Euclidean Era.

Although geometry was developed into a deductive science in Greece prior to Euclid, his collection, synthesis, and elaboration of this knowledge into the Elements represented one of the greatest achievements in the short history of mathematics. In fact, for more than two thousand years nothing was added to geometry to essentially change its foundations until the appearance of Lobachewsky's New Elements of Geometry (1829).

It is well known that the geometry contained in the Elements is neither a complete exposition of Euclidean Geometry nor is it a flawless presentation of what it does contain. Recognizing that the text probably appeared around 325 B.C. (although no original copy has ever been found), this certainly comes as no surprise. However, it is essential to discuss some of these logical difficulties in order to understand and appreciate the recent developments in the subject.

The necessity to accept certain primitive terms as undefined in order to develop a non-circular system was neither recognized by the Greeks nor by Euclid. The first flaw in his work, then, was his attempt to define objects such as, a straight line (a line which lies evenly with the points on itself), a boundary (something which is the extremity of anything) and a surface (that which has only length and breadth). As a result certain logical defects developed in his system. For example, if a straight line lies "evenly" with the points on itself, how do we define "evenly"?

A second fault was that his set of assumptions was incomplete. It was not possible to prove all the statements contained in the Elements from Euclid's axioms and common notions. This flaw also leads to certain well-known fallacies and inconsistencies.

As early as Proposition 1 of Book 1 of Euclid's Elements, we can find an incomplete proof. Here Euclid considers the problem of constructing an equilateral triangle given a side. This well-known construction requires that we determine the points of intersection of a certain two circles. Euclid merely asserts that they intersect, probably because the diagram strongly suggests they do. The fact is, on the basis of Euclid's definitions, axioms and postulates, it can not be proved that the circles intersect. In order to prove this construction additional postulates are needed.

Non-Euclidean Geometries.

The first postulate to undergo critical analysis was the assumption that through a point not on a line there exists one and only one line containing that point and parallel to the given line (Playfair's version). Mathematicians had a feeling that this assumption was not independent of the other postulates but could be deduced as a theorem. The theory of parallel lines, therefore, became a focal point for the energies of some of the greatest mathematicians of the past (Wallis, Saccheri, Lambert, Legendre, Gauss, Bolyai and Lobachewsky. It is a tribute to Euclid's genius that he included the statement concerning parallelism as one of his postulates (even though there is indication that he was not quite sure of its independence).

As we now know, if the postulate concerning parallel lines is replaced by the following:

> Through a point not on a line there exists more than one line containing that point and parallel to the given line. (1)

then a perfectly logical non-Euclidean geometry (hyperbolic) may be developed. This was the important contribution of Bolyai and Lobachewsky, who benefitted from the years of efforts of those who tried to show that Euclid parallel postulate was deducible from his other assumptions.

It is a matter of record, however, that Lobachewshy had begun to work on the independence of the parallel postulate as did his predecessors. His approach was to assume (1) and Euclid's other assumptions and then deduce some consequences from them. If he were to arrive at some contradiction, then Euclid's postulate would have been proved indirectly. Since no contradiction was reached, he concluded

a. that the parallel postulate is not provable from Euclid's other assumptions.

b. that a geometry may be developed which appears to contradict our intuition but is logically consistent.

The obvious implication of these two statements is:

THERE IS MORE THAN ONE GEOMETRY!

It is interesting to note that rarely is one man totally responsible for developments in a science as we have described above. Among the great mathematicians who were attracted to the problems of geometry were Johann Bolyai, Taurinus, Legendre, d'Alembert, Schweikart, Lagrange, Gauss . and Saccheri. Saccheri more than 90 years before Lobachewsky reached his conclusion, pioneered the use of the indirect method of proof in analyzing the parallel postulate. He, too, could not arrive at any logical contradiction to the axioms but did not recognize the significance of his findings. In fact, Saccheri deduced many of the theorems which eventually became a part of hyperbolic geometry.

The mathematical world did not immediately accept the conclusions of Bolyai and Lobachewsky. It wasn't until Bernhard Riemann's publication, <u>The Hypotheses Which Lie at the Foundations of Geometry</u> (originally given as a lecture in 1854 but published posthumously in 1868) that mathematicians began to recognize spaces other than Euclidean. In this article, Riemann not only generalized the concept of space by considering various n-dimensional spaces with metrics but he allowed for the creation of other non-Eudlidean synthethic geometries by replacement of the parallel postulate with the statement:

Any two straight lines in a plane intersect.

Also in some of the geometries one has to negate Euclid's postulate that two distinct lines have at most one common point.

The immediate result of Riemann's publication was a burst of activity with emphasis at first on the development of different types of geometries. A new light was thrown on these different geometries by Felix Klein in 1872. In his Erlanger Program he showed quite clearly that one of the criteria that may distinguish one geometry from another is the particular group of its transformations. Different geometries are viewed as those possessing particular properties of space that are preserved under particular groups of transformations. A geometry can be determined by a group; every group of transformations determines a geometry. (However there are geometries that do not fit into Klein's classification.) In particular, groups of similtudes and isometries lead to affine and Euclidean spaces respectively.

Moreover, Riemann extended the growing subject of differential geometry from a study of curves and surfaces in three dimensional Euclidean space to a study of quadratic forms with n coordinates. The story of the advance from Riemann to the present day global differential geometry and differential topology is well known to researchers in this field. Today the development of geometry and its counterpart topology are going on in all directions. The geometries being studied include projective space, Euclidean space, Hilbert and Banach spaces, 4, n, and infinite dimensional spaces, convex spaces, metric spaces, topological spaces, and so on. These theories are finding applications in and outside of mathematics, for example in the relativistic space of the physics of time and gravity and the quantum theory of nuclear physics. From all this activity it is plainly evident that geometry today has a tremendously different aspect than that which is still in today's high school program.

The Perfection of Euclid

Beginning a century ago, with the revelation of Euclid's flaws, a movement developed among the outstanding mathematicians really to clear Euclid of all blemishes. The search was for a minimal complete set of independent axioms which would place Euclid's synthetic geometry on a perfect logical foundation. All inconsistencies, fallacies, and hidden or unmentioned assumptions were to be eliminated.

The task was first completed by Moritz Pasch in 1882. It was followed by others, namely Peano, Pieri (members of the "Formulario" group, a forerunner of "Bourbaki") and culminated in 1899 with the publication of <u>Grundlagen der Geometrie</u> by David Hilbert. The problem of perfecting Euclid was solved for the mathematicians, However, the solution was far too complicated and abstract to be used as a high school subject on axiomatics and proof.

There then followed a period of sixty years of sporadic efforts to do something about the subject as a secondary school subject. Euclid must be saved! The first significant modification of Hilbert's axioms was given in 1929 by G.D. Birkhoff who affected a great economy in the number of axioms to be admitted by substituting the order and completeness properties of the real numbers.

Geometry Today

Today, geometry must be defined as a <u>study of spaces</u>. Each geometry is a (set, structure) where the elements of the set are called points, and the structure is a set of axioms (including definitions) which relate the points and important subsets of them. It is useless to attempt to list all these geometries (look in any library catalog under geometry), since to do so would certainly result in unintentional omissions. This new definition evolved slowly as a result of two phenomena. The first was purely mathematical--the discovery and description of non-Euclidean geometries and of "spaces" such as topological, vector, Banach, Hilbert, metric, etc. The second phenomenon, and more influential, occurred as a result of advances in science and technology. The advent of relativity was most significant. After Einstein showed that the existence of matter in a space-time relationship is actually described by a four-dimensional model of a Riemannian space, other spaces found application in physics, astronomy, biology, and economics. Euclid's geometry is just one of many, and to imply otherwise would be to deny all that has happened in mathematics and science during the last 100 years.

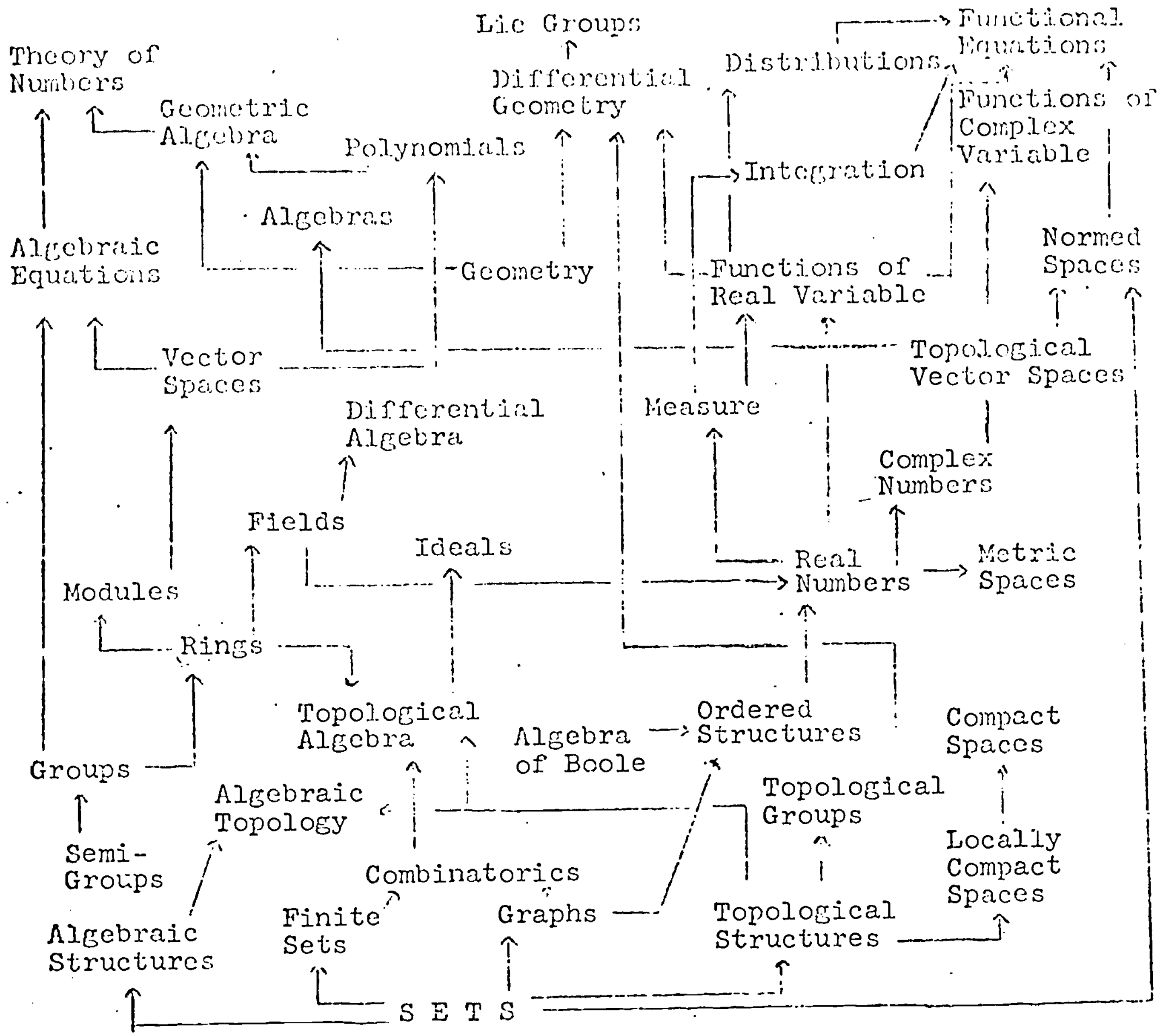
How geometry may be viewed today is epitomized in the following description by Seymour Schuster:

"Geometry is to be regarded as a body of knowledge that had its origins in the study of physical space and physical objects, but concerned itself with abstractions that derived from such study. Hence early geometry dealt with concepts such as points, lines, curves, surfaces, distance, area and volume. Over the past few centuries, the imagination and creativity of mathematicians (influenced considerably by the changing ideas in physics) have produced many extensions of this study. They have developed higher levels of abstractions, variations of axiomatic systems, and many different techniques for the analysis of geometric problems. Thus we have different geometries: Euclidean and non-Euclidean geometries, projective geometries, n-dimensional and infinite dimensional geometries, and a host of others; and we have different analytical techniques that are exhibited by some of the following familiar labels; analytic geometry, vector geometry, transformation geometry, differential geometry, algebraic geometry, combinatorial geometry and still others."

Unification of Mathematics.

The foregoing historical development of algebra and geometry indicates the sort of unity that mathematics has achieved. One cannot help but notice that there are common concepts and strands namely -- sets, relations, mappings, operations, structures, and logic that pervade all the branches. These ideas occur in all the branches with exactly the same conceptualization and mode of use. While the fundamental elements -- points, numbers, functions, or what you will, may take on a specific meaning, according to the structure placed upon them, the same structures appear in all the branches. These fundamental ideas become unifying agents. This unity and inter-relatedness can be observed in Figure 1 which illustrates, in part, the organization created by Bourbaki. We shall refer to

BOURBAKI STRUCTURE OF CONTEMPORARY MATHEMATICS



Source: IBM France

Figure 1

it as "Mathematics in a Contemporary Setting."

This unity of mathematics has been well expressed by Andre Lichnerowicz, a distinguished professor of Applied Mathematics at the College de France, as follows:

"What Piaget calls deductive sciences -- logic, mathematics in the usual sense of the word, and also information theory -- I shall now baptize "Mathematic" because it all concords with the mathematical process and its ambition to build up a type of discourse "without background noise" which is coherent and compelling for others and able by its very form to prevent rejection of its content.

Logic may mean either the mathematical study of certain forms of algebra or what is also often called Metamathematic. But we have learned with Godel that mathematics is not only infinite downwards -- this we have already known for some time --- but also upwards and that it is pure convention which now and then makes us put up the sign "Mathematics country begins here". Recently, for specifically mathematical reasons with the appearance of the notion of category, the sign was moved upwards above the concept of sets.

Mathematics has been studying itself for a century and a half and has become aware of its real ambitions and the limits imposed on those ambitions; it throws an aseptic light everywhere on the workings of our minds and on the

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Mathematics and Transdisciplinarity. Address by Andre Lichnerowicz at O.E.C.D. Conference Nice, 1970.

conditions of communication, the essential point being as follows: any would-be unequivocal discourse without misunderstandings or background noise can only be a discourse subject to mathematical severity, i.e. in fact, a mathematical discourse. But the irony of mathematics has supplemented this by the following: it is impossible to prove mathematically that the mathematical discourse is really unequivocal. Anyone who studies contemporary mathematics' view of itself will observe three major features.

One is first struck, I think by the absence of a privileged plan of mathematical beings. A set (or a category) is, I venture to say, a set of anything -- numbers or functions certainly, but also a set of sentences in a language, of elementary tasks in a project or of exhchanges within an economy. Various structures can be defined from these sets, the actual concept of structure lending itself to a technical definition which has no place here and is based on two fundamental operations concerning sets: taking the product of several sets, taking the set of the parts of a set. Perfect dictionaries can exist between sets, respecting or transporting structures, which leads us to the concept of isomorphism between structures.

At the same time, there is no idolatry of the thing for itself, no working of miracles, within the mathematical process. The mathematician always works to the nearest perfect dictionary and often unscrupulously identifies objects of different nature when a perfect dictionary or isomorphism assures him that he would only be saying the same thing twice in two different languages. Isomorphism takes the place of identity. The Being is put between brackets and it is precisely this non-ontological characteristic which gives mathematics its power, its fidelity and its polyvalence. In truth, any fact can be regarded as mathematifiable so long as it submits to this singular treatment of isomorphism or rather insofar exactly as what we overlook in this way is not important to us. We can always weave a mathematical net with an arbitrarily close mesh but from which the ontological wave will necessarily flow away.

The third feature of contemporary mathematics is its unity. By making a common language and finding common elementary structures it has cast aside the old historical framework which would have broken it up into disciplines evolving in different ways. That is why we can speak of the Mathematic."

This is a beautiful and highly literate expression of mathematics today, which, to place in more common language would be to destroy its effectiveness. Such a viewpoint however was made possible by three outstanding contributions among many other of lesser importance. The first of these was Georg Cantor's Set Theory (1874-1897). Its acceptance at the start was delayed by certain paradoxes which were later removed, but for the first time in the history of mathematics, the elements under any discussion were singled out into a domain of discourse. These elements were given a mode or symbolic process for representation, and a meaning by the structure placed upon the elements. In this way, precision was given to the extended study of the (set, structure) through the use of formal logic. For the first time in history, the word "abstract" had a new reference; it meant, essentially, that the elements were without any sense of esistence or meaning except that imposed by the postulates and definitions which constituted the structure. This was David Hilbert's contribution, which came to be known as "Formalism".

From small structures there evolved larger structures, and it was natural to suppose that one could find a complete structure for all that came under the name of mathematics. In fact Hilbert attempted to develop such a program of mathematics, and it adds more to his fame that he did not succeed, for later meta-mathematics showed that this is not possible. (See paragraph 3 of Lichnerowicz above). However, formalism transcended the other schools of thought such as intuitionism, and logicism, and today it is the basic setting of all mathematics.

It remained for the group of mathematicians who were organized in the mid 1930's and called themselves N. Bourbaki to take the last step so far as mathematics today is concerned. Using sets as a basis, and the formalism of Hilbert, they reorganized and rewrote all the essential existing mathematics -- traditional and modern -- into a contemporary setting. Their work did not constitute new research or discoveries in mathematics but it was an attempt to organize existing mathematics in a form for basic study by prospective mathematicians. They broke down the walls separating arithmetic, algebra, geometry and analysis and made a great overall open structure. In years past a mathematician used to refer to himself as a geometer, or an algebraist, or an analyst. Today he is more apt to say "I am an algebraic topological analyst."

This is the contemporary setting of mathematics. While at the secondary school we do not attempt to teach mathematics just to procure future mathematicians, yet for general culture we should reflect in our program the contemporary nature of our subject and its use. How this can be done is the subject of the next chapter.

Chapter II

SIGNIFICANT REFORM EFFORTS IN SCHOOL MATHEMATICS

It was only natural that the traditional separation of mathematics into the four branches - arithmetic, algebra, geometry, analysis, - would set the pattern for teaching mathematics. In the 13th and 14th centuries arithmetic was a university study pursued by the leading intellectual leaders of that era. By the 17th century it had already or descended into schools preparing for entrance to universities for business, governmental and maritime careers. During the 18th and 19th centuries it found a place in upper elementary school and lower secondary school instruction. In the 20th century all arithmetic of whole and rational (positive) numbers is mastered by the majority of students before leaving elementary school. Only the application of ratio, proportion and percent to business, banking and other social uses remains a part of lower secondary school study.

In the 14th and 15th centuries only top researchers were engaged in algebraic study confined mostly to the solution of equations and systems of equations. In the 16th and 17th century, this algebra became a university study exemplified by Euler's treatise of algebra as a study of calculation with quantities. At the end of the 18th century and during the 19th century, this algebra entered lower university and upper secondary school study while the more theoretical parts remained as college algebra. At the beginning of the 20^{th} century, elementary algebra - quadratics and beyond had established itself as secondary school study - (Grades 9 - 12).

If a mathematical tree had been drawn four hundred years ago, it would have put arithmetic and mensuration (geometric measures) as its roots, algebra as its trunk, both supporting the crown of geometry. Geometry, the type developed by Euclid and expanded on by great scholars down through the centuries, was researched by the mathematicians, and taught as the only axiomized structure of knowledge in university institutes of geometry (The Cremona Institute of Geometry, the University of Bologna, e.g. which still exists today). It remained a university subject until the middle of the 19^{th} century when the first few books of Euclid descended into the last years of high school study. Today, Euclid's plane geometry in modified form is generally a tenth year school study.

The calculus was conceived by Newton and Leibnitz during the period 1665-1700. It was post-university study until 1750 or later. From 1800 to 1870 it was an upper university study, but by 1900 it had become, in the larger established universities, a sophomore - junior year study. In 1908 Felix Klein urged that calculus be taught in the last year of the German gymnasuim (13^{th}

school year) and after World War II, the calculus moved down to become a first year university study. The tree of mathematics moved all of its previous subjects down into the roots and trunks and calculus became the crowning glory. In the 1950's for capable students, an advanced placement program moved secondary (9 - 12) instruction in algebra , geometry, and trigonometry to grades 8 - 10, a pre-calculus study of algebra, trigonometry and analytic geometry to grade 11, and a year of calculus study to grade 12.

The only goal of all this sequential arrangement was preparation for the study of calculus. The program was established by the College Entrance Examination Board in 1901 and is almost universally in existence in the schools of the U.S.A. today. It is illustrated by the Figure 2. The use of this mathematics was largely or totally confined to the study of physics and astronomy. In countries other than the U.S.A., the branches had become fragmented, that is, algebra, geometry, and trigonometry were taught in each of the school years 7 through 12 but each remained a separate branch, mostly unrelated until the study of calculus was begun. Calculus, for those who managed to continue mathematics study, became the great unifying agent.

Today the U.S.A. is the only developed nation that continues the 1800 A.D. conception of separation of the

Traditional U.S A. Mathematics Sequence

	Subject	School Year: Regular	School Year: Advanced
	CALCULUS	14	12-13
	ANALYTIC GEOMETRY	13	11-12
Pre-Calculus	ALGEBRA 3	12-13	11
	TRIGONOMETRY	12	11
	SOLID GEOMETRY	12	11
	ALGEBRA 2	11	10
	PLANE SYNTHETIC GEOMETRY	10	9
	ALGEBRA 1	9	8
	BUSINESS ARITHMETIC and PHYSICAL GEOMETRY	7-8	7
	ARITHMETIC	1-6	1-6

Figure 2

branches. A whole year of first course in algebra is followed by a whole year of synthetic Euclidean geometry; this is followed by a whole years study of a second course in algebra which is about one-third merely review of the first course. This is followed by a terminal year study of solid geometry, trigonometry and sometimes more study of algebra. The students are left with an 1850 conception of mathematics as though nothing has happened in mathematics in the last 150 years.

However in the 1950's dissatisfaction with the status quo gave rise to reform movements - at first in the U.S.A. and later world-wide. The strongest force for reform came from the universities where there was close relationship with the graduate and research activities of on-going mathematics. The criticism that arose resulted in a radical change of presentation both in content and formality of undergraduate mathematics courses and a strong disapproval of the high school preparatory mathematics.

The first two efforts at school reform were the University of Illinois Committee on School Mathematics (UICSM), and the Commission on Mathematics of the College Entrance Examination Board (CEEB). The UICSM began by listing 104 skills the Engineering School expected of entrants from the traditional program. However the later study switched to developing concepts as well as skills.

The work and report of the Commission resulted in the formation of the School Mathematics Study Group (SMSG), which while maintaining the traditional separation of the branches in high school study, developed a curriculum for school years 1-12 that did update the concepts of algebra and geometry.

While many people ascribe the impetus for reform of the study of mathematics in schools to the successful launching of sputnik, the professional attack on the problem came at the 1958 Edinburgh International Congress of Mathematicians. In the Educational section of this Congress the first paper revealed the lack of any genuine mathematics being taught around the world to students ages 6 to 16, the most formative cognitive years of life. A third session was given over to U.S.A. efforts to improve mathematical study (the UISMC, CUPM, and CEEB and SMSG with speakers Tucker, Allendoerfer, Price, and Begle among others), At the close of this session, mathematicians and educators of all European Countries and the U.S.A. remained for more than 2 hours to discuss what their respective nations should do to reform school mathematics instruction. Among these persons was the educational director of the Organization for European Economic Cooperation (OEEC), later to become the Organization for Economic Cooperation and Development (OECD).

In November, 1958, following the ICM meeting, the

OEEC organized a two-week seminar held at the International Center of Pedagogy, Sevres, France. It was attended by representatives of all the NATO countries and concentrated on a thorough examination of the mathematics program in effect in France from the L'Ecole Maternelle through L'Ecole Normal Superieur. The program was found to be formal and severe, but of a vintage of 1850. As Lichnerowicz said later:

> "In the whole world, and particularly in France our education had adopted a psuedo-historical style, in which history is artificially reconstructed in a generally mistaken manner. Each mathematical notion is embedded, in our instruction, in the philosophy of the time in which was created. In studying geometry, we have been obligated to become Greek, and Arabians in studying classical algebra, and Western people in in 17th and 18th century in studying analysis or mechanics and that was the end of [illegible] education. Thus, the ordinary man had to ask, [illegible] is it possible to invent something in mathematics?" this question seems to me the most serious protest against a certain type of education. Such non-unified concept of mathematics brings the students into a stage of deconditioning followed by a painful stage of reconditioning. The students have been obliged to think about the same thing over and over again by means of different concepts which appear strange to them and like different languages."

As a result of this seminar, OEEC arranged for an international seminar to discuss what is new in mathematical thinking, and what means should be taken to reform mathematical education so as to reflect contemporary notions and uses of the subject. This seminar took place at Royaumont, France November 23 - December 4, 1959 and is reported in New Thinking in School Mathematics, OECD, Paris 1961. It was this seminar and its report that gave rise to the Nordic Committee on Mathematics Reform (Iceland, Norway, Sweden, Denmark, Finland), the School Mathematics Project of England, the Chambery Plan of France, and National Committees in Italy, Greece and Spain.

The report made a strong case for reform, a *unified* syllabus, the use of modern terminology and symbolism, the elimination of most synthetic geometry and much obsolete algebra, the introduction of vectors and vector algebra including vector spaces, the inclusion of differential and integral calculus through a conceptual approach, and the teaching of probability and statistical inference. (It is significant that the teaching of flow-charting and computer programming was not mentioned at this seminar. Computers had not yet arrived in the world of school mathematics.)

How to implement these new concepts and topics, how to educate and re-educate teachers in the newer concepts and how to create teaching materials were problems that required further study. To prepare materials, OEEC convoked 17 mathematicians and educators for a one month working session in Zagreb and Dubrovnik in August and September 1960. Their work is given in *Synopses for Modern Secondary School Mathematics*, OECD 1961. The synopses proposed topics for the first cycle of secondary school (Grades 7-8-9) and for the second cycle (10-11-12), and indicated the way each topic was to be treated in the classroom, but still left open the procedure of unifying the instruction to classroom experimentation in the years ahead.

After observing the emerging programs of mathematics study in the nordic countries, Belgium, France, Germany and England, there was evidence as early as the fall of 1963 that there was shaping up a general conception of unified mathematics. The conception was based on the algebraic structures that had first been brought into prominence by Van de Waerden's Modern Algebra (1930) and Birkhoff and MacLane's Survey of Modern Algebra (1941) The work of Georg Papy and Willy Servais of Belgium were especially significant for their spiral approach to the learning of the important strands wound around the structures The OECD organized a final conference of all its member countries, held in Athens November 17-23, 1963. The report of this conference appears in Mathematics Today, A Guide for Teachers, OECD, 1964

With the publication of this report, there could be no doubt that the immediate future of mathematics study should take the form of a single unified subject of instruction. This conclusion was further strengthened by the Cambridge (Massachusetts) Conference, Report on Goals for School Mathematics. This report pointed out the limitations under which the UICSM and SMSG operated with respect to a classical curriculum framework, and the lack of adequately prepared teachers to teach contemporary mathematics proposed by the Conference. The programs (two of them) proposed by the Cambridge group, were concerned only with the teaching of a contemporary mathematics that would be required of all scientifically minded persons in the decades ahead. It, in essence, reconstructed a 1963 contemporary major mathematics program through the junior year of a top-rated university in a form that was to be mastered in 1990 A.D. at the end of the senior high school study (by capable students).

It is now established that much of what these mathematicians called for can be learned in the secondary school. One way is through a unified mathematics program that has been developed and is set for th in the next chapter.

Chapter III

AN EMERGING CURRICULUM IN SECONDARY SCHOOL MATHEMATICS

The vast accumulated and growing body of mathematical knowledge that is capable of being learned by secondary school students can no longer serve as a principal basis for building an educational program in the subject. There is just too much of it. Of course there is a core of mathematical understandings and skills, not too large, that is essential for carrying out everyday activities--arithmetic computation with numbers in decimal notation, geometrical explanations of size and shape, and the use of formulas and graphs. This knowledge also appears to be useful in the generation ahead, and it must be learned in schools. Hence it must remain in the curriculum. But if we attempt to construct a curriculum based solely on special utilitarian objectives we shall make a number of mistakes. First, viewing the changes in technological development and social behavior in the past decade, and still continuing, we can expect many utilitarian needs of today to become obsolete and their practice useless ten years from now. We need only cite the rule of three, the rule of five, and the six-percent rule for finding interest, once taught to every junior high school pupil. If the reader does not recognize these, he need not worry--they are obsolete and not used.

Secondly, utilitarian training in mathematics is generally taught through rote learning without comprehension of the underlying theory thus constraining the mind to act in a narrow behavioral associative manner rather than being taught as a basis for aiding the intellect to form generalizations for wider application. Thirdly, it is an attempt to make of the human mind a cog in a technocracy -- rather than to create a flexible mind that can adapt itself to new situations -- and indeed to be the creator of them. Lastly, it is evident that it is impossible during the school years seven to twelve to determine which students will need what special mathematical knowledge and skills in their future careers. Hence, special utilitarian aims can play only a minor role in the school mathematics program.

In chapter one it was shown that formal mathematics can be organized in terms of the fundamental ideas of sets, relations, functions, operations, and structures, both the algebraic -- group, rings, fields, vector spaces -- and topological -- compact spaces, metric spaces, and others. This fact was established by Bourbaki as early as the 1940's. What was not known, until quite recently, was how a similar type of unified organization of subject matter could be presented to secondary school students in a teachable and learnable form. Guide lines for such a construction have become available in the form of syllabus conferences and recent classroom experimentation in the United States and Europe.

The programs that are emerging, as revealed by these conferences and experiments, while differing in degree of formalism and abstractness reveal a strikingly common structure. The development has a kind of double helix organization in which abstract concepts and structures as a central core develop in coordination with the most important realizations of these strucuters namely the several number systems, matrices, and synthetic, coordinate, vector and transformation geometry. Probability and statistics as pure and applied mathematics, calculus as real and applied analysis, and numerical analysis related to the use of digital computers are futher realizations of the structures. At all times informal logic provides a precise coordinating language.

To show this substance in more detail, it will be convenient to consider separately how each of the topics--algebra, geometry, analysis, probability, logic, applications and numerical analysis, are entering in the school program. Then an overview will be presented on how these topics merge into one continuous unified study.

Algebra Instruction.

Our purpose in this section is to answer the question--how can algebra be organized and presented to high school students in a manner to be useful, but which will reflect both in content and spirit the contemporary viewpoint of the subject? A preliminary guide line for presenting all secondary school mathematical study is the following:

1. Develop the concepts of set, set manipulations, relations, mappings and operations.

2. Use notions from the theory of sets and the properties of operational systems to introduce the concepts of the structures of group, ring, and field.

3. The teaching of important topics from classical arithmetic, algebra, and geometry should arise from the study of properties possessed by these fundamental structures.

4. Develop the concept of vector space and its realizations as one of the strong unifying elements of all mathematical study at the secondary school level.

As remarked earlier, algebra is no longer merely calculation with numerical and literal quantities, nor just a science of the solution of equations. It is a more encompassing study of structures, realizations and all the activities derived from both of these entities. At the secondary school level, we shall mean by structures essentially those of operational system, group, ring, field, and vector space. By realizations we shall mean the various systems of numbers: whole numbers, integers, rational numbers, real numbers, complex numbers, finite (clock) number systems, conformable matrices, polynomials, and vectors (or ordered n-tuples), all of which exemplify or serve as models for the various structures. By activities we shall mean the use of variables and the study of expressions, functions, solution of conditional sentences (equations and inequations), absolute value, and application to problems both within and outside of mathematics.

These activities include all the important skills and computations learned in the traditional algebraic instruction but now treated from the contemporary viewpoint as well as many important new ideas. The use of the fundamental structures lies not in their formalism or rigor, both of which need not be emphasized, but in their dominant role of unification of mathematics, of permitting efficient study and giving genuine understanding of the basic concepts.

Secondary school algebra involves the following content: operations and operational systems: groups, rings, field, vector spaces, number systems, functions including polynomial, rational, irrational, exponential, logarithmic and trigonometric, their construction and their graphical representation, the solution of systems of equations and inequations including those with absolute value expressions, applications including isomorphisms and models.

To visualize a possible curriculum sequence, we state in more detail the topics that must be considered. To bridge any existing gap between elementary and secondary school study the seventh year can begin with the investigation of finite, so-called clock operation systems, which can be designated by

$(Z_n, +, .) \longleftrightarrow$ (n clock numbers, addition, multiplication)

and contrast their properties with those of familiar whole numbers. This study initiates the concept of a variable, the idea of domain, the search for an identity element and existence of inverse elements, the commutativity, associativity, and distributivity properties, the construction of expressions and the solution of open sentences. Thus in $(Z_6, +, .)$, $2x + 3 = 1$ has two solutions, namely 2 and 5; but $2x + 3 = 2$ has no solutions.

This study can set a base for a general study of a binary operation defined on a set E, as one that assigns to each ordered pair of the Cartesian produce E x E one element of the set E. The usual operations addition, multiplication, exponentation (or power), subtraction and division are re-examined in this light as well as new binary operations such as maximum, minimum, constant, first element, second element, lcm, gcf, and others created by the students. These operations are examined for the properties listed in the foregoing paragraph, and then used for calculation in numerical expressions that can be simplified to a single element. The work culminates in singling out the properties of a group which are then stressed as fundamental to all subsequent study. Later, after the study of integers, rationals, reals, and transformations of a plane, the group will have sufficient examples to warrant the study of itself as an important structure.

The concept of function is usually introduced by mapping a set into itself, or into another set, using arrow diagrams. These diagrams are now finding use in elementary school instruction. The role of domain and codomain now enter the instruction. Relations are contrasted with functions or mappings, the latter being a special type of relation. This initiation to function paves the way for later concepts of ordered pair definitions, and ultimately the definition in terms of subsets of the Cartesian product. Right from the start, the contemporary notation of function should be used namely with f, g, h as functions, for example

$f: x \longleftrightarrow x + 3.$ The function that takes each x(of a domain) into $x + 3$.

$g: x \longleftrightarrow x^2.$ The function that maps each x into x^2.

The notation $f(x)$ or $g(x)$ is used only for naming the assigned element, or the second element in the ordered pair definitions. Finally the operation <u>composition</u> on a set of functions extends the idea of operation beyond that defined on number sets only. The composition is designated by $f \circ g$, read "f following g". All the above development is limited at first to whole and finite numbers systems.

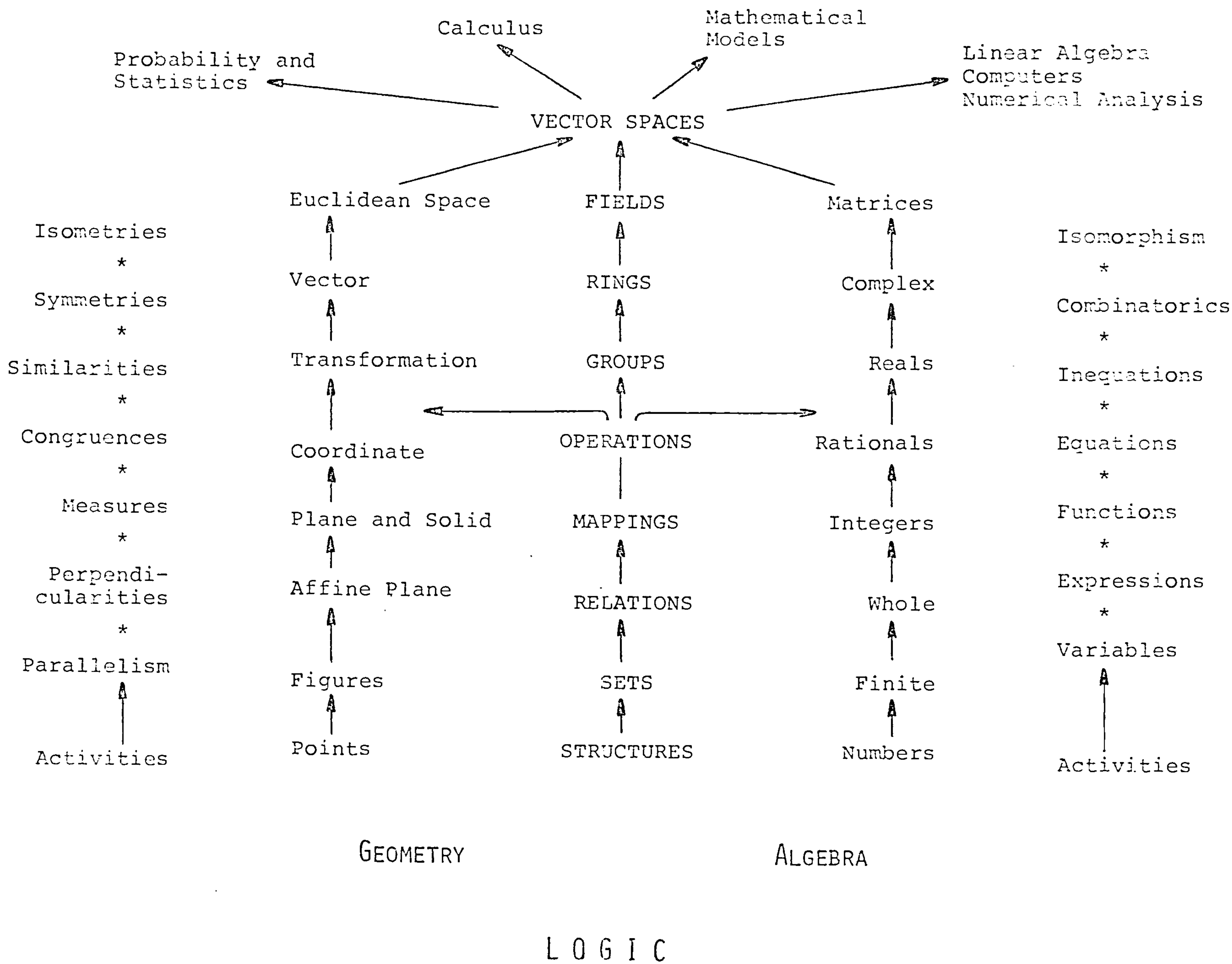
When integers are introduced (there are many ways of doing this) their group structure under addition is stressed. Now all the previous work on expressions, functions, solution to conditional sentences, mappings, etc. can be applied to the set of integers, revealing new possibilities, e.g. subtraction is an operation on the integers. When multiplication is introduced as the operation, the group structure does not exist. With the two operations we have a new system $(Z, +, .)$ which is a ring. This ring of integers can be ordered. In a similar sequence of study, the rationals under addition form a group. Under both operations they form a field, and to be sure an ordered field symbolized by $(Q, +, ., <)$. This field has gaps, for example there is no rational number to solve $X^2 = n$, where n is not a perfect square, that is $\frac{a^2}{b^2}$ or $(\frac{a}{b})^2$, $b \neq 0$. To remedy this situation real numbers are introduced in an informal manner, by adjoining all irrational numbers to the rationals to give a completely new ordered field structure. A study can be made of operations on real functions, that is of $f + g$, $f \cdot g$, $f - g$, and $f \div g$ (the last with domain restricted to those real x for which $g(x) \neq 0$), with graphical illustrations of all these cases.

The polynomials can be constructed by the use of the constant and identity functions and operations of addition and multiplication upon them. The polynomials present a new ring structure. All the usual traditional study -- factoring, special products, graphs, quadratic equations, and rational functions -- arise naturally as activities but always in a setting that demands a specification of the domain of discourse in each activity.

Similar approaches can be made to the study of matrices, complex numbers, and exponential and logarithmic functions. Thus the entire study of school algebra grows from the simple to the more complex, and from the concrete to the more abstract over a period of six years of study. The parallel and interwoven development of structures, realizations and activities is illustrated in Figure 3 on the center and right-hand columns.

Figure 3

A UNIFIED APPROACH TO SCHOOL MATHEMATICS



Geometry Instruction.

The most debated subject during recent years from the point of view of school instruction, is geometry. In 1959 when the famous mathematician Jean Dieudonne said "Down with Euclid", he did not imply that geometry instruction should be abolished. To the contrary he enhanced the study of geometry, but from an entirely different approach than that of a long synthetic study of Euclidean geometry with its global set of axioms. Recent conferences at an international level likewise all point to the necessity of teaching geometry from other points of view in addition to the synthetic approach, and to make the subject an integral part of all mathematics. In general the past method of the study of geometry isolated it from all the other parts of mathematics.

The essential elements of geometry -- points, lines, planes, space, polygons, circles, rays, segments, angles, common solids, etc. are now taught in the elementary school as physical elements in the world about us. At the start of secondary school it is only necessary to review these concepts, to develop some skill in making representative drawings, and to secure some sort of abstraction of these elements as ideals of what they represent.

Today, secondary school geometry must be conceived of as study of spaces. Each geometry is a (set, structure) where the elements of the set are called points and the structure is a set of axioms, including definitions, which relate the points and their important subsets. With this conception, instruction in geometry must be brought more and more into relation with algebra and its structures, and thus it must be developed so as to permit and exhibit the use of algebraic structures and techniques. This is the spirit of the times. In this respect, a very important objective should be to develop geometry so that it leads to a basic understanding of vector spaces and linear algebra.

There are a number of ways, all valid, to study spaces. One can use intuition alone and study physical objects in 2- and 3-space and, by abstracting shape, position, and metric properties where they exist, develop a practical geometry or at least a useful set of geometrical relations. One can proceed synthetically, as Euclid did, choosing a convenient (but small) set of sxioms. One can coordinatize space and make use of the real numbers, as Descartes indicated could be done. Perpendicularity and a distance function can then be used to obtain the Euclidean coordinatized plane. One can also follow the Erlanger Program of Klein, studying mappings, transformations and groups and the resulting geometries. One can also use vectors, first as sensed line segments, and then as points in a space with a fixed origin, leading to an algebra of points as n-tuples. Then one can go from affine to Euclidean vector space by way of an inner product of vectors. Since in the secondary school we are not, and do not need to be, solely concerned with teaching future professional mathematicians, none of these approaches should be used to the exclusion of the others. It appears that a contemporary view of geometry for the educated layman is best achieved by a study that contains all these approaches.

With the educational goals stated at the start of this report and considering the mathematical content feasible for secondary school instruction, the following objectives should guide the development of the secondary school geometry instruction:

1. Develop the concept of space as a set with special subsets, having structures that are linked to others -- especially vector, affine, and Euclidean space.

2. Develop the knowledge of precise relationships between the line and the set of real numbers. This leads to coordinatized space.

3. Develop an understanding of the principal transformations, groups of transformations, and their application, especially in a coordinatized space.

4. Develop an understanding of an axiomatic structure by this sequence of study: the affine line, the affine plane, affine space, metric space, Euclidean space as a vector space.

5. Develop skill in applying the several methods of

geometric development to the solution of original problems -- both mathematical and applied.

6. Unify the mathematical study of algebra and geometry in the concept and application of vector spaces and linear algebra.

Geometry instruction should be included in every year of study beginning in grade seven and continuing through grade twelve. It should grow in complexity and abstraction and at all times be related to those algebraic methods that enable it to become imbedded in a vector-space structure. At all times it should be applied so that it becomes a way of thinking. As Willmore (1970) has said, "What is important is a geometrical way of looking at a mathematical situation; geometry is essentially a way of life."

There are many sequences in which the geometric instruction outlined above can be organized to achieve desired objectives. One need only study the official syllabi of European countries to recognize how many different approaches, with different emphases, reach the same goal. The following sequence, integrated into a six-year unified study, is one reasonable proposal:

1. Start with a physical, informal study, using drawings, paper folding, measuring, and physical objects to gain an intuitive feeling for figures in Euclidean 2- and 3-space, especially for lines, rays, segments and angles.

2. Develop the number line as a mapping of real numbers

into the set of points on a line, preserving order. Scale the line many ways to develop the linearity of relations of the scales, $x' = ax + b$. Compare yard with meter, different temperature scales, and so forth.

3. Develop lattice points as intersection points of two directions. Use the coordinatized affine plane (parallelism only). Introduce perpendicularity and develop transformations of the plane that are isometries. Use translations and groups of translations in connection with vectors, equipollence of vectors, and addition of vectors. Use both transformations and vectors to prove relations in the plane.

4. Introduce dilations with a fixed point and elementary ideas of similitude.

5. Introduce axiomatic affine plane geometry with a minimum of axioms. Develop ideas of proof and prove theorems. Apply them to finite models, then to lattice points, and finally to the continuous plane.

6. Using further axioms (or better, informally), introduce the coordinatized affine plane.

7. Introduce perpendicularity and distance to obtain the Euclidean plane. Examine the group of transformations constituting isometries; treat congruence by isometry. Do linear equations and inequalities with respect to the intersection of lines and half planes. Relate them to matrices. Relate 2×2 matrices to transformations in the plane.

8. Introduce 3-space, both affine and Euclidean, informally

Study relations of lines and planes in space. Consider the measure of length, area, and volume.

9. Introduce (informally or with axioms) coordinatized affine 3-space.

10. Do the algebra of points in an affine plane. Develop the notion of a localized vector and the equation of a vector line and an affine line. Apply this to geometric properties in a plane.

11. Develop the vector-space structure and its linear algebra; apply this to the plane using the concepts of basis, linearity, dependence, and independence; give many other illustrations and applications of vector space.

12. Introduce the concept of inner product; develop affine 3-space as a vector space, define perpendicularity and Euclidean 3-space, and develop theorems in Euclidean 3-space.

13. Develop the conic sections, either by vectors or by rectangular coordinate geometry. Generalize transformations in the affine and the Euclidean plane.

14. Use matrices, transformations, and complex numbers to develop and relate all mathematics in developing trigonometric analysis.

The geometry program suggested in the brief outline above is an eclectic one, to be sure. It attempts to show to some extent what geometry is today; it gives important geometrical knowledge; and above all, it shows how the subject gives clarity and understanding to all other branches. Further, it develops a tool for genuine use for all those who continue their study of mathematics and science. How this geometric study may be organized in some structured way that unifies the various aspects with the 'large' mathematical structures is shown in Figure 3, the center and left hand columns.

Probability and Statistics.

Advances in science and technology have made it necessar to include new topics in secondary school mathematics and science. Probability, in its contemporary setting is one of these topics. Today, the educated citizen must understand the concepts, language, interpretation, and application of this aspect of mathematical knowledge in order to carry on everyday affairs intelligently. The following sequence of instruction is one frequently cited for the secondary schools.

1. The empirical background of probability theory through the exhibition of statistical regularity in everyday life, in nature, in games, in science, etc.

2. A development of the mathematical theory of probability.

3. A host of application of this theory to the description and prediction of random phenomena.

4. A brief overview of the historical development of the theory.

To meet these goals requires a continuous development of the subject from grade 7 through senior high school, as its concepts are developed in relation to, and along with, the corresponding set, algebraic and geometric study that is needed. Using both a-priori and a-posteriori procedures, the subject starts in grade 7 to build the empirical knowledge of outcome, outcome space, trial, event, relative frequency and probability measure. The gathering of data and the study of the distribution of outcomes leads to graphical representations -- histograms and frequency polygons -- and the measures of central tendency and dispersion, for example, the arithmetic mean, mode, median, range, and standard deviation.

At the next level of instruction, the study can be related to sophisticated counting, tree diagrams, permutation, combinations and the binomial distribution. From this empirical and algebraic study, a formal set of axioms can be abstracted to define a probability space with a probability measure. With this elementary treatment which relates events to the simple set theoretic algebra of union, intersection and complementation, a host of interesting and genuine problems can be examined.

The culminating aspect of probability study for the great of students lies in its extension to conditional probability, out of which the ideas of dependent and independent events can be drawn. This leads to the consideration of random variables and expectation, two ideas that are pervasive in the affairs of modern living. All this knowledge can give genuine and necessary insight into insurance, lotteries, and applications to biology, genetics, and physics that abound in the daily news and affect the behavior of every citizen.

Further study, for those with interest and ability can be spent in examining independent and dependent random variables, Markov chains, transition matrices and mathematical distributions. Problems properly selected, can show the wide application of probability theory. In the last year, with the tools of calculus at hand, the extension to continuous probability can be made. Here one sees the fundamental ideas underlying the use of polls, sampling, and quality control techniques.

It should be evident that this program in probability and statistics should not be presented in isolation from the other parts of mathematics and it should not be deferred as a separate study until the last year of secondary school. This is too late for by this time many students have left the study of mathematics for other interests. Unfortunately some of these other interests, sociology, economics, business, for example, now require an elementary knowledge of probability and statistics for their comprehension. But more important is the fact that presenting the subject as an integral unified part of all the rest of mathematics lends efficiency and greater understanding to the learning of it.

The Calculus.

Until quite recently, the calculus was considered a university level subject of instruction. In order to study the subject, students were expected to have acquired a rather comprehensive knowledge of algebra, especially algebraic manipulation, geometry, trigonometry, especially trigonometric analysis, and coordinate or analytic geometry. This expectation has changed radically, and good treatments of the calculus now include the study of much that yesterday was considered pre-requesite.

It is true, however, that one of the reasons for knowing algebra, coordinate geometry, and trigonometry is the basic foundation they give for the study of infinite processes. Here, in calculus, the student sees the total intersection of all these subjects as a unified mathematics study. The student also finds a great number of genuine problems that can be solved by techniques of the calculus, but that can not be solved by any mathematics previously studied. The calculus is, in this sense, a high point in the apprehending of what mathematics is today, and what its power is in the solution of problems.

The traditional introduction to calculus began with a study of sequences and limits of sequences. The existence of a limit was taught informally through the use of inequalities with merely a mention of absolute value, δ and ε, continuity of a function at a part or in any interval was then defined in terms of limits. However, the advent of metric topology has brought about a growing trend to revise the order of study of limits and continuity. This emphasis on <u>understanding</u> infinite processes, as well as being able to carry out operations involving these processes, is the more recent approach to learning the subject.

The teaching of analysis can be strengthened in the secondary school by preparing for it earlier in the school program. The following topics should occur early and be continued each year of study until the formal study of analysis is introduced.

1. Since the first course deals essentially with functions of a real variable, enrich the study of functions with a study of inverse circular functions, logarithmic and exponential functions, and special functions.

2. Familiarize the student with approximate numerical computation with problems leading to the solution x as lying in the interval $a \leq x \leq b$ or in the form of an estimated error $|x-d| \leq$.

3. Place importance on the manipulation of inequalities with absolute values.

4. Introduce linear interpolation either as approximation to a function value or to replace an incompletely known function by a piecewise linear function.

5. Stress in all teaching that the real numbers are an

ordered field with properties carefully, but informally stated and tested.

6. By introducing a concept of "distance" in the topological sense show several different distance functions by means of definitions that can be intuitively sensed.

The study of analysis itself begins with that of continuity and limits where the limit of a function at a point is the value to be given to the function at this point in order to make the function continuous at this point. The intuitive topological concepts of neighborhood, open and closed intervals, and deleted neighborhoods enables continuity to be defined in terms of δ-neighborhoods of the domain associated with ϵ-neighborhoods of the codomain -- first at a point and then over an interval. The limit of a function is then defined in terms of continuity and not in terms of a sequence.

The derivative can be introduced through linear approximations to a curve at a point and this also leads to the differential as a linear mapping. The study can then follow the traditional sequences leading to formulas for differentiation, derived functions, and applications. The notion of primitives can be introduced informally as anti-derivatives. Integration can be introduced independently of differentiation as a solution to a problem of measure. The modern presentation begins with a set of piecewise monotonic step functions on which one constructs the classical rectangles which enclose the bounding function between two step functions. This leads to the Riemann integral. The fundamental theorem of integral calculus follows, along with all the classical techniques of integration and application.

We now see analysis, as described in this section, as an indiscriminate user of all the fundamental ideas of number and space as it develops its own unique procedures for the study of infinite processes. In modern society the concepts and applications of analysis are of great use not only to scientists and technicians, but to the educated citizen as means of understanding what is happening in our scientific-technological culture.

Numerical Analysis and Computers

Perhaps the most striking and far reaching technological development of our age is the digital computer. The first commercial electronic computer, the UNIVAC dates from 1950. Today there are thousands upon thousands of highly efficient computers that have completely transformed the modus operandi of business, engineering, and science. Their use is literally changing the culture of world societies. It thus becomes socially necessary that all future citizens know the nature, purpose, and use of these machines, that there is no magic in them, and that while they may appear to perform miracles, they can do only what man tells them to do.

People must be educated in a cultural understanding of these machines. Also, a host of millions of workers must be prepared to program them by the proper use of computer languages. In addition, a large group of scientists and researchers will be needed who must have a far more extensive knowledge of the mathematical and electronic theory of the construction of computers. The use of the computer as a learning tool is growing, and its use demands a deeper understanding of algorithmic processes, as well as newer constructive theories as used in the solution of all kinds of equations, in linear programming, and in evaluation of integrals. Once again our instruction returns to numerical analysis, but with more general and deeper methods of obtaining particular numerical answers.

This study should begin early in school instruction, no later than grade seven, by making flow charts for simple arithmetic calculations, that is by understanding an algorithm as an operational process. These simple processes applied to the study of real numbers and measures, lead also to an understanding of the nature of approximations and errors. As soon as possible, flow charts should be programmed by the study of simple computer languages, of which a number are now available. As soon as a computer is available, by itself or through a console relayed to a computer center, programs should be carried out and applied to all the current study in mathematics -- this means practically all mathematics study involving numerical or logical answers. The computer should be used to solve all types of problems, for example: systems of linear equations, systems of linear inequalities, probability problems involving simulation, problems involving matrix operations, solving polynomial equations, derivatives and numerical integration. For example, instruction in the theory of equations is shifted from the classical processes to newer iterative methods for isolating and approximating the roots. The computer and its accompanying instructional material, again serve as a means of unifying the branches of mathematics.

Logic.

In the traditional program of high school mathematics, logic, as a separate study, held no place. It was limited in its use to the study of geometry where axioms, definitions and demonstrations were used as means to initiate students to formal presentations. The analysis of sentences was usually limited to those of the form of an implication: If -- then. The use of bi-implications -- "if and only if," or "necessary and sufficient" -- were seldom if ever mentioned. Likewise the use of the quantifiers "for some A" or "for all A" were never mentioned in stating theorems or propositions. If the purpose of geometry was to teach the use of logic, it did not succeed in doing so.

Formal mathematics, structures on sets, axiomatics, and demonstration, requires the explicit use of logic. However, the degree of formalization and symbolization found in mathematical logic is more than is needed by, or within the maturity and experience of, the students in secondary school. Further, the logic that is essential applies to all parts of mathematics -- number theory, algebraic structures, geometric systems, analysis and applications. It must be sufficient to give pupils a correct and usable concept of axiomatic structure, definitions, and methods of proof. Its main use is that of a tool to check the correctness of one's thinking. It has little or nothing to do with the creative or discovery aspects of learning mathematics.

After the student has had a good amount of informal, inductive, manipulative thinking about basic mathematical topics -- number theory, groups, geometric relations of figures in a plane, probability situations, and the like-- it would appear reasonable to spend some time putting these topics into an acceptable structure. To do so requires a basic knowledge of the elements of logical structure. This structure can be built up as the study of mathematics continues, but its fundamental ideas must be initiated early in the secondary school program. While the use of logic per se is not limited to the study of mathematics, and certainly occurs in everyday life situations with the use of the natural spoken language, yet for clarity and precision it should be related to mathematical situations.

The initial use of connectives "and", "or" and negation "not" can be taught in connection with sets and operations on sets. Here, also, one can teach the precise use of logical words "a", "one", "some", "each", "every", "all", "the", and relate them to the phrases they modify by the use of the universal quantifier "For all A $\mathcal{E}$".

The sentential calculus is the most important for the study of mathematics. First one clarifies the meaning of a declarative statement with its assigned true or false value. Then through the use of simple truth tables one establishes the truth values of implications, bi-implications, and contrapositives. Finally all of the foregoing study is used to develop several inference strategies which can be used to test -- prove or disprove -- stated conjectures. The indirect method of proof or contrapositive, loses its mystery when it is subjected to this type of logical analysis.

The only symbols needed in all this study are

not	and	or	if---then	iff	infer
$\neg$	$\wedge$	$\vee$	$\Longrightarrow$	$\Longleftrightarrow$	$\vdash$

Their use lies in the simplification of expressions, not in formalism. These symbols may be introduced gradually, but once learned they should continue to be used throughout the remaining part of mathematical study wherever proof enters. They are tools for common use in mathematics, as well as in all rational thought wherever such thought occurs.

APPLICATIONS.

If the utility of mathematics is to recognized, we must provide illustrations of applications of mathematics in our school study of the subject. Beyond illustrations, we must also, whenever possible, develop the ability to apply mathematical theory and concepts to the solutions of real problems in affairs of life. It is easy to show the "tool" value of elementary arithmetic and physical geometry in the everyday social, business, and trade life of all individuals. It is more difficult to see the higher uses, for example the mathematical explanations of scientific phenomena as it occurs in physics, chemistry, biology, or economics, or the way mathematical theory serves as a model to explain relativity theory, or the structure of an atom. However, as we develop our study of mathematics, it is incumbent upon us to show its uses.

All "word problems" may be considered as applied mathematics, and this type of problem should, insofar as possible, reflect real situations -- that is problems that actually occur in current life. These problems should go beyond a simple example of the mathematics that has been just studied. For example, the axiomatic probability space with a measure function is a model of practically all chance situations a person faces day after day. The applications are enormous, entering every field of academic study. This fact should be continuously exploited throughout each year of study of mathematics. The teaching of calculus bring with it the host of applications to physics and engineering that show the utility of the subject.

However, there are newer applications of more recent mathematical content that are interesting and desirable. Linear programming offers simple, yet genuine examples of the solution of linear inequalities subject to constraints, which can be solved graphically, explained theoretically, and programmed for electronic computer solution by the simplex method. The use of sets and logic in developing a

Boolean algebra that can be applied to simple electrical circuits is another newer application. In fact set theory can be used, not only to explain mathematical systems, but also to interpret such political phenomena as voting coalitions and blocs. The applications of groups and other structures are likewise invading the other disciplines, and one can find this for example in modern books on syntax and grammatical constructions in languages. The alert teacher will find and use problems from physics, chemistry, biology, geology, economics, space navigation and so on because all these fields make wide use of many areas of mathematics and mathematical thinking, that again reflect the unified aspect of the subject.

A UNIFIED CURRICULUM.

When one looks at the Bourbaki structure of mathematics, and the historical events leading to its development, one can sense that the universality that unites so many branches into a connected entity is acquired largely through the fundamental use of sets, relations, and structures. With this conception we can no longer afford to think of school mathematics as a collection of disjoint branches. The school mathematics must be newly constructed by making use of all this mathematical development, properly adapted to the nature and purposes of secondary school learning.

The foregoing partial listing of the content of secondary school mathematics shows to some extent the unity of mathematical study. A complete view of the interrelatedness however must come through a comprehensive study of the content and the manner in which it is taught and learned. Figure 3 can help the reader to see to some degree the way the entire subject is held firm by a central core of fundamentals and structures, for which both geometrical and algebraic realizations form the basis for all the mathematical activities derived therefrom. These activities include the study and use of operations within the structure realizations, manipulating variables, expressions, and all types of functions, the study of conditional sentences and their solutions, graphs and applications, isometries, similarities, constructions, and other geometrical applications.

For efficient teaching and learning, the school curriculum should be as unified as possible. The program discussed above may be described as one that constitutes an advance over the traditional organization by embedding all of mathematics, traditional and new, into a contemporary setting of the subject. The learning is designed to educate our youth for orientation to change and to avoid anachronism as they enter the university or the adult working world. The goal is to provide:

(1) a contemporary viewpoint of algebra as a study of structures, their realizations in the several number systems,

and all the derived activities. It will be a body of knowledge that includes much that is now in collegiate programs. It will prepare students to begin a rigorous abstract algebra and vector space study at college entrance.

(2) a modern viewpoint of geometry as a study of spaces -- eventually related to the algebraic structure of vector space. This is a point of view not available in many college programs today, but it must become the common knowledge of all educated people.

(3) a unified approach to the study of mathematics with the concepts of sets, relations, mappings, operations, and structures binding all the instruction into a continuous spiral approach to learning.

(4) applications of mathematics, not only to physics, but to new areas in the behavioral sciences, where probability and finite mathematics are extremely important, perhaps more so than analysis.

(5) an intuitive, non-rigorous, but correct introduction to analysis via continuity and limits.

This program provides a genuine liberal education in mathematics as it is conceived in the last quarter of the twentieth century. It forms a basis for entering the college or university at the undergraduate level for any professional study whether it be in science, engineering, law, medicine, economics, behavioral sciences, or in pure mathematics.

While in its more severe form it is for students in the upper 15 to 20% of cognitive intelligence, [illegible] a more concrete and practical modification it is a prog[illegible] for mass education. In all of America today, whi[illegible] [illegible]t evident in [illegible] newly constructed curricula, yet clearly discernible in [illegible] politi[illegible]l, social, and economic expressions, made by the govern[illegible]nt, business, and the public at large, there is growing a great concern for mass education. This concern is usually expressed as equal educational opportunity for all children, where equal opportunity is not construed as equal access but as equal acquisition of knowledge.

Whether this is possible or not, it indicates, along with the description by Lichnerowicz, that there is only one mathematics -- the same for all people. By proper modification, with more concretization of abst[illegible]ct theories, and presented at a slower pace, two-thirds of the proposed program could well give a great mass of students intellectual satisfacti[illegible] and the knowledge needed to understand, and to succeed in, a competitive technological society.

Chapter IV

THE WORK OF SSMCIS

It is a stimulating and exciting adventure to construct a mathematics curriculum that will have a completely different organization from the traditional sequence. Here one is allowed freedom of selection and structuring of the content, freedom to create new pedagogical procedures for esperimental classroom use, and is challenged to look ahead to the needs of tomorrow. All of these conditions contributed over a period of nine years to a continuous and unrestrained enthusiasm that produced the Unified Modern Mathematics textbooks, teachers commentaries, and technical reports of SSMCIS.

However, it is another matter to report the work of SSMCIS so that the reader captures the spirit of the high professional interchange and creation of ideas that led to the final product. A report tends to become a list of the activities in a summary form, devoid of the arguments, disagreements, accordances, expectances, disappointments, elations, and determiniations of all the persons involved. This chapter will attempt to reveal not only the modus operandi but also some of the elan that made the adventure a successful one.

Initial Procedures.

By June 1964, mathematics curriculum construction in the U.S.A. had reached a climax in the work of the Commission on Mathematics, the University of Illinois Committee on School Mathematics, and the School Mathematics Study Group. All these groups made their reforms within the framework of the traditional organization of year long studies of the separate branches. The only contribution beyond these three movements was that in the proposals of the Cambridge Conference on Goals for School Mathematics. However, during the period 1958 to 1964, Europeans were committed to radical reconstruction of their mathematics curricula based on recently developed movements in mathematics itself. Among these were the School Mathematics Project in England, the work of G. Papy in Belguim, the new curricula of the Nordic countries and the work of Kolmogorov and Markewschevicz in Russia. The European thinking had failed to interest any of the reform groups in the U.S.A.

Accordingly, in October 1964 a proposal was made to the Federal Office of Education, for the support of a project to construct an entirely new organization of mathematical study for the school years seven through twelve to begin in school year 1965-1966. The objective, as stated at that time was:

"It is proposed to construct a new detailed syllabus for a unified program of mathematical instruction, largely algebra oriented, for the secondary school years, grades seven through twelve. This syllabus is to encompass all that is now considered a first year university program. From the syllabus it is proposed to write complete textual material for pilot class experimental teaching by scholarly and specially trained teachers. Concomitant, it is proposed to determine the teacher education in mathematics and mathematical pedagogy necessary to teach the new program, and to develop tests and testing techniques to determine the mastery of the materials to be taught. In effect, the objective is to realize part of the "Goals for School Mathematics" recently developed by the Cambridge Conference, June, 1963."

There is a great difference between the U.S.A. updating type of reform and one in which the entire curriculum is reconstructed from a global point of view - one which eliminates the separation barriers among the several classical branches, unifies the subject through its general concepts (set, relations, mappings, functions) and builds the fundamental structures (group, ring, field, vector space). The efficiency gained through such organization (and the elimination of outmoded parts of the subject) can permit the introduction into the high school program of much that was previously considered undergraduate mathematics.

In order to initiate the project, it was proposed to bring in the thinking of European as well as U.S.A. mathematicians, all of whom would be committed to the construction of a modern unified secondary school program for the scientific line - those students comprising the upper 15 to 20% of cognitive ability.

The Office of Education after receiving the proposal favorably, suggested some modifications of the procedures and budgetary provisions. An amendment to the original proposal was made on March 9, 1965 and the inauguration of the project was delayed by one year. Appended to the proposal was a section of the OECD report of the Athens' working session on Teaching Methods of School Mathematics, pages 49 to 66 outlining a Modern Assignment of Mathematical Subject Matter for Scientific Sections of Secondary Schools.(1) This appendix formed the basis for the initial work of the project. Final approval was given to the proposal in September 1965, the initial phase to operate for one and one-half years.

The Project Commences

Since the operational scheme of this initial one-and-one-half years phase set the state for all subsequent years of operation, it is reported here in some detail. Many

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(1) Mathematics Today - A Guide for Teachers. OECD. Paris, 1964.

groups worked cooperatively. First there was an Executive Advisory Council of seven leading mathematicians and educators. This Council held its first conference in December 1965 in a two-day session to prepare the material for what became an annual June conference. A second group consisting of approximately twenty mathematicians, educators, and teachers held a two to three week conference each June for seven consecutive years preparing a d[illegible]ed syllabus for the following year of experimental [illegible]ng. At the June conference the scope and sequence of the study were put in sufficient detail to permit the writing during the summer, of textual material for classroom teaching during the following school year. A third group were the summer writers who, during the months of July and August, produced the student textbooks for the experimental teaching in the next school year. Each chapter written by one person was reviewed by another writer and also by a mathematician from the June group in order to check its mathematical correctness, its pedagogical soundness and its agreement with the June conference proposal.

A fourth group consisted of research assistants and typists who prepared the textbook manuscripts for offset printing. After each page was checked and the figures drawn, it was further edited by Meyer Jordan, one the the mathematical consultants. His careful work year after year not only eliminated errata, but insured careful rephrasing of much of the writing to make it more readable as well as correct. The complete books were ready for classroom use at the start of the school year.

A fifth group consisted of 20 to 24 classroom teachers who, in each successive summer took an intensive six week course of 100 hours of study in both related subject matter (as a background) and a pedagogical interpretation of the curriculum they would instruct during the coming school year. During the first year of classroom experimentation with each course there were two teachers in the classroom. Thereafter the usual one teacher to a class prevailed. These teachers were taught by highly competent professors from group two above.

A sixth group were students selected to study the program. All students in the program were allowed to elect it or to remain in the traditional classroom. The students were selected on the basis of their elementary school achievement, a standardized mathematical test score and an intelligence test score, to insure that they would be in the upper 20% of intellectual ability.

A final group consisted of the director, research assistants and selected members of the advisory Council who throughout the year observed the classroom teaching, conferred with the classroom teachers and students, and during the year held four all-day Saturday conferences with all the teachers involved in experimental teachings.

The overlapping of membership in these groups conduced to an understanding of the problems and difficulties in adapting the textual material to student learning. All groups - students, teachers, writers, educators and mathematicians - held high respect for each other and all listened attentively, though not always with agreement to the criticisms, suggestions and desires of each group. It was this high level of cooperativeness and willingness of compromise in controversial areas that led to the successful development of the program.

The pattern thus set became the modus operandi in all the successive years 1967-1972, namely

a) A late fall or early winter Advisory Council conference, 2-3 days.

b) A winter and early spring of preparatory work by the directors office.

c) A June planning Conference of the syllabus, 2-3 weeks.

d) A summer-writing and producing textbooks, 2 months.

e) A summer training program for 24 teachers, 6 weeks.

f) A year of experimental teaching, observation, and evaluation conferences.

The first meeting of the Advisory Council was held November [illegible]. 1[illegible]. At this meeting arrangements were made for the first planning conference to be held in June 19[illegible]. Among the important outcomes of the Advisory Council meeting was <u>the preparation of a position paper</u>.

It was agreed that a position paper should be one that would, with a minimum of debate or discussion, serve as a basis for the work of all subsequent conferences. Among other items, the position paper stated

a) The preamble to the research grant stating the objective of the experiment. (See page 80).

b) The basic knowledge expected from grades K-6 study on the part of the entrants to the experimental classes.

c) The objectives of the study of high school mathematics.

d) The scope and sequence of the subject matter to be taught.

e) The group of students for whom the curriculum is intended.

f) The training and quality of teachers in the program.

g) The stand we shall take with regard to geometry instruction.

h) The presentation of number theory and polynomials, as well as an early introduction to the calculus.

i) The nature of evaluation of the curriculum - the preparation of tests.

j) Meet controversy that might be expected, with reasoned compromise.

All members agreed that the paper should attempt to avoid the unnecessary use of sophisticated language. Among the suggestions for preparing the position paper, the following remarks seemed pertinent:

"I think that we should emphasize the concept of operation (mapping, function, transformation) as basic in algebra, geometry, and analysis, treating algebra as the theory of operations and emphasizing groups and homomorphisms. In geometry I think we should develop the concept of geometric transformation (operation) and use it systematically as a tool in making geometrical constructions and geometrical proofs. As for axiomatic geometry I am convinced that we should follow Artin (for the affine structure) and Choquet (for the specialization of Euclidean geometry). The theory of real numbers would be developed on the basis of algebraic and geometric results and should not be divorced from geometry or simply parachuted into the axioms for geometry."

"We are writing for a highly capable and interested body of students. At a subsequent period, after the initial experiment, we can consider what modifications would be made for the less capable. The classes must be taught by highly capable and well-trained teachers. There must be no binding whatsoever to any traditional program. We are free agents, and should not attempt to modify the program to admit students at a subsequent level who have not pursued the experimental program. In this manner the experiment can be tightly controlled."

"To evaluate the experiment, we should not attempt control against a traditional achievement test. We should develop new special tests, which test the aims and objectives of the material to be taught; which test concept development and related problem solving, as well as the acquisition of manipulative techniques. We must use teacher reports based on questionnaires, interviews and attitude measures."

"Elementary Number Theory should be included in the program. It goes a long way toward developing the concept of theorem and proof, which traditionally was ascribed to teaching geometry. In this connection the role of proof in the program should be spelled out. Are we to teach formal mechanical logic, or to develop thinking about proof? Informal logic ties up nicely with set theory. First proofs should not be of the obvious, but of unexpected results."

"We must take a firm stand on geometry. It should be headed toward coordinates and vector spaces. One purpose in geometric instruction should be the development of intuition. While the Pythagorean theorem is important, there is a great deal of pre-mathematical value of relations in a plane which can be introduced without a formal proof, to give the notion of what a plane is like."

"The limit concept and steps toward the calculus should be spelled out so that ideas of the calculus can be introduced early. Thus continuity should enter the program at an early stage. If emphasis is to be placed on operations and transformations, then notions of mappings, relations, functions and permutations should be stressed. Numerical Analysis should thus be introduced before the Calculus, and not be delayed until after the Calculus is taught.

By May 15, 1966, not only was an acceptable position paper compiled, but also a detailed flow chart had been prepared by Professor Vincent Haag, showing the possible content of mathematics distributed over the 6 years of study from grade 7 through grade 12. These two items set the stage for the June 6 - June 26 conference (18 working days). The work proceeded as follows.

First day: Discussion, amendment and agreement on the position paper and flow chart.

Second day: Discussion and detailing of content with respect to grades seven and eight, that is, Courses I and II.

Third to fifteenth day: Preparation of detailed syllabi as follows: Total program, 3 days; Detailed syllabus for grades 7 and 8, 3 days each; Scope and sequence of Course I by chapters, 4 days.

Sixteenth day: Sequence and scope by chapters and topics for grade 7 and 8.

Seventeenth day: Detailed chapter outlines and assignment of writers for Course I.

Eighteenth day: Evaluation and procedures for extending the project in subsequent years.

Each writer reviewed his assigned chapters with the mathematician who was principally engaged in developing the chapter, in order to have complete understanding on the objectives, nature of presentation, types of exercises, and general format of the chapter. At adjournment on June 25 everything was set for two months of intensive preparation for the actual teaching.

The persons who took part in the Advisory Council and June planning conferences are listed in Appendix A.

Experimental Schools, Writing, Teacher Training.

The Metropolitan School Study Council is a collaboration of about 125 school districts in the New York metropolitan area. Twenty of these districts were selected as possible places for the SSMCIS experimentation because they had a relatively stable population, sufficient number of academically capable students, and teachers favorable to attempting new materials. By June 1, 9 districts (one with two junior high schools) had accepted the invitation to experiment in the project. All of these districts, with the exception of one at Carbondale, Illinois, were in close enough proximity of New York City to permit classroom visitation and observation by the SSMCIS staff. The textbooks and teachers commentaries were supplied free by the project.

Office space and library facilities were provided for the writers. This made it possible for the writers and reviewers to work at home, in their personal offices, or at Teachers College, as well as to keep constantly in communication with each other. At regular intervals all the writers convened in a plenary session to raise and answer questions with regard to sequence, repetition of the same material in different chapters and to omissions that would handicap the teaching of subsequent topics. These sessions also acted as a time-guide on completing the writing so as to have the textbook ready for the opening of school in September of each year. Each course, from the beginning contained two or three added chapters so as to be prepared for any misjudgment on the amount of material that could be learned in a given year of study. In subsequent years, as adjustments were made the revised courses contained only those chapters which could reasonably be covered in the regular school year. The exercises in the textbooks contained a sufficient number of more challenging problems to accomodate the students who could cover the course in less than the expected time.

The training of the teachers to teach the unified program was at first a difficult task. In the first place, before their summer training began they had no notion of the nature of a unified program. The ideas of sets, relations, mappings, and matrices had never been in their traditional training courses. They tended, as teachers in general do even today, to teach the same subjects in the same way they were taught. It was difficult for these adult experts to adjust to new concepts, new operations, and new organization of the subject matter. In the second place, it was difficult to adjust to new goals or outcomes to be expected from their students. With greater stress put on conceptual teaching and with less stress on memorizing stereotyped operational algorithms, a new pedagogical approach to learning had to be acquired. In the unified program, learning by discovery, exploration, seeking patterns and making abstractions and generalizations from situations called for a larger share of intellectual activity on the part of the students.

It was necessary then, that the instructors of the experimental classroom teachers be selected so as to exhibit the same characteristics that would be demanded of the experimental classroom teachers in handling their students. Fortunately such professors were available, one of whom was an internationally outstanding educator in mathematics, Professor Willy Servais of Belgium. He demanded a high degree of scholarship, rigor and comprehension in his instruction, while showing great sympathy with the mental strain to which his teachers were put. Year after year he won their confidence and respect as he molded them from junior high school teachers into senior high school teachers by presenting university level unified mathematics for their study.

Besides the advanced mathematical knowledge, it was necessary to develop within the classroom teachers a pedagogical theory for presenting the new unified concepts to their students. There was a real need to break away from traditional goals and methods, and to establish new targets through the use of the textbooks their students would use. The teachers had to study the texts and work all the exercises which in the following year they would expect their students to do. Among the writers, there were several college professors of mathematical education who year after year in the summer teacher training programs developed the theory of inquiry, learning by discovery, examining situations, and the building of strucutral concepts.

It was in this manner that the SSMCIS brought about a dual reform in mathematics education: (1) complete reorganization of the scope and sequence of mathematics to be learned so as to conform to the late twentieth century concept of the nature of the subject, and (2) complete switch around from purely skill-centered learning, learning on behavioristic and rote memorization characteristics, to one based strongly on structured and conceptual approaches to learning.

The foregoing account may make it appear that everything ran smoothly and in harmony. This was not the case. Many times there were very serious disagreements among the mathematicians, between mathematician and writers, between writers and the classroom teachers (and presumable between teachers and their students). These differences had to be resolved. Thus there was a difference of judgment on whether to present a topic in rigorous and formal style, or to be informal, but correct; there were quite contradictory view-points on the teaching of logic-truth tables for example. The affine geometry program was at first considered too difficult by the teachers, but by the third time of teaching, it was accepted as a first rate mathematical study. The teaching of the calculus brought into clash three opposing theories - the traditional limit approach, the use of infinite series and approximations approach, and the continuity-first approach, the last of which was agreed upon.

The approaches to topics were correct, and could be made as rigorous as desired. But pedagogically, to the extent that it could be determined in these experimental classes, some approaches were more suitable to certain age groups than others. For example, during its first year the program attempted to develop a fairly rigorous approach to the integers as equivalence classes of ordered pairs of natural numbers. At the end of 4 weeks of teaching the students (age 12 years) were still mystified. The next year the integers were introduced informally by motion on the number line and in two days the students were operating on them in a meaningful manner. Again it is apparent that mathematics is indeed a discipline but mathematics education is not. It is a practical and theoretical study of what to teach and how it can be effectively learned.

The procedure described above was repeated each year from 1966 to 1972. The first year was given over to Course I, the second year to revising Course I and creating Course II, the third year (1968-69) to finalizing Course I, revising Course II and creating Course III, and so on. By 1971, the first three courses were in final form and placed in the public domain, by 1973 all six courses for grades 7 through 12 had been established as a superior program of mathematics for capable students. The title of these tests is Unified Modern Mathematics. A chapter by chapter list of contents for each course is given in Appendix B.

Evaluation

The success or failure of an experimental program cannot be based on observational judgment alone. There must be some form of testing outcomes against pre-determined targets. Since the explicit goals of SSMCIS differed greatly from the traditionally oriented programs, it is evident that the use of standarized tests in algebra and geometry would not provide the measures of attainment expected of SSMCIS students. It became necessary to create tests geared to the content and goals of unified mathematics.

These tests were used for two purposes. The initial and continued activity was content-formative in nature, that is they were used basically to improve the product of the project. These tests were constructed by the staff of SSMCIS with the help of the classroom teachers, and administered at the end of each unit or chapter of study. These tests revealed the difficult material, the easy subject matter, the needed revisions etc. and aided in making the revisions of each of the courses.

After three years of experimentation a new phase of testing - summative in nature, that is assessing the quality of the outcome - became necessary. In 1969 and 1970 the SSMCIS students were entering the senior high school and competing with students in the established advanced placement programs. They were obliged to take the College Entrance

Examination Board Mathematics and Scholastic Aptitude Tests as well as Regents tests if they were New York State students. The New York State Regents cooperated by allowing the SSMCIS staff and teachers to create a special 10th year Regents Examination. The results were excellent and the next year (1972) the Mathematics Department of the N.Y. Regents worked with the teachers of the SSMCIS program to create a 10th and 11th year test in unified mathematics. Again the results of the tests were excellent, and in fact so high in quality that beginning in 1973, all teachers of SSMCIS classes in New York State were allowed to make their own tests, to rate them, and to give Regent credit to all students doing satisfactorily on the tests.

To determine the ability of SSMCIS students in C.E.E.B. examinations, the SSMCIS staff, cooperating with the Educational Testing Service's Mathematics Department undertook three studies: (1) the administration of a pre-test form of level II Mathematics Achievement Test in Spring 1971; (2) the administration of the Preliminary Scholastic Aptitude Test, Mathematics Section, in October 1971; (3) the comparison of 339 SSMCIS 11th grade students with the same number of comparable advanced placement group students taking the PSAT in the Fall of 1972. In all three studies the SSMCIS students had superior results, in the last one at the 0.001 level of confidence. All the foregoing evaluation proved the quality of SSMCIS program. It enabled our students to compare favorably on traditional type examinations with superior students who had received a traditional high school mathematics education. In addition the SSMCIS students learned a great deal more mathematics that is not as yet included in CEEB examinations.

The question of arithmetic computational skills may take on a new interpretation with the advent of the electronic mini-calculators. However in 1970-71 the Stanford Advanced Arithmetic Achievement Tests were administered to SSMCIS and comparable non-SSMCIS classes in grades seven and eight in Montgomery County Schools of Maryland. Using pre- and post-testing, gains were found for all students but gains favored the SSMCIS students. Thus the stress on concepts and reasoning showed no bar to gaining requisite skill in computation. The quality of the SSMCIS product was validated.

Alongside of the cognitive aspect of development in school learning, there is always the affective domain that is to be considered. Right from the start SSMCIS was concerned with the attitude of its students toward the subject in general, toward the courses (books) they were studying, and toward the type of teaching they were experiencing. The difficulty of measuring attitudes was recognized at the start. At first there was concern for the formative aspect. It was necessary to present those topics that students liked and in a manner that attracted their interest. Capitalizing on this, it would be easier to present that mathematics which was essential for further application and further study.

A preliminary Student Opinion Survey (SOS) was developed and administered in all experimental classes in 1969-1970. The test contained the following section: (1) a list of chapter headings in each of the Course I through IV for which the students were asked to rank the three they enjoyed most, and the three they enjoyed least, (2) a set of statements concerning the courses and the textbooks to be rated agree or disagree; (3) a list of statements concerning attitudes toward mathematics to be rated 1 to 5 from strongly disagree to strongly agree. The results of these tests enabled the writers and teachers to see the relative judgments of the students' affectation to mathematics.

During the year 1972-1973 the original SOS was modified for readministration to both SSMCIS classes and non SSMCIS classes. Two forms A and B were constructed of items used in the original SOS, along with items from Aiken's Revised Math Attitude scale and the scales used in the International Study of Achievement in Mathematics. The test was administered to 1160 SSMCIS students and 862 non-SSMCIS students in accelerated classes. In general, there was no difference between SSMCIS and non-SSMCIS attitudes toward mathematics. However the SSMCIS students appear to have formed a more favorable and more mature outlook toward specific aspects of the subject. In many ways, their viewpoint is more modern then that of their compeers.

The complete results of all the evaluation studies appear as technical reports 1 through 14, issued by SSMCIS. The conclusions permit it to be said that the Unified Modern Mathematics of SSMCIS is well within the complete mastery of the upper 10 percent of academic ability, and in a somewhat modified form can form the basis of a mathematical education for the upper 20 percent of mental ability. The attitude of students in these courses is slightly more favorable toward the subject than that of equal ability students in the traditional program. It is significantly more favorable to modern and structural aspects of the subject.

Innovation.

In a country with a strong central Ministry of Education, introduction of a new curriculum is either impossible or easy. If the ministry does not favor the curriculum it is simply not permitted in the schools. If it favors the curriculum it is introduced into all schools by edict and becomes the program of the state. All book publishers in the state create a new series of publications to meet the new curriculum requirements. But in a democracy with no central or federal ministry, as in the United States of America, introduction of a new school program is more complex and difficult. Even though most states have a Department of Education, these departments usually act as advisory agencies on matters of external management, and seldom if ever impose a fixed educational program on all of their school districts. Each school district in the U.S.A. (and there are tens of thousands of them) is a law unto itself in fixing the mathematics program for its school constituents.

Most districts follow the programs set by the major publishers of textbooks. These publishing firms, select authors, or teams or writers, to produce a program in mathematics education that (1) follows a national trend- not too radical - that teachers will most likely feel at home with and (2) will make the sales of the books and required materials a successful financial adventure. Hence introduction of new programs by publishers is a rare event, all the more so since in the near past, a few publishers have not been financially rewarded by such publications. The classroom teachers tend to teach that content that they were taught and frequently in the same manner in which it was taught to them. Thus traditionalism tends to be entrenched in school programs. Any innovation in school mathematics needs the financial support of foundations (or government) as well as the professional support of the mathematical world.

A number of procedures were used by SSMCIS - Informational Bulletins sent to leaders (about 1500) in mathematical education; observation of the program in action in the experimental classrooms; informational conferences, lasting from 1 to 5 days, at various parts of the country; placing the first three courses in the public domain, using NSF and other workshops and institutes to train the teachers; capitalizing on large districts that were willing to show leadership; urging publishers to use the material in the public domain with their own team of authors.

Success in One School. In 1966, SSMCIS invited a number of schools on Long Island, N.Y. to participate in its experimental program. The only school to accept was the Alva T. Stanforth Junior High School at Elmont. The experimental class had about 35 students. Every year, in the spring, a Long Island Mathematics Fair is held for students in grades 8 through 12. In 1968 the students from 7th and 8th grades SSMCIS competed for entry among themselves and the school entered 25 candidates at 8th grade level. At the first elimination contest in Nassau County from 125 8th grade contestants 21 bronze medals were awarded and 16 went to Stanforth students. At the second elimination of bronze medalists from both Nassau and Suffold counties in a field of 45 contestants, 11 were awarded silver medals and allowed to enter the final round. Ten of the 11 finalists came from Stanforth. No school in the history of the Fair had ever had such an overwhelming majority in the final round. The winner of the final round, awarded the gold medal was Ronald Quartararo, a student at Stanforth. The Monday following this award, more than 30 administrators from Long Island called the SSMCIS office seeking admission to the SSMCIS program. The program spread so rapidly that by 1974, a full day SSMCIS Experience Sharing Conference was organized and financed personally by 425 teachers from the three Mathematics Teachers Associations of Long Island. The program is well established on Long Island and growing.

Information Conferences. From 1970 on a number of conferences of short duration were organized to inform superintendents, supervisors, and mathematics teachers of the nature of the SSMCIS program, and the needs and manner for innovating the program. At these conferences there were lectures on the nature and purpose of the SSMCIS program; class demonstrations of the teaching, the conferees doing study and examples from sample textbooks, discussion groups on what teachers must know, and the manner in which in-service teachers could be educated to innovate the program; and the like. The size of these conferences ranged from 60 to more than 100 participants. In all the regions (except one) where these conferences were held, the program has been introduced into some of the schools.

Large City Centers. In large metropolitan areas there are opportunities to organize experimental classes with qualified teachers for a number of reasons. Such areas usually have one or several outstanding supervisors, as well as some teachers highly competent in subject matter and teaching skills. The student population is large enough to find a sufficient number of classes of the type required for the new program. There are University Departments of Mathematics and Mathematics Education to which the city can look for guidance and cooperation. These conditions existed at Los Angeles, Philadelphia and New York where the program has not only been established for the better mathematics students but it is also being modified to adapt to a larger part of the student body. In Utah, it was a state venture involving both North and South Utah districts.

Publication. The final editions of the experimental textbooks were published by Teachers College Press because the cost of printing was underwritten by the project. These textbooks were published as paperbacks and at a reasonable price (to recover printing costs). However paperbacks do not last well with student handling and hard cover books are preferred by school districts. All the courses were advertised to be given to the private publisher with the best offer, but of 19 publishers approached none accepted. All publishers praised the high quality of the program, but were afraid to take the financial risk of publishing the series. Only one publisher produced a limited number of its own commercial rewriting of the first four courses. Commercial publishers, in general, tend to perpetuate the traditional and to avoid any radical change of their past successful publications.

N.S.F. Institutes and School In-Service Workshops.

The greatest source of power for innovation is a body of knowledgeable and enthusiastic teachers. Since 1968 the National Science Foundation has supported summer institutes to train teachers specifically to teach Unified Modern Mathematics. In 4 to 8 weeks the teachers were given an advanced mathematical background and a thorough study of the SSMCIS textbooks they would teach. When there were no nearby institutes, or institutes were not available, school districts established in-service workshops carrying out the same programs as the institutes. Thus in each of the six years 1970-1975, approximately 200 teachers were trained to introduce the program into their classes. From these institutes there have come forth well-trained persons to carry on workshops in their own schools or districts.

Translations. The SSMCIS program has received attention around the world. The supervisor of mathematics in the Israel Ministry of Education requested permission to translate Unified Modern Mathematics into Hebrew for use in his country. As a result it is being used by the schools in all major Israeli cities. When UNESCO began its Arab Mathematics Project, it requested the use of Courses 1 to III. As a result, the material of SSMCIS is reflected in the new Arab program. In 1973 a group of 50 Japanese mathematics teachers visited the SSMCIS for a full day of briefing on our program. As a result Courses I to III have been translated into Japanese and published in 1975 for use in junior high schools in Japan.

Outstanding Teachers. Scattered throughout the U.S.A. in each of hundreds of small school systems there exists at least one mathematics teacher, well grounded in mathematics, responsible and well prepared for directing the mathematics education of the system. These teachers attend regional and national conferences on education and bring back to their communities the latest and the best thinking. There are a number of these teachers - in states across the union - that have introduced unified mathematics for their better students. The real hope for future innovations lies in an abundance of such knowledgeable teachers. No program, no matter how good, can succeed with ignorant and uninterested teachers.

So, as of September 1975, there is estimated to be about 80,000 students studying unified mathematics as developed by SSMCIS. These classes involve about 3000 teachers. This is a relatively small number of students compared with the number having the capacity to do the study successfully. With an estimate of 10,000,000 students, the upper 20% would include 2,000,000 students in grades 7 through 12 all of whom could profit by the serious study of mathematics of an advanced contemporary sort. By keeping them in an outmoded 7th and 8th grade program and restraining them in a traditional algebra-geometry-trigcnomerty senior high school program from learning the mathematics they will need in the world of tomorrow we are placing an enormous block on the creative minds needed for the future development of our country.

Chapter V

MATHEMATICAL EDUCATION AS A SOCIAL ENTERPRISE

At every crossway
On the road that leads to the future
Each progressive spirit is opposed by
A thousand men
Appointed to Guard the Pass

Maurice Maeterlinck

The development of Unified Modern Mathematics by SSMCIS has resulted in an educational program for the upper 10 to 20 percent of academic ability, that is challenging and gains the student about two years in mathematical maturity over the traditional U.S.A. secondary school study. It is intended for a six-year sequential study through grades seven to twelve. However, not all schools are organized in a six year junior and senior high school structure. There are schools with 7 or 8 years elementary school followed by a four year high school. The teachers in the seventh and eighth year classes are frequently elementary school teachers with little training in mathematics beyond one or two years of algebra and geometry. There is need then to modify the SSMCIS program to begin in grade 8. or even in grade 9. Such modification, without loss of advance has been made in specialized high schools - e.g. the Bronx High School of Science in New York where the ability of the students is at a very high level.

There have been other successful projects for the good academic students e.g. those at the University of Illinois and the School Mathematics Study Group. Also much attention has been given to the so-called slow learner - the academically handicapped, the disadvantaged and those who have had great difficulty with computational arithmetic. However, in all these educational endeavors, the problem of creating a viable and useful mathematics study for the great majority of the students - the middle 60 percent in academic ability - has been in abeyance. This might well be the next significant undertaking for improving mathematical literacy in the U.S.A. European nations have already embarked on this study.

Outside of business mathematics, the mathematics curriculum in secondary schools has been dominated and dictated by university entrance requirements. The only consideration given to non-college bound students is to teach this college preparatory mathematics (perhaps) at a slower rate and with less accomplishment. The actual needs of these people when they enter the adult working and social world have never been investigated in a systematic manner. For example they study algebra merely as propaedeutic to studying more algebra -- but where they will ever use or need it, as contributing members to society, has not been investigated.

There can no longer be educational policy and goals or mathematical targets for school teaching that are apart from the entire life of our culture. Political goals, societal desires, industrial - economic demands must and will have great force in establishing educational goals and how they are to be achieved.

Some major issues are:

(1) Societies are looking upon their people as a reservoir of talents and skills, which are more widely distributed throughout the population as a whole then formerly believed. The schools are to select and develop these abilities in all their students. The level of education must be raised and will be raised.

(2) There must be a body of knowledge common to all people which forms a core of understandings essential to good citizenship. There must also be opportunity for individual talents to be realized in their fullness.

(3) Our education system must provide for change and reinterpretation. There must be built into our teachers and their teaching, and into the students and their learning, an orientation to change, so that learning will continue throughout the rest of their lives. This has real implication for the type of special skills we teach as opposed to the

development of conceptual thinking.

The forces compelling us to change and restructure our mathematical education for the masses are:

(1) Demographic: All children, in increasing numbers are attending school longer. We must teach more mathematics to a greater and more varied clientele. The learning and teaching require optimization for all students according to endowed or acquired ability, with little or no failure.

(2) The expansion of knowledge requires selection and continuous change in our curriculum. There is too much knowledge to teach all of it.

The explosion in knowledge continues, but our ability to organize it into the school curriculum has not grown as rapidly. We need much more intellectual activity and financial support in fundamental learning and curriculum research.

(3) Education must be related to the outside world, i.e. out of the school walls. Society with T.V, radio newspapers, organization, etc. is a powerful educational force in itself which we cannot ignore as we make our school education programs.

(4) Scientific and technological developments are transforming our societal habits, beliefs, and action. Knowledge previously restricted to the academic elite, is now demanded by all persons. Social strata are being compressed into an ever narrowing band.

There are shifts in educational philosophy that demand a complete change in the manner in which we must present our subject to the masses.

(1) There is a shift of conception of knowledge as an exploration of a fixed pre-existing domain to the concept that knowledge is limitless and capable of indefinite expansion. Knowledge is creatable. This opens the field for study (and research) to all disciplines -- especially those previously forbidden by religious, moral, and ethical authoritarianism of the past. It diverts the attention from the mere accumulation of facts (information) committed to memory, to a search for general principles and concepts for processing problematic situations.

(2) There is greater concern for utility rather than knowledge for its own sake.

(3) There is an increased orientation in education toward the future rather than toward the past. Education must be a power to bring about social, political, and economic changes while continuing to conserve the important traditional institutions.

(4) Trans-, multi-, or pluri-disciplinarity is becoming essential as the humanities, science, and mathematics unite in proposing and solving problems of modern society (ecology and pollution, for example). Such core, or integration of edu··tive material is in its infancy. Thus, not only must ou: ·thematics become a cross-branches

program -- one mathematic -- but mathematics and science must be a more integrated program.

The emerging pressure of political, social and economic desires of the people at large; the overview of how these desires are related to what goes on in the schools; the forces of educational change, and the shift in conception and purpose of knowledge, all necessitate a procedure for constructing on a more scientific manner than heretofore, a purposeful program of common core mathematics for the great majority of our students.

Factors in Building a Curriculum.

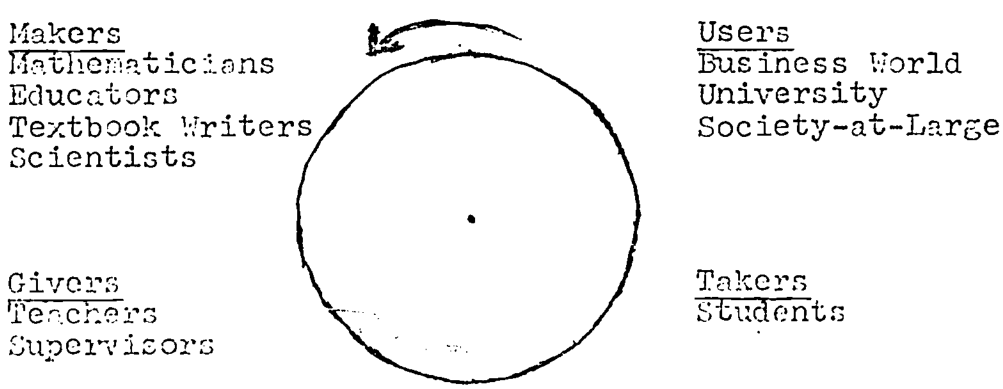
With the foregoing issues as a background we can now address ourselves to the problem under discussion -- namely, what is the role of mathematics in the general education of our youth? First of all we must ask what society expects our schools to do. In the past many idealistic goals have been set for schooling to accomplish, for example, our graduates were to have achieved worthy citizenship, worthy use of liesure, worthy home membership, etc. It did not take a long time to learn that schools could not produce this product. But there were fundamental objectives that schools could obtain, most of which were effected by intellectual development and formation. Thus the school's main target is to develop literate persons, people who can read, write, speak, and reason about life affairs, who have acquired a set of concepts and skills that are useful in engaging in the "vie courant", and who have developed the attitude and mental power to acquire new knowledge and new points of view. We can today still maintain this goal as the prime purpose of school education. It appears as tenable for the future also.

Secondly, we should clarify what meaning we give to the word "general" in referring to education. One meaning of the word is "liberal" in the sense that a general education develops a scholar or a man of letters. Another meaning is "for all persons" as a "common body of knowledge" everybody should possess. A third similar interpretation is that education which reveals all that permeates our cultural domain. In recent years, it has taken on a more or less vulgar connotation of education for the "less capable" than college preparatory students appearing in such titles as general science, general mathematics, general business practices, and so on. It is suggested here, that the word be used in relation to mathematics as that type of mathematical education which the person who has acquired it will find useful and operative in our society today, and insofar as can be seen, tomorrow also. The word "useful" is used not only in a vocational sense, but also as an instrument for interpreting and understanding a scientific, technological, industrial, and economic culture and the advancement of knowledge. If it is so used, then the word "general" takes on a different significance for each of several levels of intellectual ability found among school children, and provisions must be made to give that type of general mathematical education best suited for each level.

Thirdly, we should investigate who decides what mathematics should be taught. We could well investigate who decided our present mathematics program and on what basis it was done. Surely it was dictated largely by university mathematicians on the basis of propaedeudic material for the continued study of mathematics. This happened when almost all secondary school graduates went on to university study. So we ask, for this group, what mathematics should they study, for what purposes, and who makes the decision. It should be noted that curriculum development is not a cold, objective, scientific endeavor with correct and exact content derived from rigorous research. On the contrary it is an expression of social, political and pedagogical goals in conformity with all the goals of the education process. As such, it is greatly influenced by the value assumptions of all persons involved in it. This is reflected in the orientation of the developers as exemplified by their own attitudes toward objectives and evaluation, and who they are--mathematicians, psychologists, educators, generalists, teachers, parents, etc.

There is a growing resistance by teachers to accepting 'new' curricula handed down to them by authorities from above. All over the world, teachers are being swamped by new projects, and they are no longer satisfied to be at the end of the production line. They want to be in; they want to be heard. Where curriculum development begins is a very sensitive issue, and it should be treated as such by all those who claim to know how change takes place. It must be admitted that there is no one way to make a syllabus. No teacher should fear -- nor should anyone hope for -- a technocracy of curriculum development that would threaten the essential human relationship between a young student and his teacher, who is for him the primary interpreter of what adult society is demanding.

We can represent all persons involved in a circular diagram as <u>makers</u>, <u>givers</u>, <u>takers</u>, and <u>users</u> which returns again to the makers. The <u>makers</u>, those who decide the content, must be the pure and applied mathematicians, educators, textbook writers, and generally those who know what mathematics is and where and how it is used. The <u>givers</u> are



the school teachers and supervisors who administer the syllabus. The _takers_ are our students who learn what is presented to them. The _users_ are society at large, the business-industrial world, the universities, and all those who need mathematical competence to carry on. The circle then goes to reform by the makers in a continuous process that induces necessary change as societal demands change.

Goals.

All persons involved in the educative process must sooner or later recognize quite clearly the goals, targets, or objectives of mathematical study. This applies to the makers, givers, takers, and users, although each of these may see the goals in a different light or to a greater or lesser depth. But no one can be genuinely motivated in constructing, teaching, learning, or using mathematics for which no reasonably accepted purpose has been justified. These goals can be stated _generally_, or as in behavioral learning, _objectively_. In most cases it clarifies our thinking to have both kinds of statements. Let me first state three general objectives.

1. To the extent that it is possible for each individual, all secondary school students (grades 7 through 12) should learn to comprehend the manner in which mathematics is conceived of and used today. By the way we teach and what we teach, the human mind should be developed in its capacity

to understand and to interpret numerical, spatial, logical, and probabilistic situations occurring in our cultural milieu. When the students become adults they should approach problems with a scientific and questioning attitude -- looking at a situation. seeking a mathematical explanation, and possibly referring it to a mathematical theory. All persons must come to know what mathematics is. as conceived of in the last quarter of the twentieth century, what ideas and materials it deals with, what type of thinking and reasoning (not only axiomatics) it uses, how it accomplishes results, and particularly how it is used in other disciplines.

2. Mathematics must have an "information and skill" dimension. Our students should learn that mathematics that appears essential, inherited from one generation to the next, along with skill to apply it. This knowledge and skill can be acquired during the process of developing mathematical thinking. This goal permits us to cast out of our teaching much that, while important years ago, can no longer be considered useful.

3. Finally, it is the usefulness of subject that has maintained it as a required school discipline. Our pupils should develop the capacity to solve problems and to construct, or at least interpret, mathematical models of physical, economic, and other scientific situations.

These three general objectives can operate as a guide for more specific ones. Various groups have attempted to list such goals un "behavior outcomes" (see Bloom's taxonomy). Here I shall merely exemplify their nature.

1. To develop, so that they can be recognized, described, and exemplified, the following concepts. Here one lists every concept to be developed, for example, set, union, intersection, Venn diagram, relation, function, operation, etc.

2. To develop, so that they can be applied wherever they occur, the following skills. Then one lists in detail every computation skill of arithmetic, every algebraic manipulation, all geometric constructions, and so on.

3. To develop problem solving ability of the following type: Then one lists: to translate a word situation into a mathematical expression, to formulate a mathematical explanation, to construct a mathematical model, etc.

4. To develop mathematical thought. Here one lists: to use mathematical induction. to give a proof in an axiomatic system, to apply the rules of mathematical logic, to apply infinite processes, etc.

5. To develop the ability to do independent study: Here one assigns simple research topics, for example the numbers π, e, γ, or the properties of a group ; To make a project for a mathematics fair. to do a chapter in the textbook all on one's own.

These five behavioral items are the most frequently named -- concepts, skills, problem solving, mathematical thinking. and independent study. However, most good mathematics teachers add another.

6. To develop, to the extent possible, an appreciation of the power and beauty of mathematics. This goal may be attained by revealing the nature of symmetry, of transformations, of structures -- group, field, vector space, -- of isomorphism, periodicity, etc.

What Mathematics for General Education.

In the past -- geared to a nineteenth century culture-- the school program consisted of arithmetic, algebra, and geometry. The tremendous advances made in scientific and mathematical knowledge and their application to almost every aspect of modern living compel us to look anew at the mathematics we should be giving our students as a part of general education. For the purpose of relating mathematics to the above stated goals, it is advantageous to consider among others the following topics: arithmetic, algebra, probability and statistics, numeracy,(1) geometry, logic, and applications. There is no question that the teaching of arithmetic, especially computation with whole and rational numbers in decimal notation is an absolute essential now, and in the future, for every active citizen, far none. This is imperative in spite of the ever-increasing number of low-cost calculators and their use. Without this

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(1) "Numeracy" is a recently coined word to denote literacy in number and numerical processes.

knowledge of arithmetic and its uses in daily life, including percent, ratio and proportion, it is impossible to understand modern society regardless of its political setup. The acquisition of some minimum skill in computation and application of arithmetic to the solution of everyday like problems is essential for an individual to function in modern society. For the great majority of students, this must be acquired in early secondary school study.

The physical geometry of size, shape, position, and relations of figures both plane and solid. belong in the same category of everyday use. The ideas of parallelism, perpendicularity, congruence, similarity, length, area, volume, and distance, penetrate every description of the world we live in, no matter how small or large. They are indispensible for understanding and describing the real physical structure of the world about us.

Simple algebra, involving at most the rational numbers. occurs in the home, office, newspaper, and most occupations in the guise of formulas and graphs and again falls within the conceptual knowledge all persons should have. Here it is not special skills (as in factoring or simplifying expressions) that are important, but the ability to read algebra for the generalization or direction it igves to specific domains of work. Conceptual algebra is more important than mere manipulation. All the above content may be considered a concrete foundation upon which to build necessary mathematical content for common use. Further we must consider mathematics as needed by a citizen of the modern world.

Probability and Statistics

The number of applications of probability and statistics has grown greatly in the last 50 years. These applications are found everywhere -- in industry (quality control). commercial enterprises (decision making), agriculture (crop experimentation), politics and society (opinion polls), economics (cost of living indicies), etc. In the physical and behavioral sciences, the use of probability has greatly increased. Many countries conduct national lotteries.

In teaching science, it is no longer sufficient to develop only deterministic thinking; probabilistic thinking which dominates the phenomena of heredity, radio-active processes, astrophysics, etc. deserve attention in the school instruction of all students. In daily life one meets a number of hazardous situations (crossing traffic. contracting disease) and reports in all the news media all of which require a minimum knowledge of probability and statistics for their correct interpretation. In these media, graphs, histograms, rates, and percentages are used in reporting insurance data, incomce taxes, demography, traffic accidents, economic output and the like. In particular, opinion polls especially with regard to elections, and societal preferences, interest the entire public, which however, has little or no knowledge of the methods used in making, or of inferences drawn from, polls.

Misinterpretation of statistics is well known and it is dangerous. We must teach sufficient probability and sampling statistics to all students to make them literate in the subject. This includes the concomitant knowledge of sets, set manipulations, functions, and counting procedures, that are essential for the understanding and application of elementary probability theory, as well as the graphic and algebraic exposition of statistics.

Numeracy.

A hundred years ago commercial enterprises needed employees who could perform written and mental computation with extraordinary speed. Today there is no need at all for such performers -- the work is done automatically. But what is needed, and the need is growing, are persons who understand the theory (the algorithms) underlying a host of new uses of numerical caculation. The flowchart, the languages of the computers, the programming of problems become required knowledge for all persons -- partly because a significant sector of the mass of workers will find themselves engaged in such work for a lifelihood, and more to the point, it will be a part of the literacy of all people to understand a computer-automated era of civilization.

In this study one will require the concomitant mathematical study of number systems (whole, integers, rationals and perhaps the reals), of algorithms and iterative processes, of approximations, and of matrices as they are now, and will continue to be used in many disciplines. Every student must know with what, and how, computers operate, appreciate their services, and certainly understand what they cannot do. They should know that technology is man-made for the use of man in improving his condition of living. It is not made to enslave man in a technocracy. This is another point illustrating that mathermatical literacy is a must for mass education.

Geometry.

It was a tenet of traditional teaching of geometry that it taught the nature of axiomatic structure and of logical demonstration. All evidence points to the fact that these logical goals were rarely achieved, the most that occurred was some understanding and a repetition of so-called demonstrations of theorems. To achieve these ideals, for those students who may need them, we have simpler algebraic structures as well as local-axiom systems for study in secondary school. But the study of geometry from a modern point of view, has other social value.

The informal study of transformation (mapping geometry explains mirror images (reflection), enlargements and shrinkages (dilations). and makes symmetry, which is found in almost all organisms -- animal or plant -- stand out as one of the most descriptive of both concrete and abstract constructions. It is easily related to the arts -- music, dance. painting, sculpture, architecture -- and the total esthetic aspect of life.

It is intuitive geometric knowledge rather then rigorous axiomatic development of the subject that has the greater social significance. Thus graphs, coordinates, and geometric paradigms of physical, biological, and behavioristic phenomena contribute greatly to the common understanding of these great fields of explanation of the man and his world. In secondary school a small formal axiomatic presentation has value in showing the highest form of human reasoning in the sense that "Euclid alone has looked on beauty bare". All future voters whould experience some study of an axiomatic structure to know what it is that mathematicians do to check discoveries.

Logic.

Perhaps one of the greatest contributions to misunderstandings, and through it, failure to solve serious political economic, and humane problems is the lack of clear, precise, correct communication. To say what we mean, and mean what we say, and to do it without fear of misinterpretation is a difficult task. The logic expressed by natural language is frequently clumsy, and it can be aided by giving attention to simple mathematical logic and its application to expression in all the other disciplines -- especially in the use of the communication media.

The words -a. an, the, one, all, some, each, and every-quantify our statements.

Of greater impact is the use of implications in relation to statements Here simple truth tables give a graphic illustration as to what is meant by implication, bi- implication and the connectives "and", "or" and the negation "not". Finally, simple inference schemes are all that is required for drawing correct conclusions.

All this logic could be applied in structuring mathematical knowledge, but this is not the key point Logic is of value because it can be applied to any field in which rational thought occurs -- the many affairs in the life of every person -- as a means of checking decisions. Thus, it should also be a part of the instruction in the natural language -- where at present too much attention is given to literature and emotions and not enough to grammar, syntax, and rational discourse.

It should be evident from the foregoing that we are living in a society where mathematics can no longer be considered of value merely as a tool kit of special skills. It has become a cultural subject for all citizens who must use this knowledge as a means of realizing what the world is all about. If humanity is to advance to a higher state of living on this earth, it can no longer afford to have a huge segment of its population as scientific illiterates left far behind a small group of elite authoritarians. The population must be enabled to some degree, to understand what it is that society as a whole is doing in its uncertain quest for a better life. For the mass of students, to learn this mathematics will not be easy -- will not be play-- but for them it will be a hard study, one that contributes to mental formation, but it will accomplish for them especially a closer tie with those who create.

Applications.

All students down through the ages have demanded an answer to "What use is there for the mathematics we learn?" Certainly they deserve an answer to this legitimate question, if for no other reason than to motivate the continued study of our subject. Our teaching should usually begin with, and culminate in, some genuine or quasi-genuine applications to the world of reality. To this end the usual problems on ages of persons, the number of different coins in a collection, rowing up and down stream and so on must give place to a more current concept of the nature of application.

Mathematics enters into almost all the other disciplines to various degrees of depth and service. At least four stages can be recognized.

1. Computation-Recording. Here a table of numerical values is compiled, and a graph is constructed, or a given formula is used to calculate specific values of the function involved.

2. Descriptive. This is using mathematical language to describe a scientific event as in the use of the formula $Q = Q_0 e^{r}$ to indicate the decay of radio-active material. Generally, this stage uses mathematical formulae and graphs (e.g. the normal curve) to describe scientific phenomena without any attempt to show an isomorphism between the two disciplines.

3. Explanatory. Here mathematics is used to create a model of a scientific theory. There is an isomorphism established between elements of mathematics and of the science so that the mathematical model may be used to investigate further properties of science. It is this stage that is most important for secondary and university teaching.

4. Structural. This is the highest stage and marks the evolution of a discipline into scientific maturity. It is represented by research that is done by logico-mathematical abstract structures. An example is Minkowski's 4-dimensional

space as a means of explaining the nature of relativiistic space.

Conclusions.

What mathematics then shall be studied for general education? First, there is a fundamental knowledge of arithmetic, physical geometry, and algebraic formulation including matrices that each citizen must know and apply more or less routinely in his every work-a-day life. Today this knowledge must be augmented to include probability, statistics, rational discourse. numerical processes, programming and the computer. All of this knowledge should be presented as a unified body of knowledge based on common fundamental topics underlying all mathematics -- namely sets, relations, functions and operations.

It is the usefulness of mathematics that has maintained it, next to the mother tongue. as the principal discipline of school study. The content, its organization. and methods of teaching must exhibit this usefulness at every possible opportunity. For the great majority of students we must obtain this type of mathematical literacy if we are to avoid harmful social conflict in the years ahead. Today, a great chasm has arisen between those few persons who know. use and speak a scientific language and the vast majority who do not understand mathematics and even fear it. The chasm must be bridged. The future citizenry must come, with a modest degree of understanding, to know how mathematics operates in scientific explanation. A small segment of the population has always said of mathematics:

C'est plus belle parce-que c'est inutile!

If we teach proper mathematics in an understandable manner to the great majority they will respond by saying of mathematics:

C'est belle parce-que c'est si utile!!

While the substance of this chapter indicated a type of mathematical education for the great majority of students in our schools, it did not to set down a detailed program. For this purpose we need a national committee composed of mathematicians, mathematics educators, teachers, scientific and business personnel,educational psychologists, all concerned with this great section of the school population This committee must set goals of desirable outcomes, establish a fixed sequence (for a very mobile population), contact and coerce publishers to produce books for teachers as well as students, and create an evaluation procedure to measure progress - perhaps through existing organizations. It must be done if we are to improve the intellectural accomplishment of our next generation